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The shifted fourth moment of automorphic L-functions of prime power level

by

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1. Introduction. Let $L(s,f)$ be an automorphic L-function associated to $f\in H_k^*(q)$, where $H_k^*(q)$ denotes the set of primitive forms of weight k and level q. An important subject in analytic number theory is the behavior of such L-functions near the critical line. Of particular interest are subconvexity bounds and proportion of non-vanishing L-values. A possible way to analyze these problems is the method of moments. This technique proved to be very effective in the recent years; see [D], [DFI1], [IS], [KM], [KMV] for details and examples.

In 1995 Duke [D] proved an asymptotic formula for the first moment and an upper bound for the second moment when q is prime and $k=2$. Four years later, Akbary [A] generalized this result to the case of prime q and $k>2$. In 2011 Ichihara [Ich] found an asymptotic formula for the first moment when q is a power of a prime and $2\le k\le 10$, $k=14$. In the same year, Rouymi [R] computed the asymptotics of the first, second and third moments when q is a prime power and k is an arbitrary fixed even integer.

In order to break the convexity barrier, one needs to evaluate the limit moment of order

$$\kappa_0:=\liminf_{q\to\infty}4\frac{\log|H_k^*(q)|}{\log q}=4. \tag{1.1}$$

At the same time, starting from the κ_0th moment the main term of the asymptotic formula contains a nontrivial nondiagonal contribution (see [M] for details).

The moment of order $\kappa_0=4$ for prime $q\to\infty$ was studied by Duke, Friedlander & Iwaniec [DFI1] and Kowalski, Michel & VanderKam [KMV].

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The main term of the fourth moment splits into diagonal M^{D}, off-diagonal M^{OD} and off-off-diagonal M^{OOD} parts. Therefore, it requires three different stages of analysis.

THEOREM 1.1 ([KMV, Corollary 1.3]). *Let q be a prime number and $k=2$. For all $\epsilon>0$,*

$$\sum_{f\in H_k^*(q)}^{\mathrm{h}} L(1/2,f)^4 = R(\log q) + O_\epsilon(q^{-1/12+\epsilon}), \tag{1.2}$$

where R is a polynomial of degree 6 with leading coefficient $\frac{1}{60\pi^2}$.

In this paper, the result of Theorem 1.1 is extended as follows.

- We consider the level of the form $q=p^\nu$, where p is a fixed prime number and $\nu\to\infty$.
- We assume that the weight $k\geq 2$ is an arbitrary even integer.
- We slightly shift each L-function in the product from the critical line $\Re s=1/2$,

$$M_4(\boldsymbol{t},\boldsymbol{r}) = \sum_{f\in H_k^*(q)}^{\mathrm{h}} |L(1/2+t_1+ir_1,f)|^2|L(1/2+t_2+ir_2,f)|^2,$$

  where $\boldsymbol{t}=(t_1,t_2)$, $\boldsymbol{r}=(r_1,r_2)$, $t_1,t_2,r_1,r_2\in\mathbb{R}$ and $|t_1|,|t_2|<1/\log q$.

The shifts simplify the analysis of the off-off-diagonal term, reveal more clearly the combinatorial structure of mean values and allow us to verify the random matrix theory conjectures involving all lower order terms by Conrey, Farmer, Keating, Rubinstein and Snaith [CFKRS].

We introduce the following notation:

$$\hat{q} = \frac{\sqrt{q}}{2\pi}, \tag{1.3}$$

$$\zeta_q(s) = \zeta(s)(1-p^{-s}). \tag{1.4}$$

CONJECTURE 1.2 (analog of [CFKRS, Conjectures 4.5.1 and 4.5.2]). *Let $q=p^\nu$, where p is a fixed prime and $\nu\geq 3$. Let $k>0$ be an even integer. Up to an error term, we have*

$$M_4(\boldsymbol{t},\boldsymbol{r}) = \frac{\phi(q)}{q}\hat{q}^{-2t_1-2t_2} \sum_{\substack{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4=\pm1\\ \epsilon_1\epsilon_2\epsilon_3\epsilon_4=1}} \hat{q}^{t_1(\epsilon_1+\epsilon_2)+t_2(\epsilon_3+\epsilon_4)+ir_1(\epsilon_1-\epsilon_2)+ir_2(\epsilon_3-\epsilon_4)}$$
$$\times\left(\frac{\Gamma(-t_1-ir_1+k/2)\Gamma(-t_1+ir_1+k/2)\Gamma(-t_2-ir_2+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)\Gamma(t_2+ir_2+k/2)}\right)^{1/2}$$

$$\times \left(\frac{\Gamma(-t_2+ir_2+k/2)\Gamma(\epsilon_1(t_1+ir_1)+k/2)\Gamma(\epsilon_2(t_1-ir_1)+k/2)}{\Gamma(t_2-ir_2+k/2)\Gamma(-\epsilon_1(t_1+ir_1)+k/2)\Gamma(-\epsilon_2(t_1-ir_1)+k/2)}\right)^{1/2}$$
$$\times \left(\frac{\Gamma(\epsilon_3(t_2+ir_2)+k/2)\Gamma(\epsilon_4(t_2-ir_2)+k/2)}{\Gamma(-\epsilon_3(t_2+ir_2)+k/2)\Gamma(-\epsilon_4(t_2-ir_2)+k/2)}\right)^{1/2}$$
$$\times \frac{\zeta_q(1+t_1(\epsilon_1+\epsilon_2)+ir_1(\epsilon_1-\epsilon_2))\zeta_q(1+t_2(\epsilon_3+\epsilon_4)+ir_2(\epsilon_3-\epsilon_4))}{\zeta_q(2+t_1(\epsilon_1+\epsilon_2)+t_2(\epsilon_3+\epsilon_4)+ir_1(\epsilon_1-\epsilon_2)+ir_2(\epsilon_3-\epsilon_4))}$$
$$\times \zeta_q(1+\epsilon_1(t_1+ir_1)+\epsilon_3(t_2+ir_2))\zeta_q(1+\epsilon_1(t_1+ir_1)+\epsilon_4(t_2-ir_2))$$
$$\times \zeta_q(1+\epsilon_2(t_1-ir_1)+\epsilon_3(t_2+ir_2))\zeta_q(1+\epsilon_2(t_1-ir_1)+\epsilon_4(t_2-ir_2)).$$

MAIN THEOREM 1.3. *Let $\theta = 7/64$, $q = p^\nu$, where p is a fixed prime and $\nu \geq 3$. Let $k > 0$ be an even integer. For all $\epsilon > 0$ the fourth moment can be decomposed as follows:*

$$M_4(\boldsymbol{t},\boldsymbol{r}) = M^{\mathrm{D}} + M^{\mathrm{OD}} + M^{\mathrm{OOD}} + O_{\epsilon,p,k}\big(q^\epsilon\big(q^{-\frac{k-1-2\theta}{8-8\theta}} + q^{-1/4}\big)\big),$$

where the implied constant depends polynomially on r_1, r_2. Furthermore,

$$(1.5)\quad M^D + M^{\mathrm{OD}} = \frac{\phi(q)}{q}\sum_{\epsilon_1,\epsilon_2=\pm 1} \hat{q}^{-2t_1-2t_2+2\epsilon_1 t_1+2\epsilon_2 t_2}\frac{\Gamma(\epsilon_1 t_1+ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)}$$
$$\times \frac{\Gamma(\epsilon_1 t_1-ir_1+k/2)}{\Gamma(t_1-ir_1+k/2)}\frac{\Gamma(\epsilon_2 t_2+ir_2+k/2)\Gamma(\epsilon_2 t_2-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}$$
$$\times \zeta_q(1+2\epsilon_1 t_1)\zeta_q(1+2\epsilon_2 t_2)\frac{\prod_{\epsilon_3,\epsilon_4=\pm 1}\zeta_q(1+\epsilon_1 t_1+\epsilon_2 t_2+i\epsilon_3 r_1+i\epsilon_4 r_2)}{\zeta_q(2+2\epsilon_1 t_1+2\epsilon_2 t_2)}$$

and

$$(1.6)\qquad M^{\mathrm{OOD}} = \frac{\phi(q)}{q}\sum_{\epsilon_1,\epsilon_2=\pm 1}\hat{q}^{-2t_1-2t_2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}$$
$$\times \frac{\Gamma(k/2-t_1+i\epsilon_1 r_1)\Gamma(k/2-t_2+i\epsilon_2 r_2)}{\Gamma(k/2+t_1-i\epsilon_1 r_1)\Gamma(k/2+t_2-i\epsilon_2 r_2)}$$
$$\times\ \zeta_q(1+2i\epsilon_1 r_1)\zeta_q(1+2i\epsilon_2 r_2)\frac{\prod_{\epsilon_3,\epsilon_4=\pm 1}\zeta_q(1+\epsilon_3 t_1+\epsilon_4 t_2+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}.$$

REMARK 1.4. The condition $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4 = \pm 1$, $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1$ in Conjecture 1.2 implies that there are eight terms in the sum. Four of them,

$$(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4) = (1,1,1,1),(1,1,-1,-1),(-1,-1,1,1),(-1,-1,-1,-1),$$

coincide with the summands of (1.5), and the other four,

$$(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4) = (-1,1,-1,1),(-1,1,1,-1),(1,-1,-1,1),(1,-1,1,-1)$$

with the summands of (1.6).

By letting the shifts tend to zero in Theorem 1.3, we obtain an asymptotic formula for the fourth moment at the critical point $s = 1/2$.

COROLLARY 1.5. *Let $\theta = 7/64$, $q = p^\nu$, where p is a fixed prime and $\nu \geq 3$. Let $k > 0$ be an even integer. For all $\epsilon > 0$,*

$$M_4(\mathbf{0},\mathbf{0}) = \sum_{f \in H_k^*(q)}^{\mathrm{h}} L(1/2, f)^4 \tag{1.7}$$

$$= R(\log q) + O_{\epsilon,k,p}\big(q^\epsilon \big(q^{-\frac{k-1-2\theta}{8-8\theta}} + q^{-1/4}\big)\big),$$

where R is a polynomial of degree 6 with leading coefficient

$$\left(\frac{\phi(q)}{q}\right)^7 \frac{p^2}{p^2-1} \frac{1}{60\pi^2}. \tag{1.8}$$

The paper is organized as follows. In Section 2 we recall some definitions and fundamental results. Section 3 provides an explicit formula for the diagonal, off-diagonal and off-off-diagonal main terms. The asymptotics of the diagonal and off-diagonal terms is derived in Section 4. Sections 5 and 6 are devoted to proving the asymptotic formula for the off-off-diagonal term. Corollary 1.5 is proved as a limit case at the end of Sections 4 and 6.

2. Background information. The purpose of this section is to recall some results on automorphic forms and related objects.

2.1. Automorphic L-functions. A holomorphic function f on the Poincaré upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ is called a *cusp form* of weight k and level q if

$$f(\gamma z) = (cz+d)^k f(z) \tag{2.1}$$

for all γ in

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : c \equiv 0 \pmod{q} \right\},$$

and

$$(\Im z)^{k/2} |f(z)| \text{ is bounded on } \mathbb{H}. \tag{2.2}$$

Let $S_k(q)$ be the space of cusp forms of weight $k \geq 2$ and level q. It is equipped with the Petersson inner product

$$\langle f, g \rangle_q := \int_{F_0(q)} f(z)\overline{g(z)} y^k \frac{dx\,dy}{y^2}, \tag{2.3}$$

where $F_0(q)$ is a fundamental domain of the action of $\Gamma_0(q)$ on $\mathbb{H}$. Any $f \in S_k(q)$ has a Fourier expansion at infinity

$$f(z) = \sum_{n \geq 1} a_f(n) e(nz). \tag{2.4}$$

According to Atkin–Lehner theory [AL], the space $S_k(q)$ can be decomposed into two subspaces

$$S_k(q) = S_k^{\mathrm{new}}(q) \oplus S_k^{\mathrm{old}}(q). \tag{2.5}$$

The space of old forms contains cusp forms of level q coming from lower levels:

$$S_k^{\mathrm{old}}(q) = \mathrm{span}\{f(lz) : lq' \mid q,\, q' < q,\, f(z) \in S_k(q')\}, \tag{2.6}$$

and *the space of new forms* is the orthogonal complement to $S_k^{\mathrm{old}}(q)$.

We let $H_k(q)$ denote an orthogonal basis of the space of cusp forms $S_k(q)$, and $H_k^*(q)$ an orthogonal basis of $S_k^{\mathrm{new}}(q)$. Elements of $H_k^*(q)$ with normalized Fourier coefficients

$$\lambda_f(n) := a_f(n) n^{-(k-1)/2}, \tag{2.7}$$

$$\lambda_f(1) := 1 \tag{2.8}$$

are called *primitive forms*. Accordingly,

$$\lambda_f(n) \in \mathbb{R}, \tag{2.9}$$

$$\lambda_f(n_1)\lambda_f(n_2) = \sum_{\substack{d \mid (n_1,n_2) \\ (d,q)=1}} \lambda_f\left(\frac{n_1 n_2}{d^2}\right). \tag{2.10}$$

Let $\Re s > 1$. Then for $f \in H_k^*(q)$ we define the *automorphic L-function* as

$$L(s,f) = \sum_{n \geq 1} \lambda_f(n) n^{-s}. \tag{2.11}$$

The *completed L-function*

$$\Lambda(s,f) = \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right) L(s,f) \tag{2.12}$$

can be analytically continued onto the whole complex plane and satisfies the functional equation

$$\Lambda(s,f) = \epsilon_f \Lambda(1-s,f), \tag{2.13}$$

where $s \in \mathbb{C}$ and $\epsilon_f = \pm 1$. We define the *harmonic average* over the set of primitive newforms by

$$\sideset{}{^{\mathrm{h}}}\sum_{f \in H_k^*(q)} A(f) := \sum_{f \in H_k^*(q)} \frac{\Gamma(k-1)}{(4\pi)^{k-1} \langle f, f \rangle_q} A(f). \tag{2.14}$$

2.2. Kloosterman sums. Consider the sum

$$S(m,n,c) = \sum_{\substack{d \,(\mathrm{mod}\, c) \\ (c,d)=1}} e\left(\frac{m\overline{d} + nd}{c}\right), \tag{2.15}$$

where $d\overline{d} \equiv 1 \pmod{c}$ and $e(z) = \exp(2\pi i z)$.

The value of $S(m,n,c)$ is always a real number because

$$\overline{S(m,n,c)} = S(m,n,c). \tag{2.16}$$

Further,

$$S(m,n,c) = S(n,m,c), \tag{2.17}$$

$$S(ma,n,c) = S(m,na,c) \quad \text{if } (a,c)=1. \tag{2.18}$$

Another important property is the twisted multiplicity [Iw, formula (4.12)]. Suppose $(c_1,c_2)=1$, $c_2\overline{c_2}\equiv 1 \pmod{c_1}$, $c_1\overline{c_1}\equiv 1 \pmod{c_2}$. Then

$$S(m,n,c_1c_2) = S(m\overline{c_2}, n\overline{c_2}, c_1)S(m\overline{c_1}, n\overline{c_1}, c_2). \tag{2.19}$$

Lemma 2.1 (Weil's bound, [We]). *One has*

$$|S(m,n,c)| \le (m,n,c)^{1/2}c^{1/2}\tau(c). \tag{2.20}$$

Lemma 2.2 (Royer, [Ro, Lemma A.12]). *Let m,n,c be three strictly positive integers and p be a prime number. Suppose $p^2 \mid c$, $p \mid m$ and $p \nmid n$. Then $S(m,n,c)=0$.*

2.3. Large sieve inequality

Theorem 2.3 (Deshouillers, Iwaniec, [DI, Theorem 9]). *Let r and s be positive coprime integers, C, M, N be positive real numbers and g be a real-valued function of class $\mathbf{C^6}$ (first and second derivatives are continuous in each variable) with support in $[M,2M]\times[N,2N]\times[C,2C]$ such that*

$$\left|\frac{\partial^{j+k+l}}{\partial m^j \partial n^k \partial c^l} g(m,n,c)\right| \le M^{-j}N^{-k}C^{-l} \quad \text{for } 0\le j,k,l\le 2. \tag{2.21}$$

Then for any $\epsilon>0$ and complex sequences $\boldsymbol{a}=\{a_m\}$, $\boldsymbol{b}=\{b_n\}$,

$$\begin{aligned} &\sum_{(c,r)=1}\sum_m a_m \sum_n b_n g(m,n,c) S(m\bar{r}, \pm n, sc) \\ &\ll_\epsilon \Big(\sum_{M<m\le 2M}|a_m|^2\Big)^{1/2}\Big(\sum_{N<n\le 2N}|b_n|^2\Big)^{1/2} C^\epsilon \left(1+\frac{s\sqrt{r}\,C}{\sqrt{MN}}\right)^{2\theta} \\ &\quad\times \frac{(s\sqrt{r}\,C+\sqrt{MN}+\sqrt{sM}C)(s\sqrt{r}\,C+\sqrt{MN}+\sqrt{sN}\,C)}{s\sqrt{r}\,C+\sqrt{MN}}. \end{aligned} \tag{2.22}$$

Here

$$\theta = \theta_{rs} := \sqrt{\max(0, 1/4-\lambda_1)}$$

and $\lambda_1 = \lambda_1(rs)$ is the smallest positive eigenvalue for the Hecke congruence subgroup $\Gamma_0(rs)$. Currently the best known bound on λ_1 is due to Kim and Sarnak [KS]. Accordingly, we can take $\theta = 7/64$.

2.4. Petersson's trace formula. The key ingredient of our proof is the Petersson trace formula. It allows expressing an average of Fourier coefficients of cusp forms in terms of Kloosterman sums weighted by J-Bessel functions.

