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A regularity criterion for 3D micropolar fluid flows in terms of one partial derivative of the velocity

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Abstract. We prove a regularity criterion for micropolar fluid flows in terms of one partial derivative of the velocity in a Morrey–Campanato space.

1. Introduction and the main result. In this paper, we consider the following Cauchy problem for the incompressible micropolar fluid equations in $\mathbb{R}^3$ [9]:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla \pi - \nabla \times \omega = 0, \\ \partial_t \omega - \Delta \omega - \nabla \operatorname{div} \omega + 2\omega + u \cdot \nabla \omega - \nabla \times u = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \omega(x,0) = \omega_0(x), \end{cases} \tag{1.1}$$

where u, ω and π denote the unknown velocity vector field, the micro-rotational velocity and the unknown scalar pressure of the fluid at the point $(x,t) \in \mathbb{R}^3 \times (0,T)$, respectively, while u_0, ω_0 are given initial data with $\nabla \cdot u_0 = 0$ in the sense of distributions.

When the micro-rotation effects are neglected or $\omega = 0$, the micropolar fluid flows (1.1) reduce to the incompressible Navier–Stokes flows (see, for example, [25, 39]). Much effort has been devoted to establish the global existence and uniqueness of smooth solutions to the Navier–Stokes equations. Different criteria for regularity of weak solutions have been proposed. The Prodi–Serrin condition (see [16, 34, 38]) shows that any solution u for the 3D Navier–Stokes equations satisfying

$$u \in L^p(0,T; L^q(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} \le 1 \text{ and } 3 \le q \le \infty, \tag{1.2}$$

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is regular. Notice that the limiting case $u \in L^{\infty}(0,T;L^{3}(\mathbb{R}^{3}))$ was covered by Escauriaza et al. [10] in 2003. Later on, Beirão da Veiga [2] established another regularity criterion by replacing (1.2) with the following condition:

$$\nabla u \in L^{\beta}(0,T;L^{\alpha}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{3}{\alpha}+\frac{2}{\beta}\leq 2 \text{ and } \frac{3}{2}<\alpha\leq\infty. \tag{1.3}$$

In 2004, Penel and Pokorný [33] obtained a different type regularity criterion, which says that if

$$\partial_{3} u \in L^{\beta}(0,T;L^{\alpha}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{3}{\alpha}+\frac{2}{\beta}\leq 1 \text{ and } 2\leq\alpha\leq\infty, \tag{1.4}$$

then the solution u to the Navier–Stokes equations is regular. The same result can be found in [41]. Penel and Pokorný's work has been improved by some other authors (see, e.g., [5, 8, 24] and the references cited therein). It was already known that if one component of the velocity is bounded in a suitable space, then the solution is smooth (see Penel and Pokorný [33] and Zhou [40, 41, 43, 44]). Some of these regularity criteria can be extended to the 3D MHD equations by making assumptions on both u and b [4]. Moreover, He and Xin [17] derived some regularity criteria for the 3D MHD equations only in terms of the velocity field u, and they proved that if u satisfies either (1.2) or (1.3), then the solution is regular. Recently, Cao and Wu [7] proved that the condition

$$\partial_{3} u \in L^{\beta}(0,T;L^{\alpha}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{3}{\alpha}+\frac{2}{\beta}\leq \frac{3}{2} \text{ and } \alpha>3 \tag{1.5}$$

also implies regularity of the solution (u,b) to the 3D MHD equations. Later, Jia and Zhou [19, 20, 22] showed that if

$$\partial_{3} u \in L^{\beta}(0,T;L^{\alpha}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{3}{\alpha}+\frac{2}{\beta}= \frac{3}{4}+\frac{1}{\alpha} \text{ and } \alpha>2, \tag{1.6}$$

then the solution is regular. For more interesting component reduction results for the regularity criterion, we refer to e.g. [21, 40, 41, 43, 44].

Inspired by the above-mentioned works on regularity criteria of Navier–Stokes and MHD equations, particularly those of Penel and Pokorný [33], Cao and Wu [7] and Jia and Zhou [19, 20, 21, 22], we want to investigate a similar problem for the micropolar fluid flows (1.1). Very recently, Jia et al. [18] (see also [8]) proved the following regularity criterion:

$$\partial_{3} u \in L^{\beta}(0,T;L^{\alpha,\infty}(\mathbb{R}^{3})) \quad \text{with} \quad \frac{3}{\alpha}+\frac{2}{\beta}= 1 \text{ and } 3<\alpha\leq\infty.$$

Here $L^{\alpha,\infty}$ is the Lorentz space.

The purpose of this work is to improve the result in [18], and to prove that if the derivative of the velocity in one direction belongs to $L^{2/(1-r)}(0,T;\mathcal{M}_{2,3/r}(\mathbb{R}^{3}))$ with $0<r<1$, then the weak solution is actually regular and unique. This work is motivated by the recent results [19]–[44] on the Navier–Stokes equations and MHD equations.

2. Preliminaries and main result. Now, we recall the definition and some properties of the space that we will use. These spaces play an important role in studying the regularity of solutions to partial differential equations; see e.g. [13, 29] and the references therein.

DEFINITION 2.1. For $0 \le r < 3/2$, the space $\dot{X}_r$ is defined as the space of $f \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \le 1} \|fg\|_{L^2} < \infty,$$

where we denote by $\dot{H}^r(\mathbb{R}^3)$ the completion of the space $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\dot{H}^r} = \|(-\Delta)^{r/2}u\|_{L^2}$.

We have the following homogeneity properties: for all $x_0 \in \mathbb{R}^3$,

$$\|f(\cdot + x_0)\|_{\dot{X}_r} = \|f\|_{\dot{X}_r}, \qquad \|f(\lambda\,\cdot)\|_{\dot{X}_r} = \frac{1}{\lambda^r}\|f\|_{\dot{X}_r}, \qquad \lambda > 0.$$

The following imbedding holds:

$$L^{3/r} \subset \dot{X}_r, \qquad 0 \le r < 3/2.$$

Now we recall the definition of Morrey–Campanato spaces (see e.g. [23]):

DEFINITION 2.2. For $1 < p \le q \le \infty$, the *Morrey–Campanato space* $\dot{\mathcal{M}}_{p,q}$ is defined by

$$\dot{\mathcal{M}}_{p,q} = \Big\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^3) : \|f\|_{\dot{\mathcal{M}}_{p,q}} = \sup_{x\in\mathbb{R}^3} \sup_{R>0} R^{3/q-3/p}\|f\|_{L^p(B(x,R))} < \infty \Big\}. \tag{2.1}$$

It is easy to check that

$$\|f(\lambda\,\cdot)\|_{\dot{\mathcal{M}}_{p,q}} = \frac{1}{\lambda^{3/q}}\|f\|_{\dot{\mathcal{M}}_{p,q}}, \qquad \lambda > 0.$$

We have the following comparison between Lorentz spaces and Morrey–Campanato spaces: for $p \ge 2$,

$$L^{3/r}(\mathbb{R}^3) \subset L^{3/r,\infty}(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3).$$

Other useful comparisons are contained in [36], [35] and [37]. The relation

$$L^{3/r,\infty}(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3)$$

is shown as follows. Let $f \in L^{3/r,\infty}(\mathbb{R}^3)$. Then

$$\begin{aligned}
\|f\|_{\dot{\mathcal{M}}_{p,3/r}} &\le \sup_E |E|^{r/3-1/2}\Big(\int_E |f(y)|^p\,dy\Big)^{1/p}\\
&= \Big(\sup_E |E|^{pr/3-1}\int_E |f(y)|^p\,dy\Big)^{1/p}\\
&\cong \Big(\sup_{R>0} R|\{x\in\mathbb{R}^3 : |f(y)|^p > R\}|^{pr/3}\Big)^{1/p}\\
&= \sup_{R>0} R|\{x\in\mathbb{R}^p : |f(y)| > R\}|^{r/3} \cong \|f\|_{L^{3/r,\infty}}.
\end{aligned}$$

For $0 < r < 1$, we use the fact that

$$L^2\cap \dot{H}^1 \subset \dot{B}^r_{2,1} \subset \dot{H}^r.$$

Thus we can replace the space $\dot{X}_r$ by the pointwise multipliers from the Besov space $\dot{B}^r_{2,1}$ to L^2. Then we have the following lemma given in [28].