THEOREM 2.4 ([IK, Proposition 14.5]). *For $m, n \geq 1$ we have*

$$\Delta_q(m,n) := \sideset{}{^{\mathrm{h}}}\sum_{f\in H_k(q)} \lambda_f(m)\lambda_f(n) = \delta_{m,n} + 2\pi i^{-k} \sum_{q\mid c} \frac{S(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{2.23}$$

If q is a prime number and $k < 12$, the Petersson trace formula also works for moments of L-functions associated to primitive forms because the space of old forms is empty.

When q is composite, one needs to exclude the contribution of old forms. Iwaniec, Luo and Sarnak constructed a special basis in order to find an analog of Petersson's trace formula for primitive forms of square-free level.

THEOREM 2.5 ([ILS, Proposition 2.8]). *Let q be square-free, $(m,q) = 1$ and $(n,q^2) \mid q$. Then*

$$\Delta_q^*(m,n) := \sideset{}{^{\mathrm{h}}}\sum_{f\in H_k^*(q)} \lambda_f(m)\lambda_f(n) = \frac{k-1}{12} \sum_{LM=q} \frac{\mu(L)M}{(n,L)\prod_{p\mid(n,L)}(1+p^{-1})} \sum_{l\mid L^\infty} l^{-1}\Delta_M(ml^2,n). \tag{2.24}$$

This result was extended to the case of a prime power level by Rouymi.

THEOREM 2.6 ([R, Remark 4]). *Let $q = p^\nu$ with $\nu \geq 3$. Then*

$$\Delta_q^*(m,n) := \sideset{}{^{\mathrm{h}}}\sum_{f\in H_k^*(q)} \lambda_f(m)\lambda_f(n) = \begin{cases} \Delta_q(m,n) - \Delta_{q/p}(m,n)/p & \textit{if } (q,mn)=1, \\ 0 & \textit{otherwise.} \end{cases} \tag{2.25}$$

2.5. Poisson type summation formula connected with the Eisenstein–Maass series. Let

$$\tau_v(n) = |n|^{v-1/2}\sigma_{1-2v}(n) = |n|^{v-1/2} \sum_{d\mid n,\, d>0} d^{1-2v}. \tag{2.26}$$

If $v = 1/2$, then $\tau_v(n)$ reduces to the divisor function $\tau(n)$. Furthermore,

$\tau_v(n)$ has the multiplicity property (see [K, p. 74])

$$\tau_v(n)\tau_v(m) = \sum_{d|(n,m)} \tau_v\left(\frac{nm}{d^2}\right). \tag{2.27}$$

LEMMA 2.7 (Ramanujan's identity, [T, p. 8]). *Let* $\Re s > 1 + |\Re v - 1/2| + |\Re \mu - 1/2|$. *Then*

$$\begin{aligned} \zeta(2s) \sum_{n \geq 1} \frac{\tau_v(n)\tau_\mu(n)}{n^s} \\ = \zeta(s+v-\mu)\zeta(s-v+\mu)\zeta(s+v+\mu-1)\zeta(s-v-\mu+1). \end{aligned} \tag{2.28}$$

If $v = \mu = 1/2$, *this reduces to*

$$\sum_{n \geq 1} \frac{\tau(n)^2}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)}. \tag{2.29}$$

Consider the Bessel kernels expressed in terms of J- and K-Bessel functions:

$$k_0(x, v) := \frac{1}{2\cos \pi v}(J_{2v-1}(x) - J_{1-2v}(x)), \tag{2.30}$$

$$k_1(x, v) := \frac{2}{\pi} \sin(\pi v) K_{2v-1}(x). \tag{2.31}$$

THEOREM 2.8 ([K, Theorem 5.2, p. 89]). *Let* ϕ *be a smooth, compactly supported function on* $\mathbb{R}^+$. *Then for every* v *with* $\Re v = 1/2$, $(c, d) = 1$, $c \geq 1$ *one has*

$$\begin{aligned} \frac{4\pi}{c} \sum_{m \geq 1} e\left(\frac{md}{c}\right) \tau_v(m) \phi\left(\frac{4\pi\sqrt{m}}{c}\right) \\ = 2\frac{\zeta(2v)}{(4\pi)^{2v}} \hat{\phi}(2v+1) + 2\frac{\zeta(2-2v)}{(4\pi)^{2-2v}} \hat{\phi}(3-2v) \\ + \sum_{m \geq 1} \tau_v(m) \int_0^\infty \left[e\left(-\frac{ma}{c}\right) k_0(x\sqrt{m}, v) + e\left(\frac{ma}{c}\right) k_1(x\sqrt{m}, v) \right] \phi(x) x \, dx, \end{aligned} \tag{2.32}$$

where $ad \equiv 1 \pmod c$ *and* $\hat{\phi}$ *is the Mellin transform of* ϕ.

2.6. Quadratic divisor problem. Applying formula (2.32), we generalize [DFI2, Theorem 1] as follows.

THEOREM 2.9. *Let* $a, b \geq 1$, $(a, b) = 1$, $h \neq 0$ *and* $r_1, r_2 \in \mathbb{R}$. *Let*

$$D_f(a, b; h) = \sum_{am \mp bn = h} \tau_{1/2+ir_1}(m) \tau_{1/2+ir_2}(n) f(am, bn)$$

with

$$x^i y^j f^{(ij)}(x, y) \ll (1 + x/X)^{-1} (1 + y/Y)^{-1} Q^{i+j}. \tag{2.33}$$

Assume that

$$ab < Q^{-5/4}(X+Y)^{-5/4}(XY)^{1/4+\epsilon}. \tag{2.34}$$

Then

$$D_f(a,b;h) = \int_0^\infty g(x, \pm x \mp h)\,dx + O(Q^{5/4}(X+Y)^{1/4}(XY)^{1/4+\epsilon}),$$

where the implied constant depends polynomially on r_1, r_2. *Here* $g(x,y) = f(x,y)\Lambda_{a,b,h}(x,y)$ *with*

$$\Lambda_{a,b,h}(x,y) := \sum_{w=1}^{\infty} S(0,h,w) \sum_{\epsilon_1,\epsilon_2=\pm 1} \frac{(ab,w)(a,w)^{2i\epsilon_1 r_1}(b,w)^{2i\epsilon_2 r_2}}{a^{1+i\epsilon_1 r_1} b^{1+i\epsilon_2 r_2} w^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \times \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2) x^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}. \tag{2.35}$$

3. The fourth moment: preliminary steps

3.1. Approximate functional equation. Let $P_r(s)$ be an even polynomial vanishing at all poles of $\Gamma(s+ir+k/2)\Gamma(s-ir+k/2)$ in the range $\Re s \ge -L$ for some large constant $L > 0$. For $t, r \in \mathbb{R}$ we define

$$W_{t,r}(y) := \frac{1}{2\pi i} \int_{(3)} \frac{P_r(s)}{P_r(t)} \zeta_q(1+2s) \times \frac{\Gamma(s+ir+k/2)\Gamma(s-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)} y^{-s} \frac{2s\,ds}{s^2-t^2}. \tag{3.1}$$

LEMMA 3.1. *Suppose* $y > 0$ *and* $|t| < 1/2$. *For any* $C > |t|$,

$$W_{t,r}(y) = O_{C,t,r}(y^{-C}) \quad \text{as } y \to \infty, \tag{3.2}$$

$$W_{t,r}(y) = \zeta_q(1-2t) y^t \frac{\Gamma(-t+ir+k/2)\Gamma(-t-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)} + \zeta_q(1+2t) y^{-t} + O_{C,t,r}(y^C) \quad \text{as } y \to 0. \tag{3.3}$$

The implied constants depend polynomially on r.

Proof. Asymptotic expansion for the ratio of gamma functions gives

$$\frac{\Gamma(C+ir+k/2)\Gamma(C-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)} = |r|^{2(C-t)}(1+O(1/|r|)).$$

First, without crossing any poles, we can shift the integration contour to $\Re s = C$ with $C > |t|$. This implies (3.2). Second, we move the contour to $\Re s = -C$, meeting two simple poles at $s = \pm t$. Therefore, as $y \to 0$, we get (3.3). ∎

LEMMA 3.2. *For $t, r \in \mathbb{R}$ with $|t| < 1/2$ we have*

$$|L(1/2+t+ir,f)|^2 = \hat{q}^{-2t} \sum_{n\geq 1} \tau_{1/2+ir}(n) \frac{\lambda_f(n)}{\sqrt{n}} W_{t,r}\left(\frac{n}{\hat{q}^2}\right). \tag{3.4}$$

Proof. Consider

$$I_t := \frac{1}{2\pi i} \int_{(3)} \Lambda(1/2+s+ir,f)\Lambda(1/2+s-ir,f)\frac{P_r(s)}{s-t}\,ds.$$

Moving the integration contour to $\Re s = -3$, we pick up a simple pole at $s = t$. The functional equation (2.13) implies that

$$\begin{aligned} I_t + \epsilon_f^2 I_{-t} &= \mathrm{Res}_{s=t}\left(\Lambda(1/2+s+ir,f)\Lambda(1/2+s-ir,f)\frac{P_r(s)}{s-t}\right) \\ &= P_r(t)\Lambda(1/2+t+ir,f)\Lambda(1/2+t-ir,f). \end{aligned}$$

Observe that for $s > 1/2$, the property (2.10) yields

$$|L(1/2+s+ir,f)|^2 = \zeta_q(1+2s)\sum_{n\geq 1}\frac{\lambda_f(n)}{n^{1/2+s}}\tau_{1/2+ir}(n).$$

Finally,

$$|L(1/2+t+ir,f)|^2 = \hat{q}^{-2t}\sum_{n\geq 1}\tau_{1/2+ir}(n)\frac{\lambda_f(n)}{\sqrt{n}}$$
$$\times \frac{1}{2\pi i}\int_{(3)} \frac{P_r(s)}{P_r(t)}\zeta_q(1+2s)\frac{\Gamma(s+ir+k/2)\Gamma(s-ir+k/2)}{\Gamma(t+ir+k/2)\Gamma(t-ir+k/2)}\left(\frac{n}{\hat{q}^2}\right)^{-s}\frac{2s\,ds}{s^2-t^2}. \quad \blacksquare$$

COROLLARY 3.3. *The fourth moment can be written as follows:*

$$\begin{aligned} M_4(\boldsymbol{t},\boldsymbol{r}) = \hat{q}^{-2t_1-2t_2} &\sum_{m,n\geq 1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) \\ &\times \frac{1}{\sqrt{mn}} W_{t_1,r_1}\left(\frac{m}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{n}{\hat{q}^2}\right)\Delta_q^*(m,n). \end{aligned} \tag{3.5}$$

3.2. Applying the Petersson trace formula. Here we apply Theorem 2.6 for $\nu \geq 3$. The case $\nu = 2$ can be treated similarly, but does not seem to be of particular interest since the final goal is $\nu = \infty$. Let

$$\begin{aligned} T(c) := c &\sum_{\substack{m,n\geq 1\\(q,mn)=1}} \frac{\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)}{\sqrt{nm}} \\ &\times W_{t_1,r_1}\left(\frac{m}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{n}{\hat{q}^2}\right) S(m,n,c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \end{aligned} \tag{3.6}$$

By using the trace formula (2.25), the fourth moment (3.5) can be written as a sum of diagonal and nondiagonal parts.

PROPOSITION 3.4. *The following decomposition holds:*

$$M_4(\boldsymbol{t},\boldsymbol{r}) = M^{\mathrm{D}} + M_1^{\mathrm{ND}} + M_2^{\mathrm{ND}}, \tag{3.7}$$

where

$$M^{\mathrm{D}} = \frac{\phi(q)}{q}\hat{q}^{-2t_1-2t_2} \times \sum_{\substack{n\ge 1\\ (q,n)=1}} \frac{\tau_{1/2+ir_1}(n)\tau_{1/2+ir_2}(n)}{n} W_{t_1,r_1}\left(\frac{n}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{n}{\hat{q}^2}\right), \tag{3.8}$$

$$M_1^{\mathrm{ND}} = 2\pi i^{-k}\hat{q}^{-2t_1-2t_2}\sum_{q\mid c}\frac{1}{c^2}T(c), \tag{3.9}$$

$$M_2^{\mathrm{ND}} = -\frac{2\pi i^{-k}}{p}\hat{q}^{-2t_1-2t_2}\sum_{\frac{q}{p}\mid c}\frac{1}{c^2}T(c). \tag{3.10}$$

REMARK 3.5. For any $\epsilon > 0$ we have $M^{\mathrm{D}} \ll_{\epsilon,\boldsymbol{r}} (\phi(q)/q)q^{\epsilon}$. The asymptotics of this term will be evaluated in Section 4.2.

3.3. Smooth partition of unity and restriction of summations. Assume that $F_X(x)$ is a compactly supported function in $[X/2, 3X]$ such that for any integer $j \ge 0$,

$$x^j F_X^{(j)}(x) \ll_j 1. \tag{3.11}$$

We make a smooth dyadic partition of unity (see [RR, Appendix A] for details). Accordingly,

$$\frac{1}{\sqrt{mn}} W_{t_1,r_1}\left(\frac{m}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{n}{\hat{q}^2}\right) = \sum_{M,N\ge 1} F_{M,N}(m,n),$$

where the sums over M, N are over powers of 2 and

$$F_{M,N}(m,n) := f_{M,t_1,r_1}(m) f_{N,t_2,r_2}(n), \tag{3.12}$$

$$f_{X,t,r}(x) := \frac{1}{\sqrt{x}} W_{t,r}\left(\frac{x}{\hat{q}^2}\right) F_X(x). \tag{3.13}$$

The term (3.6) can be written as

$$T(c) = \sum_{M,N\ge 1} T_{M,N}(c), \tag{3.14}$$

where

$$T_{M,N}(c) = c \sum_{\substack{m,n \\ (q,mn)=1}} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) \times S(m,n,c)F_{M,N}(m,n)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{3.15}$$

LEMMA 3.6. *For any* $\alpha \geq |t|$,

$$x^i \frac{\partial^i}{\partial x^i} f_{X,t,r}(x) \ll_{\alpha,t,r} \frac{1}{\sqrt{X}}\left(\frac{x}{\hat{q}^2}\right)^{-\alpha} \qquad \text{if } X \gg q^{1+\epsilon}, \tag{3.16}$$

$$x^i \frac{\partial^i}{\partial x^i} f_{X,t,r}(x) \ll_{t,r} \frac{1}{\sqrt{X}}\left(\frac{x}{\hat{q}^2}\right)^{-|t|} \qquad \text{if } X \ll q^{1+\epsilon}. \tag{3.17}$$

Proof. If $X \gg q^{1+\epsilon}$ we use (3.2) to get

$$x^i \frac{\partial^i}{\partial x^i} W_{t,r}\left(\frac{x}{\hat{q}^2}\right) \ll_{\alpha,t,r} \left(\frac{x}{\hat{q}^2}\right)^{-\alpha}.$$

If $X \ll q^{1+\epsilon}$ we use (3.3) to get

$$x^i \frac{\partial^i}{\partial x^i} W_{t,r}\left(\frac{x}{\hat{q}^2}\right) \ll_{t,r} \left(\frac{x}{\hat{q}^2}\right)^{-|t|}.$$

Finally, estimate (3.11) and Leibniz's rule yield the result. ■

PROPOSITION 3.7. *For any* $\epsilon > 0$, *any* $A > 0$ *and* $l = 0, 1$,

$$\sum_{\max(M,N)\gg q^{1+\epsilon}} \sum_{\frac{q}{p^l}|c} \frac{1}{c^2} T_{M,N}(c) \ll_{\epsilon,A,\boldsymbol{r}} q^{-A}. \tag{3.18}$$

Proof. Since $\max(M,N) \gg q^{1+\epsilon}$, there are three cases to consider:

- $M \gg q^{1+\epsilon}$, $N \ll q^{1+\epsilon}$;
- $M \ll q^{1+\epsilon}$, $N \gg q^{1+\epsilon}$;
- $M \gg q^{1+\epsilon}$, $N \gg q^{1+\epsilon}$.

We prove only the first case:

$$\sum_{\substack{M\gg q^{1+\epsilon} \\ N\ll q^{1+\epsilon}}} \sum_{\frac{q}{p^l}|c} \frac{1}{c^2} T_{M,N}(c) = \sum_{\substack{M\gg q^{1+\epsilon} \\ N\ll q^{1+\epsilon}}} \sum_{\substack{c,m,n \\ (q,mn)=1 \\ \frac{q}{p^l}|c}} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) \times \frac{S(m,n,c)}{c} F_{M,N}(m,n) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

The sum over c can be decomposed into two cases:

$$\sum_{\frac{q}{p^l}|c} \frac{S(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) = \sum_{\substack{c<\sqrt{mn}\\ \frac{q}{p^l}|c}} \frac{S(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) + \sum_{\substack{c\geq\sqrt{mn}\\ \frac{q}{p^l}|c}} \frac{S(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

By (A.4) and (2.20) for any $\delta > 0$ we have

$$\sum_{\frac{q}{p^l}|c} \frac{S(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \ll (mn)^{3/4+\delta}.$$

We apply Lemma 3.6 with $i = j = 0$:

$$\sum_{\frac{q}{p^l}|c} \frac{1}{c^2} T_{M,N}(c) \ll_{\alpha_1,\boldsymbol{r}} (MN)^{1/4+\delta} \left(\frac{q}{M}\right)^{\alpha_1} \left(\frac{q}{N}\right)^{|t_2|}.$$

Taking α_1 sufficiently large, we find that for any $\epsilon > 0$, any $A > 0$ and $l = 0, 1$,

$$\sum_{\substack{M\gg q^{1+\epsilon}\\ N\ll q^{1+\epsilon}}} \sum_{\frac{q}{p^l}|c} \frac{1}{c^2} T_{M,N}(c) \ll_{\epsilon,A,\boldsymbol{r}} q^{-A}. \blacksquare$$

COROLLARY 3.8. *The range of summation in* (3.14) *can be restricted to* $M, N \ll q^{1+\epsilon}$.