Lemma 2.3. *For $0 \le r < 3/2$, define $\dot{Z}_r$ to be the space of $f \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ such that*

$$\|f\|_{\dot{Z}_r} = \sup_{\|g\|_{\dot{B}^r_{2,1}}\le 1} \|fg\|_{L^2} < \infty.$$

Then $f \in \dot{\mathcal{M}}_{2,3/r}$ if and only if $f \in \dot{Z}_r$, with equivalence of norms.

To prove our main result, we need the following lemma due to [32] (see also [42]).

Lemma 2.4. *For $0 < r < 1$, we have*

$$\|f\|_{\dot{B}^r_{2,1}} \le C\|f\|^{1-r}_{L^2}\|\nabla f\|^r_{L^2}.$$

Additionally, for $2 < p \le 3/r$ and $0 \le r < 3/2$, we have the following inclusion relations [27], [28]:

$$\dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3) \subset \dot{X}_r(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3) = \dot{Z}_r(\mathbb{R}^3).$$

The relation $\dot{X}_r(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$ is shown as follows. Let $f \in \dot{X}_r(\mathbb{R}^3)$, $0 < R \le 1$, $x_0 \in \mathbb{R}^3$ and $\phi \in C_0^\infty(\mathbb{R}^3)$, $\phi \equiv 1$ on $B(x_0/R, 1)$. We have

$$\begin{aligned}
R^{r-3/2}\Big(\int_{|x-x_0|\le R} |f(x)|^2\,dx\Big)^{1/2} &= R^r\Big(\int_{|y-x_0/R|\le 1} |f(Ry)|^2\,dy\Big)^{1/2}\\
&\le R^r\Big(\int_{y\in\mathbb{R}^3} |f(Ry)\phi(y)|^2\,dy\Big)^{1/2}\\
&\le R^r\|f(R\,\cdot)\|_{\dot{X}_r}\|\phi\|_{H^r} \le \|f\|_{\dot{X}_r}\|\phi\|_{H^r}\\
&\le C\|f\|_{\dot{X}_r}.
\end{aligned}$$

Before stating our result, let us recall the definition of Leray–Hopf weak solution.

DEFINITION 2.5 ([31]). Let $(u_0, \omega_0) \in L^2(\mathbb{R}^3)$ and $\nabla \cdot u_0 = 0$. A measurable function $(u(x,t), \omega(x,t))$ is called a *weak solution* to the 3D micropolar flow equations (1.1) on $(0,T)$ if (u,ω) has the following properties:

(1) $u, \omega \in L^\infty(0,T; L^2(\mathbb{R}^3)) \cap L^2(0,T; H^1(\mathbb{R}^3))$ for all $T > 0$;
(2) $(u(x,t), \omega(x,t))$ satisfies (1.1) in the sense of distributions;
(3) the following energy inequality holds:

$$\|u\|_{L^2}^2 + \|w\|_{L^2}^2 + 2\int_0^t (\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2)\, ds + 2\int_0^t \|\nabla \cdot w\|_{L^2}^2\, ds + 2\int_0^t \|w\|_{L^2}^2\, ds \le \|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2 \quad \text{for } 0 < t \le T.$$

By a *strong solution* we mean a weak solution (u,ω) such that

$$u, \omega \in L^\infty(0,T; H^1(\mathbb{R}^3)) \cap L^2(0,T; H^2(\mathbb{R}^3)).$$

It is well known that strong solutions are regular (say, classical) and unique in the class of weak solutions.

More precisely, we will prove

THEOREM 2.6. *Suppose that $(u_0, \omega_0) \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ in $\mathbb{R}^3$. If the velocity u satisfies*

$$\partial_3 u \in L^{2/(1-r)}(0,T; \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)) \quad \text{with } 0 < r < 1, \tag{2.2}$$

then the solution remains smooth on $(0,T]$. Therefore,

$$(u,\omega) \in L^\infty(0,T; H^1(\mathbb{R}^3)) \cap L^2(0,T; H^2(\mathbb{R}^3)).$$

The following two lemmas will be used in the proofs of our main results (see, e.g., [1, 15, 26]):

LEMMA 2.7. *Let μ, λ and γ satisfy*

$$1 \le \alpha, \lambda, \gamma < \infty, \qquad \frac{1}{\lambda} + \frac{2}{\alpha} > 1 \quad \text{and} \quad 1 + \frac{3}{\gamma} = \frac{1}{\lambda} + \frac{2}{\alpha}.$$

Then there exists a constant $C = C(\alpha, \lambda)$ such that for all $f \in H^1(\mathbb{R}^3)$ with $\partial_1 f, \partial_2 f \in L^\alpha(\mathbb{R}^3)$ and $\partial_3 f \in L^\lambda(\mathbb{R}^3)$,

$$\|f\|_{L^\gamma} \le C\|\partial_1 f\|_{L^\alpha}^{1/3} \|\partial_2 f\|_{L^\alpha}^{1/3} \|\partial_3 f\|_{L^\lambda}^{1/3}. \tag{2.3}$$

LEMMA 2.8. *Let $2 \le \beta \le 6$. Then there exists a constant $C = C(\beta)$ such that for all $f \in H^1(\mathbb{R}^3)$,*

$$\|f\|_{L^\beta} \le C\|f\|_{L^2}^{\frac{6-\beta}{2\beta}} \|\partial_1 f\|_{L^2}^{\frac{\beta-2}{2\beta}} \|\partial_2 f\|_{L^2}^{\frac{\beta-2}{2\beta}} \|\partial_3 f\|_{L^2}^{\frac{\beta-2}{2\beta}} \le C\|f\|_{L^2(\mathbb{R}^3)}^{\frac{6-\beta}{2\beta}} \|f\|_{\dot{H}^1(\mathbb{R}^3)}^{\frac{3(\beta-2)}{2\beta}}.$$

Proof of Theorem 2.6. We differentiate the first and the second equation in (1.1) with respect to x_3, we take the scalar product with $\partial_3 u$ and $\partial_3 \omega$, respectively, and integrate over $\mathbb{R}^3$, to get

$$\frac{1}{2}\frac{d}{dt}\|\partial_3 u\|_{L^2}^2 + \|\nabla\partial_3 u\|_{L^2}^2 = -\int_{\mathbb{R}^3}(\partial_3 u\cdot\nabla)u\cdot\partial_3 u\,dx + \int_{\mathbb{R}^3}\partial_3(\nabla\times\omega)\cdot\partial_3 u\,dx \tag{2.4}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\partial_3\omega\|_{L^2}^2 + \|\nabla\partial_3\omega\|_{L^2}^2 + \|\nabla\cdot(\partial_3\omega)\|_{L^2}^2 \leq -\int_{\mathbb{R}^3}(\partial_3 u\cdot\nabla)\omega\cdot\partial_3\omega\,dx - 2\|\partial_3\omega\|_{L^2}^2 + \int_{\mathbb{R}^3}\partial_3(\nabla\times u)\cdot\partial_3\omega\,dx. \tag{2.5}$$

Now, combining (2.4) and (2.5), after suitable integration by parts (recall that $\nabla\cdot u = 0$) one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[\|\partial_3 u\|_{L^2}^2 + \|\partial_3\omega\|_{L^2}^2] + \|\nabla\partial_3 u\|_{L^2}^2 + \|\nabla\partial_3\omega\|_{L^2}^2 &\leq \int_{\mathbb{R}^3}\partial_3(\nabla\times\omega)\cdot\partial_3 u\,dx + \int_{\mathbb{R}^3}\partial_3(\nabla\times u)\cdot\partial_3\omega\,dx - 2\|\partial_3\omega\|_{L^2}^2\\ &\quad - \int_{\mathbb{R}^3}(\partial_3 u\cdot\nabla)u\cdot\partial_3 u\,dx - \int_{\mathbb{R}^3}(\partial_3 u\cdot\nabla)\omega\cdot\partial_3\omega\,dx\\ &= A_1 + A_2 + A_3 + A_4 + A_5.\end{aligned} \tag{2.6}$$