Finally, we restrict the range of summation over c via a large sieve inequality.

LEMMA 3.9. *Let* $l = 0, 1$. *Assume that* $M, N \ll q^{1+\epsilon}$. *For any* $C > \sqrt{MN}$ *we have*

$$\sum_{\substack{c\geq C\\ \frac{q}{p^l}|c}} \frac{1}{c^2} T_{M,N}(c) \ll_{\epsilon,\boldsymbol{r}} \left(\frac{\hat{q}^2}{M}\right)^{|t_1|} \left(\frac{\hat{q}^2}{N}\right)^{|t_2|} q^{\epsilon} \left(\frac{\sqrt{MN}}{C}\right)^{k-1-2\theta}. \tag{3.19}$$

REMARK 3.10. If we take $C = \min(q^{\frac{1}{2-2\theta}} M^{1/2} N^{\frac{1-4\theta}{8-8\theta}}, q^{\frac{9-8\theta}{8-8\theta}})$, the error term is bounded by $q^{\epsilon - \frac{k-1-2\theta}{8-8\theta}}$. See the proof of Lemma 5.3 for the explanation of this choice.

Proof of Lemma 3.9. We are going to apply Theorem 2.3. In order to do so, we make a dyadic partition of the interval $[C, \infty)$ and assume that

$c \in [C, 2C]$. By definition

$$\sum_{\frac{q}{p^l}|c} \frac{1}{c^2} T_{M,N}(c) = \sum_{\substack{n,m \\ (q,nm)=1}} \sum_{\frac{q}{p^l}|c} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)\frac{1}{c}S(m,n,c)$$

$$\times J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)F_{M,N}(m,n)$$

$$= \frac{p^l}{q} \sum_{\substack{n,m \\ (q,nm)=1}} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)$$

$$\times \sum_{c_1} \frac{1}{c_1} S(m,n,c_1q/p^l) J_{k-1}\left(\frac{4\pi\sqrt{mn}\,p^l}{c_1 q}\right)F_{M,N}(m,n).$$

Here $m \in [M/2, 3M]$, $n \in [N/2, 3N]$ and $c_1 \in [C_1, 2C_1]$ with $C_1 := Cp^l/q$. Let

$$X := \left(\frac{\hat{q}^2}{M}\right)^{-|t_1|}\left(\frac{\hat{q}^2}{N}\right)^{-|t_2|}\sqrt{MN}\,C_1\left(\frac{\sqrt{MN}}{C}\right)^{-k+1}.$$

As a test function we choose

$$g(m,n,c_1) := \frac{X}{c_1}F_{M,N}(m,n)J_{k-1}\left(\frac{4\pi\sqrt{mn}\,p^l}{c_1 q}\right).$$

It satisfies condition (2.21), and Theorem 2.3 can be applied with $r = 1$ and $s = q/p^l$. Hence

$$\sum_{\substack{\frac{q}{p^l}|c \\ c\geq C}} \frac{1}{c^2} T_{M,N}(c) \ll_{\epsilon,\boldsymbol{r}} \left(\frac{\hat{q}^2}{M}\right)^{|t_1|}\left(\frac{\hat{q}^2}{N}\right)^{|t_2|} q^{\epsilon}\left(\frac{\sqrt{MN}}{C}\right)^{k-1-2\theta}. \blacksquare$$

3.4. Removing the coprimality condition. In order to apply Theorem 2.8, we have to exclude the coprimality condition in $T_{M,N}(c)$. This can be done using the criterion of vanishing of classical Kloosterman sums given by Lemma 2.2. Let

$$f(m,n,c) := F_{M,N}(m,n)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{3.20}$$

PROPOSITION 3.11. *Let m, n, c be three strictly positive integers and p be a prime number. Suppose $p^2 \mid c$. Then*

$$\begin{aligned}
&\sum_{(q,mn)=1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n,c)f(m,n,c)\\
&\quad = \sum_{m,n\geq 1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n,c)f(m,n,c)\\
&\qquad - \tau_{1/2+ir_2}(p)\sum_{m,n\geq 1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,np,c)f(m,np,c)\\
&\qquad + \sum_{m,n\geq 1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,np^2,c)f(m,np^2,c).
\end{aligned}$$

Proof. Recall that $q = p^\nu$. Therefore,

$$\sum_{(q,mn)=1} = \sum_{p\nmid mn} = \sum_{m,n} - \sum_{p\mid mn} = \sum_{m,n} - \sum_{p\mid n} - \sum_{p\mid m,p\nmid n}.$$

We have

$$\sum_{p\mid m,p\nmid n} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n,c)f(m,n,c) = 0$$

since the Kloosterman sum vanishes by Lemma 2.2. Further,

$$\begin{aligned}
&\sum_{p\mid n} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,n,c)f(m,n,c)\\
&\qquad\qquad = \sum_{n} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(np)S(m,np,c)f(m,np,c).
\end{aligned}$$

The identity (2.27) implies that

$$\tau_{1/2+ir_2}(np) = \begin{cases} \tau_{1/2+ir_2}(p)\tau_{1/2+ir_2}(n) - \tau_{1/2+ir_2}\left(\dfrac{n}{p}\right) & \text{if } (p,n)=p,\\ \tau_{1/2+ir_2}(p)\tau_{1/2+ir_2}(n) & \text{if } (p,n)=1. \end{cases}$$

This yields the result. ∎

3.5. Applying the Poisson-type summation formula. By Proposition 3.11, the term (3.15) can be decomposed as

$$T_{M,N}(c) = TS(c,0) - \tau_{1/2+ir_2}(p)TS(c,1) + TS(c,2), \tag{3.21}$$

where

$$TS(c,B) = c\sum_{m,n\geq 1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)S(m,np^B,c)f(m,np^B,c) \tag{3.22}$$

with $B = 0,1,2$ and $f(m,n,c)$ defined by (3.20).

Proposition 3.12. *One has*

$$TS(c,B) = TS^*(c,B) + TS^+(c,B) + TS^-(c,B), \tag{3.23}$$

where

$$(3.24)\qquad TS^*(c,B)=\sum_{n\geq 1}\tau_{1/2+ir_2}(n)S(0,np^B,c)[G^*_{r_1}(np^B)+G^*_{-r_1}(np^B)],$$

$$(3.25)\qquad TS^{\mp}(c,B)=\sum_{m,n\geq 1}\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)\times S(0,np^B\mp m,c)G^{\mp}_{r_1}(m,np^B).$$

The functions G^*_r, G^-_r, G^+_r *are defined as follows:*

$$(3.26)\qquad G^*_r(y)=\frac{\zeta(1+2ir)}{c^{2ir}}\int_0^\infty J_{k-1}\left(\frac{4\pi\sqrt{xy}}{c}\right)F_{M,N}(x,y)x^{ir}\,dx,$$

$$(3.27)\qquad G^-_r(z,y)=2\pi\int_0^\infty k_0\left(\frac{4\pi\sqrt{xz}}{c},1/2+ir\right)\times J_{k-1}\left(\frac{4\pi\sqrt{xy}}{c}\right)F_{M,N}(x,y)\,dx,$$

$$(3.28)\qquad G^+_r(z,y)=2\pi\int_0^\infty k_1\left(\frac{4\pi\sqrt{xz}}{c},1/2+ir\right)\times J_{k-1}\left(\frac{4\pi\sqrt{xy}}{c}\right)F_{M,N}(x,y)\,dx.$$

Proof. The function f is smooth, compactly supported, and thus satisfies all conditions of Theorem 2.8. Applying the summation formula with $\phi(x):=f\big(\frac{c^2}{16\pi^2}x^2,np^B,c\big)$, we obtain

$$\begin{aligned}\sum_{m\geq 1}e\left(\frac{md}{c}\right)\tau_{1/2+ir_1}(m)f(m,np^B,c)\\ =\frac{\zeta(1+2ir_1)}{c^{1+2ir_1}}\int_0^\infty f(x,np^B,c)x^{ir_1}dx+\frac{\zeta(1-2ir_1)}{c^{1-2ir_1}}\int_0^\infty f(x,np^B,c)x^{-ir_1}\,dx\\ +\frac{2\pi}{c}\sum_{m\geq 1}\tau_{1/2+ir_1}(m)\int_0^\infty e\left(\frac{-m\overline{d}}{c}\right)k_0\left(\frac{4\pi}{c}\sqrt{xm},1/2+ir_1\right)f(x,np^B,c)\,dx\\ +\frac{2\pi}{c}\sum_{m\geq 1}\tau_{1/2+ir_1}(m)\int_0^\infty e\left(\frac{m\overline{d}}{c}\right)k_1\left(\frac{4\pi}{c}\sqrt{xm},1/2+ir_1\right)f(x,np^B,c)\,dx.\end{aligned}$$

Plugging this into (3.22) yields the assertion. ∎

The next lemma shows that the $TS^*(c)$ term contributes to the fourth moment as an error.

LEMMA 3.13. *Let $l = 0, 1$. Then*

$$\sum_{\frac{q}{p^l}|c} \sum_{M,N\le q^{1+\epsilon}} c^{-2} TS^*(c,B) \ll_{\epsilon,\boldsymbol{r}} q^{-1+\epsilon}. \tag{3.29}$$

Proof. We use Lemma 3.6 to estimate $F_{M,N}(m,n)$. The J-Bessel function can be trivially bounded by 1. Then

$$G_r^*(np^B) \ll_{\boldsymbol{r}} \left(\frac{\hat{q}^2}{M}\right)^{|t_1|} \left(\frac{\hat{q}^2}{N}\right)^{|t_2|} \left(\frac{M}{N}\right)^{1/2}.$$

Since $S(0, np^B, c) \ll (np^B, c)$, we have

$$TS^*(c,B) \ll_{\boldsymbol{r}} (MN)^{1/2} q^{\epsilon} \left(\frac{\hat{q}^2}{M}\right)^{|t_1|} \left(\frac{\hat{q}^2}{N}\right)^{|t_2|}.$$

Therefore,

$$\sum_{\frac{q}{p^l}|c} \sum_{M,N\le q^{1+\epsilon}} c^{-2} TS^*(c,B) \ll_{\epsilon,\boldsymbol{r}} q^{-1+\epsilon}. \blacksquare$$

The last two summands require a more detailed treatment. We rewrite the sums $TS^{\pm}$ in the form that is more convenient for later computations:

$$\begin{aligned} TS^-(c,B) &= \sum_{m\ge1}\sum_{n\ge1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) S(0, np^B - m, c) G_{r_1}^-(m, np^B) \\ &= \phi(c) \sum_{n\ge1} \tau_{1/2+ir_1}(np^B)\tau_{1/2+ir_2}(n) G_{r_1}^-(np^B, np^B) + \sum_{h\ne0} S(0,h,c) T_h^-(c,B) \end{aligned}$$

and

$$\begin{aligned} TS^+(c,B) &= \sum_{m\ge1}\sum_{n\ge1} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) S(0, np^B + m, c) G_{r_1}^+(m, np^B) \\ &= \sum_{h\ne0} S(0,h,c) T_h^+(c,B), \end{aligned}$$

where

$$T_h^{\mp}(c,B) = \sum_{m\mp np^B = h} \tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n) G_{r_1}^{\mp}(m, p^B n). \tag{3.30}$$

At this point, the nondiagonal term splits into the off-diagonal (corresponding to $h = 0$) and off-off-diagonal ($h \ne 0$) parts.

THEOREM 3.14. *One has*

$$M^{\mathrm{OD}} = M^{\mathrm{OD}}(0) - \tau_{1/2+ir_2}(p) M^{\mathrm{OD}}(1) + M^{\mathrm{OD}}(2), \tag{3.31}$$

$$M^{\mathrm{OOD}} = M^{\mathrm{OOD}}(0) - \tau_{1/2+ir_2}(p) M^{\mathrm{OOD}}(1) + M^{\mathrm{OOD}}(2). \tag{3.32}$$

For $B = 0, 1, 2$,

$$M^{\mathrm{OD}}(B) = 2\pi i^{-k}\Bigg(\sum_{\substack{q|c\\ c\ll C}}\frac{\phi(c)}{c^2}\sum_{\substack{n\\ M,N\ll q^{1+\epsilon}}}\tau_{1/2+ir_1}(np^B)\tau_{1/2+ir_2}(n)G^-_{r_1}(np^B,np^B)$$

$$-\frac{1}{p}\sum_{\substack{\frac{q}{p}|c\\ c\ll C}}\frac{\phi(c)}{c^2}\sum_{\substack{n\\ M,N\ll q^{1+\epsilon}}}\tau_{1/2+ir_1}(np^B)\tau_{1/2+ir_2}(n)G^-_{r_1}(np^B,np^B)\Bigg),$$

$$M^{\mathrm{OOD}}(B) = 2\pi i^{-k}\Bigg(\sum_{\substack{q|c\\ c\ll C}}\frac{1}{c^2}\sum_{M,N\ll q^{1+\epsilon}}\sum_{h\neq 0}S(0,h,c)(T^-_h(c,B)+T^+_h(c,B))$$

$$-\frac{1}{p}\sum_{\substack{\frac{q}{p}|c\\ c\ll C}}\frac{1}{c^2}\sum_{M,N\ll q^{1+\epsilon}}\sum_{h\neq 0}S(0,h,c)(T^-_h(c,B)+T^+_h(c,B))\Bigg).$$

Here $T^{\pm}_h(c,B)$ *is given by* (3.30) *and* $G^{\pm}_r(z,y)$ *by* (3.27) *and* (3.28).

4. Asymptotic evaluation of diagonal and off-diagonal terms. The main result of this section is an asymptotic formula for the diagonal and off-diagonal terms.

THEOREM 4.1. *Up to a negligible error term, we have*

$$M^{\mathrm{D}} + M^{\mathrm{OD}} = \frac{\phi(q)}{q}\sum_{\epsilon_1,\epsilon_2=\pm 1}\hat{q}^{-2t_1-2t_2+2\epsilon_1 t_1+2\epsilon_2 t_2}\frac{\Gamma(\epsilon_1 t_1+ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)}$$
$$\times\frac{\Gamma(\epsilon_1 t_1-ir_1+k/2)}{\Gamma(t_1-ir_1+k/2)}\frac{\Gamma(\epsilon_2 t_2+ir_2+k/2)\Gamma(\epsilon_2 t_2-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}$$
$$\times\zeta_q(1+2\epsilon_1 t_1)\zeta_q(1+2\epsilon_2 t_2)\frac{\prod_{\epsilon_3,\epsilon_4=\pm 1}\zeta_q(1+\epsilon_1 t_1+\epsilon_2 t_2+i\epsilon_3 r_1+i\epsilon_4 r_2)}{\zeta_q(2+2\epsilon_1 t_1+2\epsilon_2 t_2)}. \tag{4.1}$$

4.1. Extension of summations. First, we reintroduce the summation over $c > C$ and $\max(M,N) \gg q^{1+\epsilon}$ for the off-diagonal term at the cost of an admissible error.

PROPOSITION 4.2. *For any* $\epsilon > 0$,

$$\sum_{\substack{\frac{q}{p^l}|c\\ c>C}}\frac{\phi(c)}{c^2}\sum_{\max(M,N)\ll q^{1+\epsilon}}\sum_{n}\tau_{1/2+ir_1}(np^B)\tau_{1/2+ir_2}(n)G^-_{r_1}(np^B,np^B) \ll_{\epsilon,\boldsymbol{r}} q^{\epsilon-\frac{k-1}{8-8\theta}}. \tag{4.2}$$

Proof. Let

$$\eta_C(c) = \begin{cases} 0 & \text{if } c > C,\\ 1 & \text{if } c \le C.\end{cases}$$

Consider

$$T_1 := \sum_{\substack{\frac{q}{p^l}|c \\ c>C}} \frac{\phi(c)}{c^2} \sum_{\max(M,N)\ll q^{1+\epsilon}} \sum_n \tau_{1/2+ir_1}(np^B)\tau_{1/2+ir_2}(n) G^-_{r_1}(np^B, np^B)$$
$$= \sum_{\max(M,N)\ll q^{1+\epsilon}} \sum_n \tau_{1/2+ir_1}(np^B)\tau_{1/2+ir_2}(n) \int_0^\infty k_0\big(4\pi\sqrt{xnp^B}, 1/2+ir_1\big)$$
$$\times 2\pi J_{k-1}\big(4\pi\sqrt{xnp^B}\big) \sum_{\frac{q}{p^l}|c} (1-\eta_C(c))\phi(c) F_{M,N}(xc^2, np^B)\, dx.$$

We use Lemma 3.6 to bound $F_{M,N}(xc^2, np^B)$, formula (A.4) to bound the Bessel function $J_{k-1}\big(4\pi\sqrt{xnp^B}\big)$, and a trivial estimate for the Bessel kernel,

$$k_0\big(4\pi\sqrt{xnp^B}, 1/2+ir_1\big) \ll 1.$$

Then

$$T_1 \ll_{\epsilon,\boldsymbol{r}} q^\epsilon \sum_{M,N\ll q^{1+\epsilon}} \frac{1}{\sqrt{MN}} \sum_{n\sim N} \int_0^{2M/C^2} (\sqrt{xn})^{k-1} \frac{M}{qx}\, dx \ll_{\epsilon,\boldsymbol{r}} q^{\epsilon - \frac{k-1}{8-8\theta}}. \ \blacksquare$$

Proposition 4.3. *For any $\epsilon > 0$, any $A > 0$ and $l = 0, 1$,*

$$\sum_{\frac{q}{p^l}|c} \frac{\phi(c)}{c^2} \sum_{\max(M,N)\gg q^{1+\epsilon}} \sum_n \tau_{1/2+ir_1}(np^B)\tau_{1/2+ir_2}(n) G^-_{r_1}(np^B, np^B) \ll_{\epsilon,A,\boldsymbol{r}} q^{-A}. \tag{4.3}$$

Proof. This can be proved analogously to Proposition 3.7. $\blacksquare$

Now it is possible to combine all functions F_M into F and replace $\sum_{M,N} F_{M,N}$ by

$$F(x,y) := \frac{1}{\sqrt{xy}} W_{t_1,r_1}\Big(\frac{x}{\hat{q}^2}\Big) W_{t_2,r_2}\Big(\frac{y}{\hat{q}^2}\Big) F(x)F(y), \tag{4.4}$$

where $F(x)$ is a smooth function, compactly supported in $[1/2,\infty)$ such that $F(x) = 1$ for $x \geq 1$.