Integrating by parts and using Hölder's inequality and Young's inequality (as in [14]), we derive an estimate of the first three terms on the right-hand side:

$$\begin{aligned}A_1 + A_2 + A_3 &= \int_{\mathbb{R}^3}\partial_3(\nabla\times\omega)\cdot\partial_3 u\,dx + \int_{\mathbb{R}^3}\partial_3(\nabla\times u)\cdot\partial_3\omega\,dx - 2\|\partial_3\omega\|_{L^2}^2\\ &\leq 2\|\partial_3\omega\|_{L^2}^2 + \tfrac{1}{2}\|\nabla\partial_3 u\|_{L^2}^2 - 2\|\partial_3\omega\|_{L^2}^2 = \tfrac{1}{2}\|\nabla\partial_3 u\|_{L^2}^2.\end{aligned}$$

For A_4, using Lemma 2.3 together with the Hölder inequality and the Young inequality, we find that

$$\begin{aligned}|A_4| = \left|\int_{\mathbb{R}^3}(\partial_3 u\cdot\nabla)u\cdot\partial_3 u\,dx\right| &\leq \|\partial_3 u\cdot\partial_3 u\|_{L^2}\|\nabla u\|_{L^2}\\ &\leq \|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\partial_3 u\|_{\dot{B}^r_{2,1}}\|\nabla u\|_{L^2}\\ &\leq \|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla\partial_3 u\|_{L^2}^r\|\partial_3 u\|_{L^2}^{1-r}\|\nabla u\|_{L^2}\end{aligned} \tag{2.7}$$

by using the bilinear estimate (see [11, 12, 28])

$$\|fg\|_{L^2} \leq C\|f\|_{\dot{\mathcal{M}}_{2,3/r}}\|g\|_{\dot{B}^r_{2,1}}$$

and the interpolation inequality [32]

$$\|w\|_{\dot{B}^r_{2,1}} \leq C\|w\|_{L^2}^{1-r}\|\nabla w\|_{L^2}^r.$$

Similarly, we can bound

$$|A_5| = \Big|\int_{\mathbb{R}^3} (\partial_3 u\cdot\nabla)\omega\cdot\partial_3\omega\,dx\Big| \le \|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\partial_3\omega\|_{\dot{B}^r_{2,1}}\|\nabla\omega\|_{L^2}$$
$$\le \|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\partial_3\omega\|_{L^2}^{1-r}\|\nabla\partial_3\omega\|_{L^2}^{r}\|\nabla\omega\|_{L^2}.$$

From the above inequalities and (2.6), we obtain

$$\frac{1}{2}\frac{d}{dt}[\|\partial_3 u\|_{L^2}^2+\|\partial_3\omega\|_{L^2}^2]+\frac{1}{2}\|\nabla\partial_3 u\|_{L^2}^2+\|\nabla\partial_3\omega\|_{L^2}^2$$
$$\le \|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla\partial_3 u\|_{L^2}^{r}\|\partial_3 u\|_{L^2}^{1-r}\|\nabla u\|_{L^2}$$
$$+\|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}\|\partial_3\omega\|_{L^2}^{1-r}\|\nabla\partial_3\omega\|_{L^2}^{r}\|\nabla\omega\|_{L^2}.$$

By Young's inequality ($a^\alpha b^{1-\alpha}\le \alpha a+(1-\alpha)b\le a+b$ with $a,b\ge 0$ and $0\le\alpha\le 1$), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|\partial_3 u\|_{L^2}^2+\|\partial_3\omega\|_{L^2}^2]+\frac{1}{2}\|\nabla\partial_3 u\|_{L^2}^2+\|\nabla\partial_3\omega\|_{L^2}^2\\
&\le \big(\|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}}\|\partial_3 u\|_{L^2}^{2(\frac{1-r}{2-r})}\|\nabla u\|_{L^2}^{\frac{2}{2-r}}\big)^{\frac{2-r}{2}}(\|\nabla\partial_3 u\|_{L^2}^2)^{r/2}\\
&\quad+3\big(\|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}}\|\partial_3\omega\|_{L^2}^{2(\frac{1-r}{2-r})}\|\nabla\omega\|_{L^2}^{\frac{2}{2-r}}\big)^{\frac{2-r}{2}}(\|\nabla\partial_3\omega\|_{L^2}^2)^{r/2}\\
&\le C\|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}}\|\partial_3 u\|_{L^2}^{2(\frac{1-r}{2-r})}\|\nabla u\|_{L^2}^{\frac{2}{2-r}}+C\|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{2-r}}\|\partial_3\omega\|_{L^2}^{2(\frac{1-r}{2-r})}\|\nabla\omega\|_{L^2}^{\frac{2}{2-r}}\\
&\quad+\tfrac{1}{2}\|\nabla\partial_3\omega\|_{L^2}^2+\tfrac{1}{2}\|\nabla\partial_3 u\|_{L^2}^2\\
&=\tfrac{1}{2}\|\nabla\partial_3\omega\|_{L^2}^2+\tfrac{1}{2}\|\nabla\partial_3 u\|_{L^2}^2+C\|\partial_3 u\|_{L^2}^{2(\frac{1-r}{2-r})}\Big(\big(\|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{1-r}}\big)^{\frac{1-r}{2-r}}(\|\nabla u\|_{L^2}^2)^{\frac{1}{2-r}}\Big)\\
&\quad+C\|\partial_3\omega\|_{L^2}^{2(\frac{1-r}{2-r})}\Big(\big(\|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{1-r}}\big)^{\frac{1-r}{2-r}}(\|\nabla\omega\|_{L^2}^2)^{\frac{1}{2-r}}\Big)\\
&\le \tfrac{1}{2}\|\nabla\partial_3\omega\|_{L^2}^2+\tfrac{1}{2}\|\nabla\partial_3 u\|_{L^2}^2+C\|\partial_3 u\|_{L^2}^{2(\frac{1-r}{2-r})}\big(\|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{1-r}}+\|\nabla u\|_{L^2}^2\big)\\
&\quad+C\|\partial_3\omega\|_{L^2}^{2(\frac{1-r}{2-r})}\big(\|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{1-r}}+\|\nabla\omega\|_{L^2}^2\big),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(1+\|\partial_3 u\|_{L^2}^2+\|\partial_3\omega\|_{L^2}^2)+\|\nabla\partial_3 u\|_{L^2}^2+\|\nabla\partial_3\omega\|_{L^2}^2\\
&\qquad\le C(1+\|\partial_3 u\|_{L^2}^2)\big(\|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{1-r}}+\|\nabla u\|_{L^2}^2\big)\\
&\qquad\quad+C(1+\|\partial_3\omega\|_{L^2}^2)\big(\|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{1-r}}+\|\nabla\omega\|_{L^2}^2\big)\\
&\qquad\le C(1+\|\partial_3 u\|_{L^2}^2+\|\partial_3\omega\|_{L^2}^2)\big(\|\partial_3 u\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{1-r}}+\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2\big),
\end{aligned}$$

since $\frac{1-r}{2-r} < 1$. It follows from Gronwall's inequality together with the energy inequality (1.6) that

$$\begin{aligned}
&(1 + \|\partial_3 u(t,\cdot)\|_{L^2}^2 + \|\partial_3 \omega(t,\cdot)\|_{L^2}^2) \\
&\quad \le (1 + \|\partial_3 u_0\|_{L^2}^2 + \|\partial_3 \omega_0\|_{L^2}^2) \\
&\qquad \times \exp\Big(C\int_0^t (\|\partial_3 u(s,\cdot)\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{1-r}} + \|\nabla u(s,\cdot)\|_{L^2}^2 + \|\nabla \omega(s,.)\|_{L^2}^2)\, ds\Big) \\
&\quad \le (1 + \|\partial_3 u_0\|_{L^2}^2 + \|\partial_3 \omega_0\|_{L^2}^2) \\
&\qquad \times \exp\Big(C\int_0^t \|\partial_3 u(s,\cdot)\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{1-r}}\, ds + C\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2\Big) \\
&\quad = (1 + \|\partial_3 u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2) e^{C(\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2)} \exp\Big(C\int_0^t \|\partial_3 u(s,\cdot)\|_{\dot{\mathcal{M}}_{2,3/r}}^{\frac{2}{1-r}}\, ds\Big)
\end{aligned}$$

and

$$\int_0^t (\|\nabla \partial_3 u(s,\cdot)\|_{L^2}^2 + \|\nabla \partial_3 \omega(s,\cdot)\|_{L^2}^2)\, ds \le C. \tag{2.8}$$