Proposition 4.4. *Up to an error term of $O_{\boldsymbol{r},\epsilon}(q^{\epsilon-k/2})$, the product $F(x)F(y)$ can be replaced by 1 in (4.4).*

Proof. Consider

$$T_2 := \sum_{\substack{c \\ q|c}} \frac{\phi(c)}{c^2} \sum_n \tau_{1/2+ir_1}(np^B)\tau_{1/2+ir_2}(n) \int_0^1 k_0\left(\frac{4\pi\sqrt{xnp^B}}{c}, 1/2+ir_1\right) \times J_{k-1}\left(\frac{4\pi\sqrt{xnp^B}}{c}\right) \frac{1}{\sqrt{xnp^B}} W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{np^B}{\hat{q}^2}\right)(1-F(x))\,dx.$$

We estimate the kernel $k_0(4\pi\sqrt{xnp^B}/c, 1/2+ir_1)$ trivially by 1 and apply the following bound for the J-Bessel function:

$$J_{k-1}\left(\frac{4\pi\sqrt{xnp^B}}{c}\right) \ll \left(\frac{\sqrt{xn}}{c}\right)^{k-1}.$$

If $n < q$, the function W_{t_2,r_2} can be estimated using (3.3). Otherwise we apply (3.2). This gives $T_2 \ll_{\boldsymbol{r}} q^{\epsilon-k/2}$. ∎

4.2. Asymptotics of diagonal and off-diagonal terms. The off-diagonal term can be written as

$$M^{\mathrm{OD}}(B) = \hat{q}^{-2t_1-2t_2} \sum_n \frac{\tau_{1/2+ir_1}(np^B)\tau_{1/2+ir_2}(n)}{np^B} W_{t_2,r_2}\left(\frac{np^B}{\hat{q}^2}\right) Z(np^B)$$

for $B = 0, 1, 2$ with

$$Z(u) := 2\pi i^{-k} \int_0^\infty k_0(z, 1/2+ir_1) J_{k-1}(z) \times \left(\sum_{q|c} \frac{\phi(c)}{c} W_{t_1,r_1}\left(\frac{z^2c^2}{(4\pi)^2\hat{q}^2u}\right) - \frac{1}{p}\sum_{\frac{q}{p}|c} \frac{\phi(c)}{c} W_{t_1,r_1}\left(\frac{z^2c^2}{(4\pi)^2\hat{q}^2u}\right)\right) dz.$$

Note that we made the change of variables $x = \frac{z^2c^2}{(4\pi)^2u}$ in the integral. Applying (3.1), we have

$$Z(u) = 2\pi i^{-k} \int_0^\infty k_0(z, 1/2+ir_1) J_{k-1}(z) \times \frac{1}{2\pi i} \int_{(3)} P_r(s) \frac{\Gamma(s+ir_1+k/2)\Gamma(s-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)} \times \zeta_q(1+2s)\left(\frac{z^2}{(4\pi)^2\hat{q}^2u}\right)^{-s} \left[\sum_{q|c} \frac{\phi(c)}{c^{1+2s}} - \frac{1}{p}\sum_{\frac{q}{p}|c} \frac{\phi(c)}{c^{1+2s}}\right] \frac{2s\,ds}{s^2-t_1^2}\,dz.$$

The term in the brackets can be simplified to

$$\sum_{q|c} \frac{\phi(c)}{c^{1+2s}} - \frac{1}{p} \sum_{\frac{q}{p}|c} \frac{\phi(c)}{c^{1+2s}} = \frac{\phi(q)}{q^{1+2s}} \frac{1-p^{2s-1}}{1-p^{-2s}} \frac{\zeta_q(2s)}{\zeta_q(2s+1)}.$$

Lemma A.3 implies that

$$\begin{aligned}\int_0^\infty k_0(z, 1/2+ir_1) J_{k-1}(z) z^{-2s} dz = \frac{\Gamma(2s)}{2^{2s+1}\cos(\pi(1/2+ir_1))} \\ \times \bigg(\frac{\Gamma(ir_1+k/2-s)}{\Gamma(-ir_1+k/2+s)\Gamma(ir_1+k/2+s)\Gamma(ir_1-k/2+s+1)} \\ - \frac{\Gamma(-ir_1+k/2-s)}{\Gamma(-ir_1+k/2+s)\Gamma(ir_1+k/2+s)\Gamma(-ir_1-k/2+s+1)} \bigg).\end{aligned}$$

By duplication and reflection formulas,

$$\begin{aligned}\int_0^\infty k_0(z, 1/2+ir_1) J_{k-1}(z) z^{-2s}\, dz = -\frac{i^k \Gamma(s)\Gamma(s+1/2)}{2^2 \pi^{3/2} \sin(\pi i r_1)} \\ \times \frac{\Gamma(ir_1+k/2-s)\Gamma(-ir_1+k/2-s)}{\Gamma(ir_1+k/2+s)\Gamma(-ir_1+k/2+s)} [\sin \pi(-s-ir_1) - \sin \pi(-s+ir_1)].\end{aligned}$$

Observe that

$$\frac{\Gamma(1/2-s)\Gamma(1/2+s)}{2\pi \sin(\pi i r_1)} [\sin(\pi(-s-ir_1)) - \sin(\pi(-s+ir_1))] = -1.$$

Consequently,

$$\begin{aligned}Z(u) = \frac{\phi(q)}{q} \frac{1}{2\pi i} \int_{(3)} P_r(s) \frac{\Gamma(-s+ir_1+k/2)\Gamma(-s-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)} \\ \times \zeta_q(1-2s) \left(\frac{u}{\hat{q}^2}\right)^s \frac{2s\, ds}{s^2-t_1^2}.\end{aligned}$$

Shifting the integration contour to $\Re s = -3$, we cross poles at $s = \pm t_1$. Hence

$$\begin{aligned}Z(u) = \frac{\phi(q)}{q} \sum_{\epsilon_1 = \pm 1} \frac{\Gamma(\epsilon_1 t_1 + ir_1 + k/2)\Gamma(\epsilon_1 t_1 - ir_1 + k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)} \\ \times \zeta_q(1+2\epsilon_1 t_1) \left(\frac{u}{\hat{q}^2}\right)^{-\epsilon_1 t_1} \\ - \frac{\phi(q)}{q} \frac{1}{2\pi i} \int_{(3)} P_r(s) \frac{\Gamma(s+ir_1+k/2)\Gamma(s-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)} \zeta_q(1+2s) \left(\frac{u}{\hat{q}^2}\right)^{-s} \frac{2s\, ds}{s^2-t_1^2}.\end{aligned}$$

Substitution of $Z(np^B)$ into $M^{\mathrm{OD}}(B)$ gives

$$M^{\mathrm{OD}}(B)=\frac{\phi(q)}{q}\hat{q}^{-2t_1-2t_2}\sum_{n}\frac{\tau_{1/2+ir_1}(np^B)\tau_{1/2+ir_2}(n)}{np^B}W_{t_2,r_2}\left(\frac{np^B}{\hat{q}^2}\right)$$
$$\times\left(-W_{t_1,r_1}\left(\frac{np^B}{\hat{q}^2}\right)+\sum_{\epsilon_1=\pm1}\frac{\Gamma(\epsilon_1t_1+ir_1+k/2)\Gamma(\epsilon_1t_1-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)}\right.$$
$$\left.\times\zeta_q(1+2\epsilon_1t_1)\left(\frac{np^B}{\hat{q}^2}\right)^{-\epsilon_1t_1}\right).$$

The multiplicity property (2.27) implies that

$$\sum_{\substack{n\geq1\\(n,p)=1}}\tau_{1/2+ir_2}(n)f(n)=\sum_{n\geq1}\tau_{1/2+ir_2}(n)f(n)$$
$$-\tau_{1/2+ir_2}(p)\sum_{n\geq1}\tau_{1/2+ir_2}(n)f(np)+\sum_{n\geq1}\tau_{1/2+ir_2}(n)f(np^2).$$

Thus,

$$M^{\mathrm{D}}+M^{\mathrm{OD}}=\frac{\phi(q)}{q}\hat{q}^{-2t_1-2t_2}\sum_{(n,p)=1}\frac{\tau_{1/2+ir_1}(n)\tau_{1/2+ir_2}(n)}{n}W_{t_2,r_2}\left(\frac{n}{\hat{q}^2}\right)\tag{4.5}$$
$$\times\left(\sum_{\epsilon_1=\pm1}\frac{\Gamma(\epsilon_1t_1+ir_1+k/2)\Gamma(\epsilon_1t_1-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)}\zeta_q(1+2\epsilon_1t_1)\left(\frac{n}{\hat{q}^2}\right)^{-\epsilon_1t_1}\right).$$

Ramanujan's identity (2.28) yields

$$\sum_{(n,p)=1}\frac{\tau_{1/2+ir_1}(n)\tau_{1/2+ir_2}(n)}{n^{1+\epsilon_1t_1+s}}=\frac{\prod_{\epsilon_3,\epsilon_4=\pm1}\zeta_q(1+\epsilon_1t_1+s+i\epsilon_3r_1+i\epsilon_4r_2)}{\zeta_q(2+2\epsilon_1t_1+2s)}.$$

Therefore,

$$M^{\mathrm{D}}+M^{\mathrm{OD}}=\frac{\phi(q)}{q}\sum_{\epsilon_1=\pm1}\frac{\Gamma(\epsilon_1t_1+ir_1+k/2)\Gamma(\epsilon_1t_1-ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)\Gamma(t_1-ir_1+k/2)}$$
$$\times\hat{q}^{-2t_1-2t_2+2\epsilon_1t_1}\frac{1}{2\pi i}\int_{\Re s=3}\frac{P_r(s)}{P_r(t_2)}\hat{q}^{2s}\zeta_q(1+2s)\zeta_q(1+2\epsilon_1t_1)$$
$$\times\frac{\Gamma(s+ir_2+k/2)\Gamma(s-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}$$
$$\times\frac{\prod_{\epsilon_3,\epsilon_4=\pm1}\zeta_q(1+\epsilon_1t_1+s+i\epsilon_3r_1+i\epsilon_4r_2)}{\zeta_q(2+2\epsilon_1t_1+2s)}\frac{2s\,ds}{s^2-t_2^2}.$$

After shifting the integration contour to $\Re s=-1/2$, the resulting integral is bounded by $q^{\epsilon-1/2}$ plus the contribution of the simple poles at $s=\pm t_2$.

Up to an error term,

$$M^{\mathrm{D}}+M^{\mathrm{OD}}=\frac{\phi(q)}{q}\sum_{\epsilon_1,\epsilon_2=\pm1}\hat{q}^{-2t_1-2t_2+2\epsilon_1t_1+2\epsilon_2t_2}\frac{\Gamma(\epsilon_1t_1+ir_1+k/2)}{\Gamma(t_1+ir_1+k/2)}$$
$$\times\frac{\Gamma(\epsilon_1t_1-ir_1+k/2)}{\Gamma(t_1-ir_1+k/2)}\frac{\Gamma(\epsilon_2t_2+ir_2+k/2)\Gamma(\epsilon_2t_2-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}$$
$$\times\zeta_q(1+2\epsilon_1t_1)\zeta_q(1+2\epsilon_2t_2)\frac{\prod_{\epsilon_3,\epsilon_4=\pm1}\zeta_q(1+\epsilon_1t_1+\epsilon_2t_2+i\epsilon_3r_1+i\epsilon_4r_2)}{\zeta_q(2+2\epsilon_1t_1+2\epsilon_2t_2)}.$$

By letting the shifts tend to zero in (4.5), we find

$$M^{\mathrm{D}}+M^{\mathrm{OD}}=\left(\frac{\phi(q)}{q}\right)^2\sum_{(n,p)=1}\frac{\tau(n)^2}{n}W_{0,0}\left(\frac{n}{\hat{q}^2}\right)\log\left(\frac{\hat{q}^2}{n}\right).\tag{4.6}$$

The equality (2.29) gives

$$M^{\mathrm{D}}+M^{\mathrm{OD}}=\frac{1}{2\pi i}\left(\frac{\phi(q)}{q}\right)^2\int_{(3)}\frac{P_r(s)}{P_r(0)}\frac{\Gamma(k/2+s)^2}{\Gamma(k/2)^2}\zeta_q(1+2s)\hat{q}^{2s}\frac{\zeta_q(1+s)^4}{\zeta_q(2+2s)}$$
$$\times\left[\log\hat{q}^2+4\frac{\zeta_q'}{\zeta_q}(1+s)-2\frac{\zeta_q'}{\zeta_q}(2+2s)\right]\frac{2\,ds}{s}.$$

If we shift the integration contour to $\Re s=-1/2$, the resulting integral is bounded by $q^{-1/2}$ plus the contribution of the multiple poles at $s=0$. Calculation of the residue

$$\left(\frac{\phi(q)}{q}\right)^7\frac{1}{\zeta_q(2)}\operatorname{Res}_{s=0}\frac{\hat{q}^{2s}}{s^6}\left(\log\hat{q}-\frac{4}{s}\right)$$

shows that the main term is

$$\left(\frac{\phi(q)}{q}\right)^7\frac{p^2}{p^2-1}\frac{(\log q)^6}{60\pi^2}.$$

5. Off-off-diagonal term: double integral representation. In this section, we will show that the off-off-diagonal main term can be written as a double integral.

Theorem 5.1. *Up to a negligible error, we have*

$$M^{\mathrm{OOD}}=\frac{\phi(q)}{q}\sum_{\epsilon_1,\epsilon_2=\pm1}\frac{\zeta_q(1+2i\epsilon_1r_1)\zeta_q(1+2i\epsilon_2r_2)}{\zeta_q(2+2i\epsilon_1r_1+2i\epsilon_2r_2)}\hat{q}^{-2t_1-2t_2}$$
$$\times\hat{q}^{-2i\epsilon_1r_1+2i\epsilon_2r_2}\frac{1}{(2\pi i)^2}\int_{\Re t=k/2+0.7}\int_{\Re s=k/2-0.4}I_{\epsilon_1,\epsilon_2}(s,t)\frac{2s\,ds}{s^2-t_1^2}\frac{2t\,dt}{t^2-t_2^2},\tag{5.1}$$

where

$$I_{\epsilon_1,\epsilon_2}(s,t) = \frac{P_r(s)P_r(t)}{P_r(t_1)P_r(t_2)}\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2) \tag{5.2}$$

$$\times\zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta_q(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)$$

$$\times\frac{\Gamma(k/2+s+i\epsilon_1 r_1)\Gamma(k/2+t+i\epsilon_2 r_2)}{\Gamma(k/2+t_1+ir_1)\Gamma(k/2+t_1-ir_1)}\frac{\Gamma(k/2-s+i\epsilon_1 r_1)\Gamma(k/2-t+i\epsilon_2 r_2)}{\Gamma(k/2+t_2+ir_2)\Gamma(k/2+t_2-ir_2)}.$$

5.1. Estimation of $G^{\mp}_{r_1}$. The expression

$$T^{\pm}_h(c,B) = \sum_{m\pm np^B=h}\tau_{1/2+ir_1}(m)\tau_{1/2+ir_2}(n)G^{\pm}_{r_1}(m,p^B n)$$

can be evaluated using Theorem 2.9. To this end, we show that the functions $G^{\pm}_{r_1}$, defined by (3.27) and (3.28), satisfy condition (2.33).

Let $Q := 1+\sqrt{MN}/c$, $Z := Q^2c^2/M$ and $Y := N$.

LEMMA 5.2. *For all positive n_1 and n_2,*

$$z^j y^i\frac{\partial^j}{\partial z^j}\frac{\partial^i}{\partial y^i}G^{\pm}_{r_1}(z,y) \ll \left(1+\frac{z}{Z}\right)^{-n_1}\left(1+\frac{y}{Y}\right)^{-n_2}\frac{M^{1/2}}{N^{1/2}} \times\left(\frac{\sqrt{MN}}{c}\right)^{k-1}Q^{j+i-k+1/2}. \tag{5.3}$$

Proof. Consider

$$G^{-}_{r_1}(z,y) = 2\pi\int_0^\infty k_0\left(\frac{4\pi\sqrt{xz}}{c},1/2+ir_1\right)J_{k-1}\left(\frac{4\pi\sqrt{xy}}{c}\right)F_{M,N}(x,y)\,dx$$
$$= -\frac{\pi}{\sin(\pi i r_1)}\int_0^\infty\left[J_{2ir_1}\left(\frac{4\pi\sqrt{xz}}{c}\right)-J_{-2ir_1}\left(\frac{4\pi\sqrt{xz}}{c}\right)\right] \times J_{k-1}\left(\frac{4\pi\sqrt{xy}}{c}\right)F_{M,N}(x,y)\,dx.$$

Suppose that $z>Z$. Let $u := 4\pi\sqrt{xz}/c$. Then

$$G^{-}_{r_1}(z,y) = -\frac{c^2}{8\pi z\sin(\pi i r_1)}\int_0^\infty u[J_{2ir_1}(u)-J_{-2ir_1}(u)] \times J_{k-1}\left(u\sqrt{\frac{y}{z}}\right)F_{M,N}\left(\frac{c^2u^2}{16\pi^2 z},y\right)du.$$

It is sufficient to estimate

$$G_1(z,y) := -\frac{c^2}{8\pi z\sin(\pi i r_1)}\int_0^\infty uJ_{2ir_1}(u)J_{k-1}\left(u\sqrt{\frac{y}{z}}\right)F_{M,N}\left(\frac{c^2u^2}{16\pi^2 z},y\right)du.$$

Note that $F_{M,N}(x,y)$ is compactly supported on $[M/2,3M]\times[N/2,3N]$. Let

$$f(u) := g_1(u)g_2(u)u^{-2ir_1}$$

with

$$g_1(u) := J_{k-1}\left(u\sqrt{\frac{y}{z}}\right) \quad \text{and} \quad g_2(u) := F_{M,N}\left(\frac{c^2u^2}{16\pi^2 z}, y\right).$$

The recurrence relation (A.1) implies that

$$G_1(z,y) = -\frac{c^2}{8\pi z\sin(\pi i r_1)}\int_0^\infty (u^{1+2ir_1}J_{1+2ir_1}(u))' f(u)\,du.$$

Integration by parts gives

$$\begin{aligned} G_1(z,y) &= \frac{c^2}{8\pi z\sin(\pi i r_1)}\int_0^\infty u^{1+2ir_1}J_{1+2ir_1}(u)f'(u)\,du \\ &= -\frac{c^2}{8\pi z\sin(\pi i r_1)}\int_0^\infty u^{2+2ir_1}J_{2+2ir_1}(u)\left(\frac{1}{u}f'(u)\right)'du. \end{aligned}$$

Repeating the procedure n times, we obtain

$$\begin{aligned} G_1(z,y) &= (-1)^{n+1}\frac{c^2}{8\pi z\sin(\pi i r_1)}\int_0^\infty u^{n+2ir_1}J_{n+2ir_1}(u)h_n(u)\,du \\ &= (-1)^{n+1}\frac{c^2}{8\pi z\sin(\pi i r_1)}\int_{u\sim\sqrt{Mz}/c}\frac{J_{n+2ir_1}(u)}{u^{n-1-2ir_1}}u^{2n-1}h_n(u)\,du, \end{aligned}$$

where

$$\begin{aligned} &h_0(u) = f(u), \qquad h_1(u) = f'(u), \\ &h_n(u) = (u^{-1}h_{n-1}(u))' \quad \text{for } n \ge 2. \end{aligned}$$

By induction, for $n \ge 1$,

$$u^{2n-1}h_n(u) = \sum_{i=0}^{n} c(i,n)f^{(i)}(u)u^i$$

with

$$\begin{aligned} f^{(i)}(u)u^i &\ll \sum_{j+l+m=i}(g_1^{(j)}(u)u^j)(g_2^{(l)}(u)u^l)u^{2ir_1} \\ &\ll \sum_{j+m<i}(g_1^{(j)}(u)u^j)(g_2^{(i-j-m)}(u)u^{i-j-m}). \end{aligned}$$

Faà di Bruno's formula and the estimate (A.4) give

$$u^j g_1^{(j)}(u) = u^j \frac{\partial^j}{\partial u^j}\left(J_{k-1}\left(u\sqrt{\frac{y}{z}}\right)\right) = \left(\sqrt{\frac{y}{z}}u\right)^j J_{k-1}^{(j)}\left(u\sqrt{\frac{y}{z}}\right)$$
$$\ll \frac{(u\sqrt{y/z})^{k-1}(1+u\sqrt{y/z})^j}{(1+u\sqrt{y/z})^{k-1/2}}.$$

Applying Faà di Bruno's formula to the second function, we obtain

$$u^{i-j-m} g_2^{(i-j-m)}(u) = u^{i-j-m}\frac{\partial^{i-j-m}}{\partial u^{i-j-m}} F_{M,N}\left(\frac{c^2u^2}{16\pi^2 z}, y\right)$$
$$= u^{i-j-m}\sum_{\substack{(m_1,m_2)\\ m_1+2m_2=i-j-m}} \frac{(i-j-m)!}{m_1!m_2!(2!)^{m_2}}$$
$$\times F_{M,N}^{(m_1+m_2)}\left(\frac{c^2u^2}{16\pi^2 z}, y\right)\left(\frac{2c^2}{16\pi^2 z}u\right)^{m_1}\left(\frac{2c^2}{16\pi^2 z}\right)^{m_2}.$$

Lemma 3.6 implies

$$u^{i-j-m} g_2^{(i-j-m)}(u) \ll \sum_{\substack{(m_1,m_2)\\ m_1+2m_2=i-j-m}} \left(\frac{c^2}{16\pi^2 z}u^2\right)^{m_1+m_2}$$
$$\times F_{M,N}^{(m_1+m_2)}\left(\frac{c^2u^2}{16\pi^2 z}, y\right) \ll (MN)^{-1/2}.$$

And for the J-Bessel function we use the trivial bound $J_{n+2ir_1}(u) \ll 1$. Hence

$$G_1(z,y) \ll \left(\frac{Qc}{\sqrt{Mz}}\right)^n \frac{M^{1/2}}{N^{1/2}}\left(\frac{\sqrt{MN}}{c}\right)^{k-1} Q^{-k+1/2}$$

for every integer $n>0$. The same bound is valid for $G_{r_1}^-(z,y)$. So, if $z>Z$, the value of $G_{r_1}^-(z,y)$ is small.

Suppose $z\le Z$. One can estimate $G_{r_1}^-(z,y)$ directly (without integration by parts):

$$G_{r_1}^-(z,y) \ll \frac{M^{1/2}}{N^{1/2}}\left(\frac{\sqrt{MN}}{c}\right)^{k-1} Q^{-k+1/2}.$$

Since $y\in[N/2,3N]$, we can add a factor of $(1+y/Y)^{-n_2}$.

Combining the two estimates for $G_{r_1}^-(z,y)$ into one, we find that for all positive n_1 and n_2,

$$G_{r_1}^-(z,y) \ll \left(1+\frac{z}{Z}\right)^{-n_1}\left(1+\frac{y}{Y}\right)^{-n_2}\frac{M^{1/2}}{N^{1/2}}\left(\frac{\sqrt{MN}}{c}\right)^{k-1} Q^{-k+1/2}.$$

Analogously, using relation (A.3) and the bound (A.6) for the K-Bessel function, we may estimate $G_{r_1}^+(z,y)$. Finally, differentiating $G_{r_1}^{\pm}(z,y)$ j times in z and i times in y, we find

$$z^j y^i \frac{\partial^j}{\partial z^j}\frac{\partial^i}{\partial y^i} G^{\pm}_{r_1}(z,y) \ll \left(1+\frac{z}{Z}\right)^{-n_1}\left(1+\frac{y}{Y}\right)^{-n_2} \times \frac{M^{1/2}}{N^{1/2}}\left(\frac{\sqrt{MN}}{c}\right)^{k-1} Q^{j+i-k+1/2}$$

for all positive n_1 and n_2. An extra factor of Q^{i+j} is obtained by differentiating the Bessel functions under the integral. Indeed, by Faà di Bruno's formula

$$z^j \frac{\partial^j}{\partial z^j} J_{2ir_1}(\alpha\sqrt{z}) = z^j \sum \binom{j}{m_1,\ldots,m_j} \times J_{2ir_1}^{(m_1+\cdots+m_j)}(\alpha\sqrt{z}) \cdot \prod_{n=1}^{j}\left(\frac{\alpha z^{1/2-n}}{n!}\right)^{m_n},$$

where $\alpha := 4\pi\sqrt{x}/c$ and the sum is over all j-tuples $(m_1,\ldots,m_j)$ such that $1\cdot m_1 + 2\cdot m_2 + \cdots + j\cdot m_j = j$. Formula (A.9) gives

$$J_{2ir_1}^{(b)}(z) = \frac{1}{2^b}\sum_{t=0}^{b}(-1)^t\binom{b}{t} J_{2ir_1-b+2t}(z).$$

When $z > Z$, the maximum of $z^j \frac{\partial^j}{\partial z^j} J_{2ir_1}(\alpha\sqrt{z})$ is attained when $m_1+\cdots+m_j = j$. Therefore,

$$z^j \frac{\partial^j}{\partial z^j} J_{2ir_1}(\alpha\sqrt{z}) \ll (\alpha\sqrt{z})^j \sum_{t=0}^{j} J_{2ir_1-j+2t}(\alpha\sqrt{z}).$$

This gives an extra factor of $(\sqrt{Mz}/c)^j$ and

$$G^{-}_{r_1}(z,y) \ll \left(\frac{Qc}{\sqrt{Mz}}\right)^{n-j} Q^j \frac{M^{1/2}}{N^{1/2}}\left(\frac{\sqrt{MN}}{c}\right)^{k-1} Q^{-k+1/2}$$

for every integer $n > 0$.

In a similar manner

$$y^i \frac{\partial^i}{\partial y^i}[J_{k-1}(\alpha\sqrt{y})F_{M,N}(x,y)] \ll \sum_{a=0}^{i} y^a (J_{k-1}(\alpha\sqrt{y}))^{(a)} y^{i-a} F_{M,N}(x,y)^{(i-a)}$$

gives an extra factor of Q^i. ∎

5.2. Applying Theorem 2.9. According to formula (3.32), the off-off-diagonal term is equal to

$$M^{\mathrm{OOD}} = M^{\mathrm{OOD}}(0) - \tau_{1/2+ir_2}(p) M^{\mathrm{OOD}}(1) + M^{\mathrm{OOD}}(2), \tag{5.4}$$

where for $B = 0, 1, 2$,

$$M^{\mathrm{OOD}}(B) = \frac{2\pi i^{-k}}{\hat{q}^{2t_1+2t_2}} \Bigg(\sum_{M,N \ll q^{1+\epsilon}} \sum_{\substack{q|c \\ c \ll C}} \frac{1}{c^2} \sum_{h \neq 0} S(0,h,c) \big(T_h^-(c,B) + T_h^+(c,B)\big) - \frac{1}{p} \sum_{M,N \ll q^{1+\epsilon}} \sum_{\substack{\frac{q}{p}|c \\ c \ll C}} \frac{1}{c^2} \sum_{h \neq 0} S(0,h,c) \big(T_h^-(c,B) + T_h^+(c,B)\big) \Bigg).$$

Since k is even, $i^{-k} = i^k$.

Lemma 5.3. *Up to an error of $O_{\boldsymbol{r},\epsilon}\big(q^\epsilon\big(q^{-\frac{k-1-2\theta}{8-8\theta}} + q^{-1/4}\big)\big)$, we have*

$$T_h^{\mp}(c,B) = \pm \int_0^\infty \delta_{h \pm p^B y > 0} G_{r_1}^{\mp}(h \pm p^B y, p^B y) \Lambda(h \pm p^B y, p^B y)\, dy \tag{5.5}$$

with

$$\Lambda(h \pm p^B y, p^B y) := \sum_{w=1}^\infty S(0,h,w) \sum_{\epsilon_1,\epsilon_2 = \pm 1} \frac{(p^B, w)^{1+2i\epsilon_2 r_2}}{w^{2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \times \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)(h \pm p^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}. \tag{5.6}$$

Proof. We apply Theorem 2.9 to the function $T_h^{\mp}(c,B)$ and let $x = h \pm p^B y$. Then

$$T_h^{\mp}(c,B) = \pm \int_0^\infty \delta_{h \pm p^B y > 0} G_{r_1}^{\mp}(h \pm p^B y, p^B y) \Lambda(h \pm p^B y, p^B y)\, dy + O(\mathrm{ET}),$$

where

$$\Lambda(h \pm p^B y, p^B y) := \sum_{w=1}^\infty S(0,h,w) \sum_{\epsilon_1,\epsilon_2 = \pm 1} \frac{(p^B, w)^{1+2i\epsilon_2 r_2}}{w^{2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \times \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)(h \pm p^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}$$

and the error term is

$$\mathrm{ET} := \frac{M^{1/2}}{N^{1/2}} \bigg(\frac{\sqrt{MN}}{c}\bigg)^{k-1} Q^{-k+1/2} Q^{5/4} (Z+N)^{1/4} (ZN)^{1/4+\epsilon}.$$

Since $Z = Q^2c^2/M > N$,

$$\mathrm{ET} \ll M^{1/2} N^{1/4} \bigg(\frac{\sqrt{MN}}{c}\bigg)^{k-2} Q^{-k+11/4}.$$

Note that $T_h^{\mp}(c)$ is small when $|h| \gg Zq^\epsilon$ because $G^{\mp}$ is small when $z \gg Zq^\epsilon$. This allows us to add $(1+|h|/Z)^{-2}$ to the error term ET. Multiplying by $S(0,h,c)$ and summing over h, we have

$$\mathrm{ET}_1 := \sum_{h\neq 0} S(0,h,c)\Big(1+\frac{|h|}{Z}\Big)^{-2}\mathrm{ET} \ll c^2\frac{N^{1/4}}{M^{1/2}}\Big(\frac{\sqrt{MN}}{c}\Big)^{k-2}Q^{-k+2+11/4}.$$

Finally, we sum over c. For $k=2$,

$$\begin{aligned}\sum_{\substack{c\leq C\\ q\mid c}} c^{-2}\mathrm{ET}_1 &\ll q^{\epsilon}\frac{N^{1/4}}{M^{1/2}}\sum_{\substack{c\leq C\\ q\mid c}}\bigg[1+\Big(\frac{\sqrt{MN}}{c}\Big)^{11/4}\bigg]\\ &\ll q^{\epsilon}\Big(\frac{N^{1/4}}{M^{1/2}}\frac{C}{q}+\frac{N^{13/8}M^{7/8}}{q^{11/4}}\Big).\end{aligned}$$

The optimal value of C can be found by making equal the first summand and the error term in Lemma 3.9:

$$\frac{N^{1/4}}{M^{1/2}}\frac{C}{q} = \Big(\frac{\sqrt{MN}}{C}\Big)^{1-2\theta}.$$

Thus, $C := \min\big(q^{\frac{1}{2-2\theta}}M^{1/2}N^{\frac{1-4\theta}{8-8\theta}}, q^{\frac{9-8\theta}{8-8\theta}}\big)$ and

$$\sum_{M,N\ll q^{1+\epsilon}}\sum_{\substack{c\leq C\\ q\mid c}} c^{-2}\mathrm{ET}_1 \ll q^{\epsilon}(q^{-\frac{1-2\theta}{8-8\theta}}+q^{-1/4}).$$

If $k\geq 4$, then

$$\begin{aligned}\sum_{\substack{c\leq C\\ q\mid c}} c^{-2}\mathrm{ET}_1 &\ll q^{\epsilon}\frac{N^{1/4}}{M^{1/2}}\sum_{\substack{c\leq C\\ q\mid c}}\bigg[\Big(\frac{\sqrt{MN}}{c}\Big)^{k-2}+\Big(\frac{\sqrt{MN}}{c}\Big)^{11/4}\bigg]\\ &\ll q^{\epsilon}\Big(\frac{N^{1/4}}{M^{1/2}}\Big(\frac{\sqrt{MN}}{q}\Big)^{k-2}+\frac{N^{13/8}M^{7/8}}{q^{11/4}}\Big).\end{aligned}$$

Therefore,

$$\sum_{M,N\ll q^{1+\epsilon}}\sum_{\substack{c\leq C\\ q\mid c}} c^{-2}\mathrm{ET}_1 \ll q^{\epsilon-1/4}.$$

Combining the two estimates into one, we find that for any even k,

$$\sum_{M,N\ll q^{1+\epsilon}}\sum_{\substack{c\leq C\\ q\mid c}} c^{-2}\mathrm{ET}_1 \ll q^{\epsilon}\big(q^{-\frac{k-1-2\theta}{8-8\theta}}+q^{-1/4}\big). \ ■$$

5.3. Extension of summations. Analogously to the off-diagonal term, at the cost of an admissible error, we can reintroduce summation over $\max(M,N)\geq q^{1+\epsilon}$ and extend the summation over c up to some large value $C_{\max}=q^{\Omega}$.