Here C denotes a constant depending on the initial data and on

$$\|\partial_3 u(s,\cdot)\|_{L^{2/(1-r)}(0,T;\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3))}.$$

Now we establish

$$(u,\omega) \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2).$$

Taking the inner product of the equation (1.1) with $-\Delta u$ and $-\Delta \omega$ in $L^2(\mathbb{R}^3)$, respectively, after suitable integration by parts, by the same calculation as in [3], [11], [18] we obtain, for $t \in (0,T)$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 &= \int_{\mathbb{R}^3} (u\cdot\nabla)u \cdot \Delta u\, dx - \int_{\mathbb{R}^3} (\nabla \times \omega)\cdot \Delta u\, dx \\
&= -\sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u \cdot (\partial_k u \cdot \nabla u)\, dx - \int_{\mathbb{R}^3} (\nabla \times u)\cdot \Delta\omega\, dx
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla \omega(t)\|_{L^2}^2 + \|\Delta \omega(t)\|_{L^2}^2 + \|\nabla \operatorname{div} \omega(t)\|_{L^2}^2 + 2\|\nabla\omega(t)\|_{L^2}^2 \\
&\qquad = \int_{\mathbb{R}^3} (u\cdot\nabla)\omega \cdot \Delta\omega\, dx - \int_{\mathbb{R}^3} (\nabla \times u)\cdot \Delta\omega\, dx \\
&\qquad = -\sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k \omega \cdot (\partial_k u \cdot \nabla \omega)\, dx - \int_{\mathbb{R}^3} (\nabla \times u)\cdot \Delta\omega\, dx,
\end{aligned}$$

where we have used

$$\int_{\mathbb{R}^3} (\nabla \times \omega) \cdot \Delta u \, dx = \int_{\mathbb{R}^3} (\nabla \times u) \cdot \Delta \omega \, dx.$$

We sum the above equations to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\omega(t)\|_{L^2}^2) + \|\Delta u(t)\|_{L^2}^2 + \|\Delta\omega(t)\|_{L^2}^2 \\
&\quad + \|\nabla \operatorname{div} \omega(t)\|_{L^2}^2 + 2\|\nabla\omega(t)\|_{L^2}^2 \\
&\quad \le C\|\nabla u\|_{L^3}^3 + \|\nabla u\|_{L^3}\|\nabla\omega\|_{L^3}^2 + 2\|\nabla u\|_{L^2}\|\Delta\omega\|_{L^2} \\
&\quad \le C\|\nabla u\|_{L^3}^3 + (\|\nabla u\|_{L^3}^3)^{1/3}(\|\nabla\omega\|_{L^3}^3)^{2/3} + C\|\nabla u\|_{L^2}^2 + \tfrac{1}{4}\|\Delta\omega\|_{L^2}^2 \\
&\quad \le C\|\nabla u\|_{L^3}^3 + C\|\nabla\omega\|_{L^3}^3 + \tfrac{1}{4}\|\Delta\omega\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \\
&\quad \le C\|\nabla u\|_{L^2}^{3/2}\|\nabla\partial_1 u\|_{L^2}^{1/2}\|\nabla\partial_2 u\|_{L^2}^{1/2}\|\nabla\partial_3 u\|_{L^2}^{1/2} \\
&\qquad + C\|\nabla\omega\|_{L^2}^{3/2}\|\nabla\partial_1\omega\|_{L^2}^{1/2}\|\nabla\partial_2\omega\|_{L^2}^{1/2}\|\nabla\partial_3\omega\|_{L^2}^{1/2} + \tfrac{1}{4}\|\Delta\omega\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \\
&\quad \le C\|\nabla u\|_{L^2}^{3/2}\|\nabla^2 u\|_{L^2}\|\nabla\partial_3 u\|_{L^2}^{1/2} + C\|\nabla\omega\|_{L^2}^{3/2}\|\nabla^2\omega\|_{L^2}\|\nabla\partial_3\omega\|_{L^2}^{1/2} \\
&\qquad + \tfrac{1}{4}\|\Delta\omega\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \\
&\quad = (\|\nabla^2 u\|_{L^2}^2)^{1/2}(C\|\nabla u\|_{L^2}^3\|\nabla\partial_3 u\|_{L^2})^{1/2} \\
&\qquad + (\|\nabla^2\omega\|_{L^2}^2)^{1/2}(C\|\nabla\omega\|_{L^2}^3\|\nabla\partial_3\omega\|_{L^2})^{1/2} + \tfrac{1}{4}\|\Delta\omega\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \\
&\quad \le \tfrac{1}{2}\|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^3\|\nabla\partial_3 u\|_{L^2} + \tfrac{1}{4}\|\Delta\omega\|_{L^2}^2 + C\|\nabla\omega\|_{L^2}^3\|\nabla\partial_3\omega\|_{L^2} \\
&\qquad + \tfrac{1}{4}\|\Delta\omega\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \\
&\quad = \tfrac{1}{2}(\|\Delta u\|_{L^2}^2 + \|\Delta\omega\|_{L^2}^2) + C\|\nabla u\|_{L^2}^2(\|\nabla u\|_{L^2}\|\nabla\partial_3 u\|_{L^2}) \\
&\qquad + C\|\nabla\omega\|_{L^2}^2(\|\nabla\omega\|_{L^2}\|\nabla\partial_3\omega\|_{L^2}) + \tfrac{1}{2}(\|\nabla\omega\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) \\
&\quad \le \tfrac{1}{2}(\|\Delta u\|_{L^2}^2 + \|\Delta\omega\|_{L^2}^2) + C\|\nabla u\|_{L^2}^2(\|\nabla u\|_{L^2}^2 + \|\nabla\partial_3 u\|_{L^2}^2) \\
&\qquad + C\|\nabla\omega\|_{L^2}^2(\|\nabla\omega\|_{L^2}^2 + \|\nabla\partial_3\omega\|_{L^2}^2),
\end{aligned}$$