PROPOSITION 5.4. *For $l = 0, 1$, we have*

$$\sum_{\max(M,N)\ll q^{1+\epsilon}} \sum_{\substack{\frac{q}{p^l}|c \\ C<c\leq C_{\max}}} \frac{1}{c^2} \sum_{h\neq 0} S(0,h,c) T_h^{\pm}(c,B) \ll_{\epsilon,\boldsymbol{r}} q^{\epsilon - \frac{k-1}{8-8\theta}}. \tag{5.7}$$

Proof. Consider $T_h^{\pm}(c,B)$ given by (5.5). We split the sum over w in (5.6) into two parts: $w < q$ and $w \geq q$. If $w < q$ we follow [DFI3, Section 10] to show that

$$\sum_{\substack{\frac{q}{p^l}|c \\ C<c\leq C_{\max}}} \frac{1}{c^2} \sum_{h\neq 0} S(0,h,c) T_h^{\pm}(c,B) \ll_{\epsilon,\boldsymbol{r}} q^{\epsilon} \frac{(\sqrt{MN})^k}{qC^{k-1}}.$$

If $w \geq q$ we estimate the absolute value of $T_h^{\pm}(c,B)$ using

$$G_{r_1}^{\pm}(h \pm p^B y, p^B y) \ll \left(1 + \frac{M}{c^2}(h \pm p^B y)\right)^{-2} \frac{M^{1/2}}{N^{1/2}} \left(\frac{\sqrt{MN}}{c}\right)^{k-1}$$

and $S(0,w,c) \ll (w,c)$. This gives

$$\sum_{\substack{\frac{q}{p^l}|c \\ C<c\leq C_{\max}}} \frac{1}{c^2} \sum_{h\neq 0} S(0,h,c) T_h^{\pm}(c,B) \ll_{\epsilon,\boldsymbol{r}} q^{\epsilon} \frac{(\sqrt{MN})^k}{qC^{k-1}} \frac{C}{qM}.$$

Finally, taking $C = \min\left(q^{\frac{1}{2-2\theta}} M^{1/2} N^{\frac{1-4\theta}{8-8\theta}}, q^{\frac{9-8\theta}{8-8\theta}}\right)$ and performing dyadic summation over M, N, we obtain the assertion. ∎

PROPOSITION 5.5. *For any $\epsilon > 0$, any $A > 0$ and $l = 0, 1$,*

$$\sum_{\frac{q}{p^l}|c} \frac{1}{c^2} \sum_{\max(M,N)\gg q^{1+\epsilon}} \sum_{h\neq 0} S(0,h,c) T_h^{\pm}(c,B) \ll_{\epsilon,A,\boldsymbol{r}} q^{-A}. \tag{5.8}$$

Proof. The assertion follows from the rapid decay of $F_{M,N}$. See the proof of Proposition 3.7 for details. ∎

Now it is possible to combine all functions F_M into F and replace $\sum_{M,N} F_{M,N}$ by

$$F(x,y) := \frac{1}{\sqrt{xy}} W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{y}{\hat{q}^2}\right) F(x)F(y), \tag{5.9}$$

where $F(x)$ is a smooth function, compactly supported in $[1/2, \infty)$ and such that $F(x) = 1$ for $x \geq 1$.

LEMMA 5.6. *One has*

$$M^{\mathrm{OOD}}(B) = 2\pi i^k \hat{q}^{-2t_1-2t_2} \sum_{\epsilon_1,\epsilon_2=\pm 1} \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)$$
$$\times \Bigg(\sum_{g,v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{q|cg \\ cg<q^{\Omega}}} \frac{1}{c} \sum_{w} \frac{(p^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{h\neq 0 \\ [w,c]|h}} V(h)$$
$$- \frac{1}{p} \sum_{g,v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{\frac{q}{p}|cg \\ cg<q^{\Omega}}} \frac{1}{c} \sum_{w} \frac{(p^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{h\neq 0 \\ [w,c]|h}} V(h)\Bigg),$$

where

$$V(h) = -\frac{1}{(2\pi i)^2} \frac{1}{p^{B(1+i\epsilon_2 r_2)}} \int_{\Re\beta=0.7} \int_{\Re z=-0.1} \frac{\Gamma(\beta+ir_1)\Gamma(\beta-ir_1)}{\Gamma(1+z)\Gamma(k+z)}$$
$$\times \frac{(4\pi)^{k+2z-2\beta} 2^{-k-2z+2\beta}}{\sin(\pi z)} (cg)^{-k+1-2z+2\beta} h^{k/2+z-\beta+i\epsilon_1 r_1+i\epsilon_2 r_2}$$
$$\times \int_{x=0}^{\infty} x^{z-\beta+k/2} W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right) F(x) \frac{dx}{x} \int_{y=0}^{\infty} y^{z+k/2+i\epsilon_2 r_2} W_{t_2,r_2}\left(\frac{hy}{\hat{q}^2}\right) F(hy)$$
$$\times \left(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y>1} \frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y<1} \frac{\cos(\pi i r_1)}{(1-y)^{\beta-i\epsilon_1 r_1}}\right) \frac{dy}{y}\, dz\, d\beta.$$

Proof. Lemma 5.3 yields

$$T_h^-(c,B) + T_h^+(c,B) = \sum_{w=1}^{\infty} S(0,h,w) \sum_{\epsilon_1,\epsilon_2=\pm 1} \frac{(p^B, w)^{1+2i\epsilon_2 r_2}}{w^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \zeta(1+2i\epsilon_1 r_1)$$
$$\times \zeta(1+2i\epsilon_2 r_2) \int_0^{\infty} \big[\delta_{h+p^B y>0} G_{r_1}^-(h+p^B y, p^B y)(h+p^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}$$
$$+ \delta_{h-p^B y>0} G_{r_1}^+(h-p^B y, p^B y)(h-p^B y)^{i\epsilon_1 r_1} y^{i\epsilon_2 r_2}\big]\, dy.$$

We plug in the expressions for $G_{r_1}^-$ and $G_{r_1}^+$ given by (3.27) and (3.28) and use the identity

$$F(x, p^B y) = \frac{1}{(p^B xy)^{1/2}} W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right) W_{t_2,r_2}\left(\frac{p^B y}{\hat{q}^2}\right) F(x) F(p^B y).$$

This gives

$$T_h^-(c,B)+T_h^+(c,B)=\sum_{w=1}^{\infty}S(0,h,w)\sum_{\epsilon_1,\epsilon_2=\pm1}\frac{(p^B,w)^{1+2i\epsilon_2r_2}}{w^{2+2i\epsilon_1r_1+2i\epsilon_2r_2}}\times\zeta(1+2i\epsilon_1r_1)\zeta(1+2i\epsilon_2r_2)$$
$$\times 2\pi\int_0^\infty\int_0^\infty\frac{1}{(p^Bxy)^{1/2}}W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right)W_{t_2,r_2}\left(\frac{p^By}{\hat{q}^2}\right)J_{k-1}\left(\frac{4\pi\sqrt{xp^By}}{c}\right)$$
$$\times\left[\delta_{h+p^By>0}k_0\left(\frac{4\pi\sqrt{x(h+p^By)}}{c},1/2+ir_1\right)(h+p^By)^{i\epsilon_1r_1}y^{i\epsilon_2r_2}\right.$$
$$\left.+\ \delta_{h-p^By>0}k_1\left(\frac{4\pi\sqrt{x(h+p^By)}}{c},1/2+ir_1\right)(h-p^By)^{i\epsilon_1r_1}y^{i\epsilon_2r_2}\right]$$
$$\times F(x)F(p^By)\,dx\,dy.$$

The off-off-diagonal term

$$M^{\mathrm{OOD}}(B)=2\pi i^k\hat{q}^{-2t_1-2t_2}\left(\sum_{q|c}\frac{1}{c^2}\sum_{h\neq0}S(0,h,c)(T_h^-(c,B)+T_h^+(c,B))\right.$$
$$\left.-\frac{1}{p}\sum_{\frac{q}{p}|c}\frac{1}{c^2}\sum_{h\neq0}S(0,h,c)(T_h^-(c,B)+T_h^+(c,B))\right)$$

contains two Ramanujan sums $S(0,h,c)$ and $S(0,h,w)$. Applying the formulas

$$S(0,h,c)=\sum_{\substack{gc_1=c\\c_1|h}}\mu(g)c_1,\quad S(0,h,w)=\sum_{\substack{vw_1=c\\w_1|h}}\mu(v)w_1,$$

we obtain

$$M^{\mathrm{OOD}}(B)=2\pi i^k\hat{q}^{-2t_1-2t_2}\sum_{\epsilon_1,\epsilon_2=\pm1}\zeta(1+2i\epsilon_1r_1)\zeta(1+2i\epsilon_2r_2)$$
$$\times\left(\sum_{g,v}\frac{\mu(g)}{g^2}\frac{\mu(v)}{v^{2+2i\epsilon_1r_1+2i\epsilon_2r_2}}\sum_{\substack{q|cg\\cg<q^{\Omega}}}\frac{1}{c}\sum_{w}\frac{(p^B,wv)^{1+2i\epsilon_2r_2}}{w^{1+2i\epsilon_1r_1+2i\epsilon_2r_2}}\sum_{\substack{h\neq0\\ [w,c]|h}}V(h)\right.$$
$$\left.-\frac{1}{p}\sum_{g,v}\frac{\mu(g)}{g^2}\frac{\mu(v)}{v^{2+2i\epsilon_1r_1+2i\epsilon_2r_2}}\sum_{\substack{\frac{q}{p}|cg\\cg<q^{\Omega}}}\frac{1}{c}\sum_{w}\frac{(p^B,wv)^{1+2i\epsilon_2r_2}}{w^{1+2i\epsilon_1r_1+2i\epsilon_2r_2}}\sum_{\substack{h\neq0\\ [w,c]|h}}V(h)\right),$$

where

$$V(h)=2\pi\int_0^\infty\int_0^\infty\frac{1}{(p^Bxy)^{1/2}}W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right)W_{t_2,r_2}\left(\frac{p^By}{\hat{q}^2}\right)$$
$$\times\Bigg[\delta_{h+p^By>0}k_0\left(\frac{4\pi\sqrt{x(h+p^By)}}{cg},1/2+ir_1\right)(h+p^By)^{i\epsilon_1r_1}y^{i\epsilon_2r_2}$$
$$+\ \delta_{h-p^By>0}k_1\left(\frac{4\pi\sqrt{x(h+p^By)}}{cg},1/2+ir_1\right)(h-p^By)^{i\epsilon_1r_1}y^{i\epsilon_2r_2}\Bigg]$$
$$\times J_{k-1}\left(\frac{4\pi\sqrt{xp^By}}{cg}\right)F(x)F(p^By)\,dx\,dy.$$

In the expression $V(h)$ we replace negative h by their absolute value and make the change of variables $p^By/h\to y$ in the integral. As a result,

$$V(h)=2\pi\frac{h^{1/2+i\epsilon_1r_1+i\epsilon_2r_2}}{p^{B(1+i\epsilon_2r_2)}}\int_0^\infty\int_0^\infty\frac{y^{i\epsilon_2r_2}}{(xy)^{1/2}}W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right)W_{t_2,r_2}\left(\frac{hy}{\hat{q}^2}\right)$$
$$\times\Bigg[(1+y)^{i\epsilon_1r_1}k_0\left(\frac{4\pi\sqrt{xh(1+y)}}{cg},1/2+ir_1\right)+\delta_{y>1}(-1+y)^{i\epsilon_1r_1}$$
$$\times k_0\left(\frac{4\pi\sqrt{xh(y-1)}}{cg},1/2+ir_1\right)+\delta_{y<1}(1-y)^{i\epsilon_1r_1}$$
$$\times k_1\left(\frac{4\pi\sqrt{xh(1-y)}}{cg},1/2+ir_1\right)\Bigg]J_{k-1}\left(\frac{4\pi\sqrt{xhy}}{cg}\right)F(x)F(hy)\,dx\,dy.$$

Finally, we use Mellin transforms of Bessel functions (B.5), (B.9) and (B.10), so that

$$V(h)=-\frac{p^{-B(1+i\epsilon_2r_2)}}{(2\pi i)^2}\int_{\Re\beta=0.7}\int_{\Re z=-0.1}\frac{\Gamma(\beta+ir_1)\Gamma(\beta-ir_1)}{\Gamma(1+z)\Gamma(k+z)}$$
$$\times\int_{\Re\beta=0.7}\int_{\Re z=-0.1}\frac{(4\pi)^{k+2z-2\beta}2^{2\beta-k-2z}}{\sin(\pi z)}(cg)^{-k+1-2z+2\beta}h^{k/2+z-\beta+i\epsilon_1r_1+i\epsilon_2r_2}$$
$$\times\int_{x=0}^\infty x^{z-\beta+k/2}W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right)F(x)\frac{dx}{x}\int_{y=0}^\infty y^{z+k/2+i\epsilon_2r_2}W_{t_2,r_2}\left(\frac{hy}{\hat{q}^2}\right)F(hy)$$
$$\times\left(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1r_1}}+\delta_{y>1}\frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1r_1}}+\delta_{y<1}\frac{\cos(\pi ir_1)}{(1-y)^{\beta-i\epsilon_1r_1}}\right)\frac{dy}{y}\,dz\,d\beta.$$

Note that we shifted the integration contour given in (B.10) to $\Re\beta=0.7$, which is possible due to the rapid decay of the x-integral in β. The change of the order of integration in $V(h)$ is justified by absolute convergence of all integrals. ∎

5.4. Replacing $F(x)F(hy)$ by 1 on the interval $[0,\infty)^2$. This step allows us to simplify the integration and can be performed at the cost of a negligible error.

5.4.1. *y-integral.* Consider

$$\mathrm{IY} := \int_{y=0}^{\infty} y^{z+k/2+i\epsilon_2 r_2} W_{t_2,r_2}\left(\frac{hy}{\hat{q}^2}\right) F(hy) \times \left(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y>1}\frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1 r_1}} + \delta_{y<1}\frac{\cos(\pi i r_1)}{(1-y)^{\beta-i\epsilon_1 r_1}}\right)\frac{dy}{y}. \tag{5.10}$$

LEMMA 5.7. *The function $F(hy)$ can be replaced by* 1 *in* IY *with an error of $O_\epsilon(q^{-1/2+\epsilon})$.*

Proof. $F(hy)$ is a smooth function, compactly supported in $[1/2,\infty)$ and such that $F(hy)=1$ for $hy\geq 1$. Thus, we only need to estimate the integral for $y<1/h$. It is bounded by $(1/h)^{k/2+\Re z}\cos(\pi\beta)$. We are left to estimate

$$T := \sum_{g,v,w}\frac{1}{g^2v^2w}\sum_{\substack{q|cg\\ cg<q^{\Omega}}}\frac{1}{c}\sum_{[c,w]|h} h^{-\beta}\int_{\Re\beta=0.7}\int_{\Re z=-0.1}\frac{\Gamma(\beta+ir_1)\Gamma(\beta-ir_1)}{\Gamma(1+z)\Gamma(k+z)} \times \frac{\cos(\pi\beta)}{\sin(\pi z)}(cg)^{-k+1-2z+2\beta}\int_{x=0}^{\infty} x^{z-\beta+k/2}W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right)F(x)\,\frac{dx}{x}.$$

To make the sums over h and w absolutely convergent, one has to move the β-contour to the right, where $\Re\beta>1$. At the same time, integration by parts shows that the x-integral decays rapidly in β:

$$\int_0^{\infty} x^{z-\beta+k/2}W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right)F(x)\,\frac{dx}{x} = \frac{1}{(z-\beta+k/2)(z-\beta+k/2)\dots(z-\beta+k/2+n-1)} \times\int_0^{\infty}\frac{\partial^n}{\partial x^n}\left(W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right)F(x)\right)x^{z-\beta+k/2+n-1}\,dx \ll \frac{1}{|\beta|^n}q^{z-\beta+k/2}.$$

Assume that $\Re\beta>1$. We have

$$T \ll q^{z-\beta+k/2}\sum_{v,w,h}\frac{1}{v^2w^{1+\beta}h^{\beta}}\sum_{\substack{q|cg\\ cg<q^{\Omega}}}\frac{1}{c^{1+\beta}g^2}(cg)^{-k+1-2z+2\beta} \ll q^{z-\beta+k/2}\sum_{\substack{q|cg\\ cg<q^{\Omega}}}(cg)^{-k-1-2z+2\beta} \ll q^{z-\beta+k/2-1}q^{\Omega(-k-2z+2\beta)}.$$

By moving the β-contour to $\Re\beta = k/2+\delta$ and the z-contour to $-\delta$, M^{OOD} is dominated by $q^{-1}q^{4\delta\Omega-2\delta}$. Choosing $\delta = \frac{1+2\epsilon}{4(2\Omega-1)}$, we obtain the result. ■

LEMMA 5.8. *One has*

$$
\mathrm{IY} = \frac{1}{2\pi i}\int_{\Re t=k/2-0.2} \frac{P_r(t)}{P_r(t_2)}\,\frac{\Gamma(t+ir_2+k/2)\Gamma(t-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}
\times\left(\frac{\hat{q}^2}{h}\right)^t \frac{\Gamma(k/2+z-t+i\epsilon_2r_2)\Gamma(-k/2-z+t+\beta-i\epsilon_1r_1-i\epsilon_2r_2)}{\Gamma(\beta-i\epsilon_1r_1)}
\times\zeta_q(1+2t)\biggl(\cos(\pi\beta)+\frac{\cos(\pi\beta)\sin(\pi(k/2+z-t+i\epsilon_2r_2))}{\sin(\pi(\beta-i\epsilon_1r_1))}
+\frac{\cos(\pi ir_1)\sin(\pi(-k/2-z+t+\beta-i\epsilon_1r_1-i\epsilon_2r_2))}{\sin(\pi(\beta-i\epsilon_1r_1))}\biggr)\frac{2t\,dt}{t^2-t_2^2}. \tag{5.11}
$$

Proof. By Lemma 5.7, the y-integral is equal to

$$
\mathrm{IY} = \int_{y=0}^{\infty} y^{z+k/2+i\epsilon_2r_2}W_{t_2,r_2}\left(\frac{hy}{\hat{q}^2}\right)
\times\left(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1r_1}}+\delta_{y>1}\frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1r_1}}+\delta_{y<1}\frac{\cos(\pi ir_1)}{(1-y)^{\beta-i\epsilon_1r_1}}\right)\frac{dy}{y}.
$$

We plug in the expression

$$
W_{t_2,r_2}\left(\frac{hy}{\hat{q}^2}\right) = \frac{1}{2\pi i}\int_{\Re t=k/2-0.2}\frac{P_r(t)}{P_r(t_2)}\zeta_q(1+2t)
\times\frac{\Gamma(t+ir_2+k/2)\Gamma(t-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}\left(\frac{hy}{\hat{q}^2}\right)^{-t}\frac{2t\,dt}{t^2-t_2^2}.
$$

Note that we shifted $\Re t$ from 3 to $k/2-0.2$ without crossing any poles. This step is required to ensure that all poles of $\Gamma(-k/2-z+t+\beta-i\epsilon_1r_1-i\epsilon_2r_2)$ lie to the left of the t-contour. Therefore,

$$
\mathrm{IY} = \frac{1}{2\pi i}\int_{\Re t=k/2-0.2}\frac{P_r(t)}{P_r(t_2)}\,\frac{\Gamma(t+ir_2+k/2)\Gamma(t-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}\zeta_q(1+2t)
\times\left(\frac{\hat{q}^2}{h}\right)^t\int_{y=0}^{\infty} y^{z+k/2+i\epsilon_2r_2-t}\biggl(\frac{\cos(\pi\beta)}{(1+y)^{\beta-i\epsilon_1r_1}}+\delta_{y>1}\frac{\cos(\pi\beta)}{(-1+y)^{\beta-i\epsilon_1r_1}}
+\delta_{y<1}\frac{\cos(\pi ir_1)}{(1-y)^{\beta-i\epsilon_1r_1}}\biggr)\frac{dy}{y}\,\frac{2tdt}{t^2-t_2^2}.
$$

Mellin transforms (B.1)–(B.3) and Euler's reflection formula give

$$\mathrm{IY} = \frac{1}{2\pi i}\int\limits_{\Re t=k/2-0.2}\frac{P_r(t)}{P_r(t_2)}\,\frac{\Gamma(t+ir_2+k/2)\Gamma(t-ir_2+k/2)}{\Gamma(t_2+ir_2+k/2)\Gamma(t_2-ir_2+k/2)}\zeta_q(1+2t)$$
$$\times\left(\frac{\hat{q}^2}{h}\right)^t\biggl(\cos(\pi\beta)+\frac{\cos(\pi\beta)\sin(\pi(k/2+z-t+i\epsilon_2r_2))}{\sin(\pi(\beta-i\epsilon_1r_1))}$$
$$+\frac{\cos(\pi ir_1)\sin(\pi(-k/2-z+t+\beta-i\epsilon_1r_1-i\epsilon_2r_2))}{\sin(\pi(\beta-i\epsilon_1r_1))}\biggr)$$
$$\times\frac{\Gamma(k/2+z-t+i\epsilon_2r_2)\Gamma(-k/2-z+t+\beta-i\epsilon_1r_1-i\epsilon_2r_2)}{\Gamma(\beta-i\epsilon_1r_1)}\,\frac{2t\,dt}{t^2-t_2^2}. \blacksquare$$

REMARK 5.9. Consider the expression (5.11). If $\beta-i\epsilon_1r_1=0,-1,-2,\ldots$, the poles of $1/\sin(\pi(\beta-i\epsilon_1r_1))$ are canceled by the zeros of $1/\Gamma(\beta-i\epsilon_1r_1)$. The poles at $\beta-i\epsilon_1r_1=j$ with $j=1,2,\ldots$ are compensated by the vanishing numerator.

5.4.2. *x-integral*

LEMMA 5.10. *The function $F(x)$ can be replaced by 1 in the expression $V(h)$ at the cost of a negligible error of $O_{\epsilon,\boldsymbol{r}}(q^{\epsilon-k/2+0.5})$.*

Proof. We show that the contribution of $F_1(x)=1-F(x)$ is negligible. Note that $F_1(x)=0$ for $x\geq 1$ since in that case $F(x)=1$. The part of M^{OOD} which affects the x-integral can be written as follows:

$$\sum_{v,w}\frac{1}{v^2w}\sum_{\substack{c,g\\ q|cg}}\sum_{[c,w]|h} g^{-k-1-2z+2\beta}c^{-k-2z+2\beta}h^{k/2+z-\beta-t}q^t$$
$$\times\,\Gamma(-k/2-z+t+\beta-i\epsilon_1r_1-i\epsilon_2r_2)\Gamma(k/2+z-t+i\epsilon_2r_2)H_1(t,z,\beta)$$
$$\times\int_0^1 x^{z-\beta+k/2}W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right)F_1(x)\,\frac{dx}{x}.$$

Here H_1 is an analytic function. We have

$$\Re z=-0.1,\qquad \Re\beta=0.7,\qquad \Re t=k/2-0.2.$$

Without crossing any poles, we shift β-contour to $\Re\beta=0.3$. In order to make the sums over h and w absolutely convergent, we move the t-contour to $\Re t=k/2+0.7$, crossing a pole at $t=k/2+z+i\epsilon_2r_2$. Since $\Re z-\Re\beta+k/2>0$, the x-integral can be integrated by parts n times (for sufficiently large n) to make the β-integral convergent. This gives

$$\int_0^1 x^{z-\beta+k/2}W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right)F_1(x)\,\frac{dx}{x}\ll\frac{1}{|\beta|^n}.$$

Finally, all sums and integrals are absolutely convergent and $q^{-k-1+t-2z+2\beta}$ can be factored out due to divisibility conditions. In total, this gives an error of $O_{\epsilon,\boldsymbol{r}}(q^{\epsilon-k/2+0.5})$.

For the pole at $t = k/2 + z + i\epsilon_2 r_2$ another contour shift is required to make all sums absolutely convergent. We move the z-contour to $\Re z = 0.5+2\epsilon$ and β to $\Re\beta = 1 + \epsilon$. Note that the pole of $1/\sin(\pi z)$ at $z = 0$ is canceled by a zero of $P_r(t) = P_r(k/2 + z + i\epsilon_2 r_2)$. The x-integral is bounded by $1/|\beta|^n$. The power of q, corresponding to divisibility conditions on g, c, h, is $q^{-k-1+t-2z+2\beta}$. This gives an error term of $O_{\epsilon,\boldsymbol{r}}(q^{\epsilon-k/2+0.5})$. ∎

Proposition 5.11. *One has*

$$
\begin{aligned}
M^{\mathrm{OOD}}(B) = 2\pi i^k \hat{q}^{-2t_1-2t_2} \sum_{\epsilon_1,\epsilon_2=\pm 1} \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2) \\
\times \Bigg(\sum_{g,v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{q|cg \\ cg<q^{\Omega}}} \frac{1}{c} \sum_{w} \frac{(p^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{h\neq 0 \\ [w,c]|h}} V(h) \\
- \frac{1}{p} \sum_{g,v} \frac{\mu(g)}{g^2} \frac{\mu(v)}{v^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{\frac{q}{p}|cg \\ cg<q^{\Omega}}} \frac{1}{c} \sum_{w} \frac{(p^B, wv)^{1+2i\epsilon_2 r_2}}{w^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}} \sum_{\substack{h\neq 0 \\ [w,c]|h}} V(h) \Bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
V(h) = -\frac{i^k}{(2\pi i)^3} \int_{\Re t=k/2+0.7} \int_{\Re s=k/2-0.4} \int_{\Re z=-0.1} \frac{\hat{q}^{2s+2t}}{p^{B(1+i\epsilon_2 r_2)}} (cg)^{1-2s} \\
\times (2\pi)^{2s} \frac{P_r(s)P_r(t)}{P_r(t_1)P_r(t_2)} \zeta_q(1+2t)\zeta_q(1+2s) \frac{h^{s-t+i\epsilon_1 r_1+i\epsilon_2 r_2}}{\sin(\pi z)\Gamma(1+z)\Gamma(k+z)} \\
\times \frac{\Gamma(k/2+s\pm ir_1)\Gamma(k/2+t\pm ir_2)}{\Gamma(k/2+t_1\pm ir_1)\Gamma(k/2+t_2\pm ir_2)} \\
\times \Gamma(t-s-i\epsilon_1 r_1 - i\epsilon_2 r_2)\Gamma(k/2+z-s+i\epsilon_1 r_1)\Gamma(k/2+z-t+i\epsilon_2 r_2) \\
\times \Bigg(\cos(\pi(z-s)) + \frac{\cos(\pi(z-s))\sin(\pi(z-t+i\epsilon_2 r_2))}{\sin(\pi(z-s-i\epsilon_1 r_1))} \\
+ \frac{\cos(\pi i r_1)\sin(\pi(t-s-i\epsilon_1 r_1 - i\epsilon_2 r_2))}{\sin(\pi(z-s-i\epsilon_1 r_1))} \Bigg) \, dz \, \frac{2s\,ds}{s^2-t_1^2} \, \frac{2t\,dt}{t^2-t_2^2}.
\end{aligned}
$$

Remark 5.12. We do not compute the contribution of the poles at $t = k/2 + z + i\epsilon_2 r_2$ since it will be canceled by another contour shift in 5.5.1.

Proof of Theorem 5.1. By inverse Mellin transform we have

$$\int_0^\infty x^{z-1-\beta+k/2} W_{t_1,r_1}\left(\frac{x}{\hat{q}^2}\right) dx = 2\frac{P_r(z-\beta+k/2)}{P_r(t_1)}\zeta_q(1+k+2z-2\beta)$$
$$\times \hat{q}^{k+2z-2\beta}\frac{\Gamma(k+z-\beta+ir_1)\Gamma(k+z-\beta-ir_1)}{\Gamma(k/2+t_1+ir_1)\Gamma(k/2+t_1-ir_1)}\frac{k/2+z-\beta}{(k/2+z-\beta)^2-t_1^2}$$

for $\Re(z-\beta+k/2) > -1$. Setting $s := k/2+z-\beta$ yields the assertion. ∎

5.5. Shifting the z-contour. The z-integral is given by

$$\text{(5.12)} \qquad \mathrm{IZ} := \frac{1}{2\pi i}\int_{\Re z=-0.1}\frac{\Gamma(k/2+z-s+i\epsilon_1 r_1)\Gamma(k/2+z-t+i\epsilon_2 r_2)}{\sin(\pi z)\Gamma(1+z)\Gamma(k+z)}$$
$$\times\left(\cos(\pi(z-s)) + \frac{\cos(\pi(z-s))\sin(\pi(z-t+i\epsilon_2 r_2))}{\sin(\pi(z-s-i\epsilon_1 r_1))}\right.$$
$$\left. + \frac{\cos(\pi i r_1)\sin(\pi(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2))}{\sin(\pi(z-s-i\epsilon_1 r_1))}\right) dz.$$

Stirling's formula implies that the integrand decays as $|z|^{-1-s-t}$. We shift $\Re z$ to $D > 0$ and let $D \to \infty$. This leads to three types of possible poles described in the table below.

Possible poles at	Coming from
$z = t - k/2 - i\epsilon_2 r_2$	$\Gamma(k/2+z-t+i\epsilon_2 r_2)$
$z = n + s + i\epsilon_1 r_1$	$1/\sin(\pi(z-s-i\epsilon_1 r_1))$
$z = n, n \geq 0$	$1/\sin(\pi z)$

5.5.1. *Poles at $z = t - k/2 - i\epsilon_2 r_2$.* The residues at these poles cancel those mentioned in Remark 5.12 (when one shifs t to the right). Consider

$$\int_t\int_z \Gamma(k/2+z-t+i\epsilon_2 r_2) f(z,t)\, dz\, dt.$$

Shifting the t-contour to the right, we have the residue

$$-\operatorname{Res}_{z=-k/2+t-i\epsilon_2 r_2}\Gamma(k/2+z-t+i\epsilon_2 r_2) f(z,t).$$

Moving z to the right, we obtain

$$-\operatorname{Res}_{t=k/2+z+i\epsilon_2 r_2}\Gamma(k/2+z-t+i\epsilon_2 r_2) f(z,t).$$

Since z and t have different signs in $\Gamma(k/2+z-t+i\epsilon_2 r_2)$, these residues cancel each other.

5.5.2. *Poles at $z = n + s + i\epsilon_1 r_1$*

Proposition 5.13. *The integrand in* IZ *is holomorphic at $z = n + s + i\epsilon_1 r_1$.*

Proof. We write

$$\begin{aligned}\sin(\pi(z-t+i\epsilon_2 r_2)) = &-\sin(\pi(t-s-i\epsilon_1 r_1 - i\epsilon_2 r_2))\cos(\pi(z-s-i\epsilon_1 r_1)) \\ &+\cos(\pi(t-s-i\epsilon_1 r_1 - i\epsilon_2 r_2))\sin(\pi(z-s-i\epsilon_1 r_1))\end{aligned}$$

and plug this in IZ. After simplifications,

$$\begin{aligned}\mathrm{IZ} = \frac{1}{2\pi i}\int\limits_{\Re z=-0.1} &\frac{\Gamma(k/2+z-s+i\epsilon_1 r_1)\Gamma(k/2+z-t+i\epsilon_2 r_2)}{\sin(\pi z)\Gamma(1+z)\Gamma(k+z)} \\ &\times\big[\cos(\pi(z-s)) + \cos(\pi(z-s))\cos(\pi(t-s-i\epsilon_1 r_1 - i\epsilon_2 r_2)) \\ &+ \sin(\pi(z-s))\sin(\pi(z-s+i\epsilon_1 r_1))\big].\end{aligned}$$

This is holomorphic at $z = n+s+i\epsilon_1 r_1$. ∎

5.5.3. *Poles at $z = n, n \geq 0$*

PROPOSITION 5.14. *The poles at $z = n$ are simple and their contribution to* IZ *is given by*

$$\begin{aligned}-\frac{1}{\pi}\Gamma(s+t-i\epsilon_1 r_1 - i\epsilon_2 r_2)&\frac{\Gamma(k/2-s+i\epsilon_1 r_1)\Gamma(k/2-t+i\epsilon_2 r_2)}{\Gamma(k/2+s-i\epsilon_1 r_1)\Gamma(k/2+t-i\epsilon_2 r_2)} \\ &\times[\cos(\pi s) + \cos(\pi(t-i\epsilon_1 r_1 - i\epsilon_2 r_2))].\end{aligned}$$

Proof. Consider

$$\begin{aligned}P_1 := -\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{\cos(\pi n)}&\frac{\Gamma(k/2+n-s+i\epsilon_1 r_1)\Gamma(k/2+n-t+i\epsilon_2 r_2)}{\Gamma(1+n)\Gamma(k+n)} \\ &\times\left(\cos(\pi(n-s)) + \frac{\cos(\pi(n-s))\sin(\pi(n-t+i\epsilon_2 r_2))}{\sin(\pi(n-s-i\epsilon_1 r_1))}\right. \\ &\left.+\frac{\cos(\pi i r_1)\sin(\pi(t-s-i\epsilon_1 r_1 - i\epsilon_2 r_2))}{\sin(\pi(n-s-i\epsilon_1 r_1))}\right).\end{aligned}$$

Since $n \in \mathbb{Z}$, we have

$$\begin{aligned}P_1 := -\frac{1}{\pi}\sum_{n=0}^{\infty}&\frac{\Gamma(k/2+n-s+i\epsilon_1 r_1)\Gamma(k/2+n-t+i\epsilon_2 r_2)}{\Gamma(1+n)\Gamma(k+n)} \\ &\times\left(\cos(\pi s) + \frac{\cos(\pi s)\sin(\pi(t-i\epsilon_2 r_2))}{\sin(\pi(s+i\epsilon_1 r_1))}\right. \\ &\left.-\frac{\cos(\pi i r_1)\sin(\pi(t-s-i\epsilon_1 r_1 - i\epsilon_2 r_2))}{\sin(\pi(s+i\epsilon_1 r_1))}\right).\end{aligned}$$

By the Gauss hypergeometric identity,

$$\sum_{n=0}^{\infty}\frac{\Gamma(k/2+n-s+i\epsilon_1r_1)\Gamma(k/2+n-t+i\epsilon_2r_2)}{\Gamma(1+n)\Gamma(k+n)}$$
$$=\Gamma(s+t-i\epsilon_1r_1-i\epsilon_2r_2)\frac{\Gamma(k/2-s+i\epsilon_1r_1)\Gamma(k/2-t+i\epsilon_2r_2)}{\Gamma(k/2+s-i\epsilon_1r_1)\Gamma(k/2+t-i\epsilon_2r_2)}.$$

Simplifying the trigonometric part, we obtain

$$\frac{\cos(\pi s)\sin(\pi(t-i\epsilon_2r_2))}{\sin(\pi(s+i\epsilon_1r_1))}-\frac{\cos(\pi ir_1)\sin(\pi(t-s-i\epsilon_1r_1-i\epsilon_2r_2))}{\sin(\pi(s+i\epsilon_1r_1))}$$
$$=\cos(\pi(t-i\epsilon_1r_1-i\epsilon_2r_2)).$$

This implies

$$P_1=-\frac{1}{\pi}\Gamma(s+t-i\epsilon_1r_1-i\epsilon_2r_2)\frac{\Gamma(k/2-s+i\epsilon_1r_1)\Gamma(k/2-t+i\epsilon_2r_2)}{\Gamma(k/2+s-i\epsilon_1r_1)\Gamma(k/2+t-i\epsilon_2r_2)}$$
$$\times[\cos(\pi s)+\cos(\pi(t-i\epsilon_1r_1-i\epsilon_2r_2))]. \blacksquare$$