and by using Hölder's inequality and (2.3) with $\alpha = \lambda = 2$ and $\gamma = 6$, we get

$$\|f\|_{L^6} \le C\|\partial_1 f\|_{L^2}^{1/3}\|\partial_2 f\|_{L^2}^{1/3}\|\partial_3 f\|_{L^2}^{1/3}.$$

Hence

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\omega(t)\|_{L^2}^2) + \|\Delta u(t)\|_{L^2}^2 + \|\Delta\omega(t)\|_{L^2}^2 \\
&\le C(1 + \|\nabla u\|_{L^2}^2 + \|\nabla\partial_3 u\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|\nabla\partial_3\omega\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2).
\end{aligned}$$

Using Gronwall's inequality, the energy inequality (1.6) and the estimate (2.8), we conclude that

$$\begin{aligned}
&\|\nabla u(t,\cdot)\|_{L^2}^2 + \|\nabla\omega(t,\cdot)\|_{L^2}^2 + \int_0^t (\|\Delta u(s,\cdot))\|_{L^2}^2 + \|\Delta\omega(s,\cdot))\|_{L^2}^2)\,ds \\
&\qquad \le (\|\nabla u_0\|_{L^2}^2 + \|\nabla\omega_0\|_{L^2}^2)\exp\Big(C\int_0^t \|\nabla u(s,\cdot)\|_{L^2}^2 + \|\nabla\partial_3 u(s,\cdot)\|_{L^2}^2\,ds\Big) \\
&\qquad\quad \times \exp\Big(C\int_0^t \|\nabla\omega(s,\cdot)\|_{L^2}^2 + \|\nabla\partial_3\omega(s,\cdot)\|_{L^2}^2\,ds\Big) \\
&\qquad \le C
\end{aligned}$$

for all $0 \le t < T$. Hence

$$(u,\omega) \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)),$$

which shows that u and ω are smooth, completing the proof of Theorem 2.6. ∎

REMARK 2.1. Theorem 2.6 is still true for the Navier–Stokes equation with $\omega \equiv 0$, so we give an extension of Serrin's regularity criterion for the Navier–Stokes equations [30].

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