PROPOSITION 5.15. *The off-off-diagonal term can be written as follows:*

$$M^{\mathrm{OOD}}(B)=\frac{2}{(2\pi i)^2}\hat{q}^{-2t_1-2t_2}\sum_{\epsilon_1,\epsilon_2=\pm1}\zeta(1+2i\epsilon_1r_1)\zeta(1+2i\epsilon_2r_2)$$
$$\times\int_{\Re t=k/2+0.7}\int_{\Re s=k/2-0.4}\frac{P_r(s)P_r(t)}{P_r(t_1)P_r(t_2)}\zeta_q(1+2s)\zeta_q(1+2t)\frac{q^{s+t}}{(2\pi)^{2t}}$$
$$\times\Gamma(t-s-i\epsilon_1r_1-i\epsilon_2r_2)\Gamma(t+s-i\epsilon_1r_1-i\epsilon_2r_2)$$
$$\times\frac{\Gamma(k/2+s+i\epsilon_1r_1)\Gamma(k/2+t+i\epsilon_2r_2)\Gamma(k/2-s+i\epsilon_1r_1)}{\Gamma(k/2+t_1+ir_1)\Gamma(k/2+t_1-ir_1)\Gamma(k/2+t_2+ir_2)}$$
$$\times\frac{\Gamma(k/2-t+i\epsilon_2r_2)}{\Gamma(k/2+t_2-ir_2)}\Bigg(\sum_{q|cg}\sum_{g}\frac{\mu(g)}{g^{2s+1}}\mathrm{TD}(c)-1/p\sum_{q|cpg}\sum_{g}\frac{\mu(g)}{g^{2s+1}}\mathrm{TD}(c)\Bigg)$$
$$\times[\cos(\pi s)+\cos(\pi(t-i\epsilon_1r_1-i\epsilon_2r_2))]\frac{2s\,ds}{s^2-t_1^2}\frac{2t\,dt}{t^2-t_2^2},$$

where

$$\mathrm{TD}(c)=\frac{1}{c^{2s}}\sum_{v}\frac{\mu(v)}{v^{2+2i\epsilon_1r_1+2i\epsilon_2r_2}}\sum_{w}\frac{(p^B,wv)^{1+2i\epsilon_2r_2}}{p^{B(1+i\epsilon_2r_2)}w^{1+2i\epsilon_1r_1+2i\epsilon_2r_2}}$$
$$\times\sum_{c,w|h}\frac{1}{h^{t-s-i\epsilon_1r_1-i\epsilon_2r_2}}. \tag{5.13}$$

5.6. Explicit formula. We start by transforming the off-off-diagonal term.

PROPOSITION 5.16. *One has*

$$M^{\mathrm{OOD}} = \sum_{\epsilon_1,\epsilon_2=\pm 1} \zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2)\frac{1}{(2\pi i)^2} \times \int\limits_{\Re t=k/2+0.6}\int\limits_{\Re s=k/2-0.4} E(s,t)\Phi(s,t) 2s\,ds\,2t\,dt, \tag{5.14}$$

where

$$E(s,t) := \hat{q}^{-2t_1-2t_2}\frac{P_r(s)P_r(t)}{P_r(t_1)P_r(t_2)}\frac{1}{s^2-t_1^2}\frac{1}{t^2-t_2^2}\frac{\Gamma(k/2+s+i\epsilon_1 r_1)}{\Gamma(k/2+t_1+ir_1)} \times \frac{\Gamma(k/2+t+i\epsilon_2 r_2)\Gamma(k/2-s+i\epsilon_1 r_1)\Gamma(k/2-t+i\epsilon_2 r_2)}{\Gamma(k/2+t_1-ir_1)\Gamma(k/2+t_2+ir_2)\Gamma(k/2+t_2-ir_2)}, \tag{5.15}$$

$$\Phi(s,t) := 2\zeta_q(1+2t)\frac{q^{s+t}}{(2\pi)^{2t}}[\cos(\pi s)+\cos(\pi(t-i\epsilon_1 r_1-i\epsilon_2 r_2))] \tag{5.16}$$

$$\times \Gamma(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2)\Gamma(t+s-i\epsilon_1 r_1-i\epsilon_2 r_2)\sum_{A,B=0}^{2} C(A,B)\sum_{q|cp^A}\mathrm{TD}(c)$$

and the coefficients $C(A,B)$ are given in Table 1.

Proof. Consider the term $M^{\mathrm{OOD}}(B)$. The Möbius function does not vanish only if $(q,g)=1$ or $(q,g)=p$. Thus we can write

$$\begin{aligned} M^{\mathrm{OOD}}(B) = {} & \frac{2}{(2\pi i)^2}\hat{q}^{-2t_1-2t_2}\sum_{\epsilon_1,\epsilon_2=\pm 1}\zeta(1+2i\epsilon_1 r_1)\zeta(1+2i\epsilon_2 r_2) \\ & \times \int\limits_{\Re t=k/2+0.7}\int\limits_{\Re s=k/2-0.4}\frac{P_r(s)P_r(t)}{P_r(t_1)P_r(t_2)}\zeta_q(1+2s)\zeta_q(1+2t)\frac{q^{s+t}}{(2\pi)^{2t}} \\ & \times \frac{\Gamma(k/2+s+i\epsilon_1 r_1)\Gamma(k/2+t+i\epsilon_2 r_2)\Gamma(k/2-s+i\epsilon_1 r_1)\Gamma(k/2-t+i\epsilon_2 r_2)}{\Gamma(k/2+t_1+ir_1)\Gamma(k/2+t_1-ir_1)\Gamma(k/2+t_2+ir_2)\Gamma(k/2+t_2-ir_2)} \\ & \times \sum_{(q,g)=1}\frac{\mu(g)}{g^{1+2s}}\Big[\sum_{q|c}\mathrm{TD}(c)-\Big(\frac{1}{p^{1+2s}}+\frac{1}{p}\Big)\sum_{q|cp}\mathrm{TD}(c)+\frac{1}{p^{2+2s}}\sum_{q|cp^2}\mathrm{TD}(c)\Big] \\ & \times \Gamma(t-s-i\epsilon_1 r_1-i\epsilon_2 r_2)\Gamma(t+s-i\epsilon_1 r_1-i\epsilon_2 r_2) \\ & \times [\cos(\pi s)+\cos(\pi(t-i\epsilon_1 r_1-i\epsilon_2 r_2))]\frac{2s\,ds}{s^2-t_1^2}\frac{2t\,dt}{t^2-t_2^2}. \end{aligned}$$

Note that

$$\zeta_q(1+2s)\sum_{(q,g)=1}\frac{\mu(g)}{g^{1+2s}} = 1.$$

In order to simplify notation, set

$$E(s,t):=\hat{q}^{-2t_1-2t_2}\frac{P_r(s)P_r(t)}{P_r(t_1)P_r(t_2)}\frac{1}{s^2-t_1^2}\frac{1}{t^2-t_2^2}\frac{\Gamma(k/2+s+i\epsilon_1r_1)}{\Gamma(k/2+t_1+ir_1)}\times\frac{\Gamma(k/2+t+i\epsilon_2r_2)\Gamma(k/2-s+i\epsilon_1r_1)\Gamma(k/2-t+i\epsilon_2r_2)}{\Gamma(k/2+t_1-ir_1)\Gamma(k/2+t_2+ir_2)\Gamma(k/2+t_2-ir_2)}.$$

This is an even function since G is even. By (3.32),

$$M^{\mathrm{OOD}}=M^{\mathrm{OOD}}(0)-\tau_{1/2+ir_2}(p)M^{\mathrm{OOD}}(1)+M^{\mathrm{OOD}}(2).$$

Next, we introduce a parameter A corresponding to the condition $q\mid cp^A$, so that

$$M^{\mathrm{OOD}}=\sum_{A,B=0}^{2}C(A,B)M^{\mathrm{OOD}}(A,B)$$

with

$$M^{\mathrm{OOD}}(A,B)=\frac{2}{(2\pi i)^2}\sum_{\epsilon_1,\epsilon_2=\pm1}\zeta(1+2i\epsilon_1r_1)\zeta(1+2i\epsilon_2r_2)\times\int\limits_{\Re t=k/2+0.7}\int\limits_{\Re s=k/2-0.4}E(s,t)\zeta_q(1+2t)\frac{q^{s+t}}{(2\pi)^{2t}}\times[\cos(\pi s)+\cos(\pi(t-i\epsilon_1r_1-i\epsilon_2r_2))]\times\Gamma(t-s-i\epsilon_1r_1-i\epsilon_2r_2)\Gamma(t+s-i\epsilon_1r_1-i\epsilon_2r_2)\sum_{q\mid cp^A}\mathrm{TD}(c)2s\,ds\,2t\,dt,$$

where the coefficients $C(A,B)$ are given in Table 1. ■

Table 1. The values of the coefficients $C(A,B)$

	$A=0$	$A=1$	$A=2$
$B=0$	1	$-(1+p^{2s})p^{-2s-1}$	p^{-2-2s}
$B=1$	$-\tau_{1/2+ir_2}(p)$	$\tau_{1/2+ir_2}(p)(1+p^{2s})p^{-2s-1}$	$-\tau_{1/2+ir_2}(p)p^{-2-2s}$
$B=2$	1	$-(1+p^{2s})p^{-2s-1}$	p^{-2-2s}

The next lemma allows removing the divisibility condition $c,w\mid h$ in the expression $\sum_{q\mid cp^A}\mathrm{TD}(c)$.

Lemma 5.17. *One has*

$$\sum_{\substack{c,w\\ p^{\nu-A}\mid c}}f(c,w)\sum_{c,w\mid h}g(h)=\sum_{p\nmid u}\mu(u)\times\sum_{\substack{\beta\geq0\\ \delta=\max(\nu-A,\beta)}}\sum_{\substack{c\\ p\nmid d\\ p\nmid w}}f(p^{\nu-A}\,duc,p^{\beta}\,duw)\sum_{h}g(p^{\delta}u^2dcwh).\tag{5.17}$$

REMARK 5.18. Recall that $q = p^\nu$, $\nu \geq 3$ and so $\nu - A \geq 1$.

Proof of Lemma 5.17. Consider

$$S := \sum_{\substack{c,w \\ p^{\nu-A}|c}} f(c,w) \sum_{c,w|h} g(h).$$

Let us make the following change of variables:

$$\begin{aligned} c &= p^{\nu-A} c_1 = p^{\nu-A} d c_2, \\ w &= p^\beta w_1 = p^\beta d w_2, \\ d &= (c_1, w_1) \quad \text{so that} \quad (c_2, w_2) = 1 \text{ and } p \nmid d w_2, \\ h &= p^\delta d c_2 w_2 h_1 \quad \text{where} \quad \delta = \max(\nu - A, \beta). \end{aligned}$$

Then

$$S = \sum_{\substack{\beta \geq 0 \\ \delta = \max(\nu - A, \beta)}} \sum_{p \nmid d} \sum_{\substack{c_2 \\ p \nmid w_2 \\ (c_2, w_2) = 1}} f(p^{\nu-A} d c_2, p^\beta d w_2) \sum_{h_1} g(p^\delta d c_2 w_2 h_1).$$

Finally, we remove the requirement $(c_2, w_2) = 1$ by Möbius inversion:

$$S = \sum_{p \nmid u} \mu(u) \sum_{\substack{\beta \geq 0 \\ \delta = \max(\nu - A, \beta)}} \sum_{\substack{c \\ p \nmid d \\ p \nmid w}} f(p^{\nu-A} duc, p^\beta duw) \sum_h g(p^\delta u^2 dcwh). \quad \blacksquare$$

PROPOSITION 5.19. *We have*

$$\begin{aligned} \Phi(s,t) = q^{t-s} (2\pi)^{-2i\epsilon_1 r_1 - 2i\epsilon_2 r_2} &\frac{\zeta_q(1 + t + s + i\epsilon_1 r_1 + i\epsilon_2 r_2)}{\zeta_q(2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)} \\ &\times \zeta(1 - t + s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \zeta(1 - t - s + i\epsilon_1 r_1 + i\epsilon_2 r_2) \\ \times \zeta_q(1 + t - s + i\epsilon_1 r_1 + i\epsilon_2 r_2) &\sum_{\alpha \geq 0} \frac{\mu(p^\alpha)}{p^{\alpha(2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)}} \sum_{A,B=0}^{2} C(A,B)(p^A)^{2s} \\ \times &\sum_{\substack{\beta \geq 0 \\ \delta = \max(\nu - A, \beta)}} \frac{(p^B, p^{\alpha+\beta})^{1 + 2i\epsilon_2 r_2}}{p^{B(1 + i\epsilon_2 r_2)} p^{\beta(1 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)} p^{\delta(t - s - i\epsilon_1 r_1 - i\epsilon_2 r_2)}}. \end{aligned}$$

Proof. The expression $\mathrm{TD}(c)$ is given by (5.13). Consider

$$\begin{aligned} \sum_{p^{\nu-A}|c} \mathrm{TD}(c) = \sum_{p^{\nu-A}|c} \frac{1}{c^{2s}} \sum_v \frac{\mu(v)}{v^{2 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \sum_w \frac{(p^B, wv)^{1 + 2i\epsilon_2 r_2}}{p^{B(1 + i\epsilon_2 r_2)} w^{1 + 2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} \\ \times \sum_{c,w|h} \frac{1}{h^{t - s - i\epsilon_1 r_1 - i\epsilon_2 r_2}}. \end{aligned}$$

According to Lemma 5.17

$$c \to p^{\nu-A}duc, \quad w \to p^{\beta}duw, \quad h \to p^{\delta}u^2dcwh.$$

In addition, the sum over v can be decomposed as

$$\sum_{v} \frac{\mu(v)}{v^{2+2i\epsilon_1r_1+2i\epsilon_2r_2}} = \sum_{\alpha\geq 0} \frac{\mu(p^{\alpha})}{p^{\alpha(2+2i\epsilon_1r_1+2i\epsilon_2r_2)}} \sum_{(v,p)=1} \frac{\mu(v)}{v^{2+2i\epsilon_1r_1+2i\epsilon_2r_2}}.$$

Thus

$$\begin{aligned}\sum_{q|cp^A} \mathrm{TD}(c) = \left(\frac{p^A}{q}\right)^{2s} &\sum_{(u,p)=1} \frac{\mu(u)}{u^{2t+1}} \sum_{(v,p)=1} \frac{\mu(v)}{v^{2+2i\epsilon_1r_1+2i\epsilon_2r_2}} \\ &\times \sum_{c} \frac{1}{c^{t+s-i\epsilon_1r_1-i\epsilon_2r_2}} \sum_{h} \frac{1}{h^{t-s-i\epsilon_1r_1-i\epsilon_2r_2}} \sum_{(d,p)=1} \frac{1}{d^{1+t+s+i\epsilon_1r_+i\epsilon_2r_2}} \\ &\times \sum_{(w,p)=1} \frac{1}{w^{1+t-s+i\epsilon_1r_+i\epsilon_2r_2}} \sum_{\alpha\geq 0} \frac{\mu(p^{\alpha})}{p^{\alpha(2+2i\epsilon_1r_1+2i\epsilon_2r_2)}} \\ &\times \sum_{\substack{\beta\geq 0\\ \delta=\max(\nu-A,\beta)}} \frac{(p^B,p^{\alpha+\beta})^{1+2i\epsilon_2r_2}}{p^{B(1+i\epsilon_2r_2)}p^{\beta(1+2i\epsilon_1r_1+2i\epsilon_2r_2)}p^{\delta(t-s-i\epsilon_1r_1-i\epsilon_2r_2)}}.\end{aligned}$$

The asymmetric functional equation implies

$$\begin{aligned}&\frac{\Gamma(t-s-i\epsilon_1r_1-i\epsilon_2r_2)\Gamma(t+s-i\epsilon_1r_1-i\epsilon_2r_2)\prod\zeta(t\pm s-i\epsilon_1r_1-i\epsilon_2r_2)}{(2\pi)^{2t-2i\epsilon_1r_1-2i\epsilon_2r_2}} \\ &\qquad = \frac{\zeta(1-t-s+i\epsilon_1r_1+i\epsilon_2r_2)\zeta(1-t+s+i\epsilon_1r_1+i\epsilon_2r_2)}{2[\cos(\pi s)+\cos(\pi(t-i\epsilon_1r_1-i\epsilon_2r_2))]}.\end{aligned}$$

Thus,

$$\begin{aligned}\Phi(s,t) &= q^{t-s}(2\pi)^{-2i\epsilon_1r_1-2i\epsilon_2r_2}\frac{\zeta_q(1+t+s+i\epsilon_1r_1+i\epsilon_2r_2)}{\zeta_q(2+2i\epsilon_1r_1+2i\epsilon_2r_2)} \\ &\times \zeta_q(1+t-s+i\epsilon_1r_1+i\epsilon_2r_2)\zeta(1-t+s+i\epsilon_1r_1+i\epsilon_2r_2)\zeta(1-t-s+i\epsilon_1r_1+i\epsilon_2r_2) \\ &\times \sum_{\alpha\geq 0} \frac{\mu(p^{\alpha})}{p^{\alpha(2+2i\epsilon_1r_1+2i\epsilon_2r_2)}} \sum_{A,B=0}^{2} C(A,B)(p^A)^{2s} \\ &\times \sum_{\substack{\beta\geq 0\\ \delta=\max(\nu-A,\beta)}} \frac{(p^B,p^{\alpha+\beta})^{1+2i\epsilon_2r_2}}{p^{B(1+i\epsilon_2r_2)}p^{\beta(1+2i\epsilon_1r_1+2i\epsilon_2r_2)}p^{\delta(t-s-i\epsilon_1r_1-i\epsilon_2r_2)}}. \ \blacksquare\end{aligned}$$

The sums over α and β in $\Phi(s,t)$ can be evaluated by considering different cases, as we now show.

5.6.1. *Case 1: $\beta > \nu - A$*

Proposition 5.20. *The case $\beta > \nu - A$ contributes $O_{\epsilon,\boldsymbol{r}}(q^{-1+\epsilon})$ to the off-off-diagonal term.*

Proof. We have $\delta = \beta$ and

$$\begin{aligned}\Phi(s,t) = q^{t-s}(2\pi)^{-2i\epsilon_1 r_1 - 2i\epsilon_2 r_2} \frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)} \\ \times \zeta(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2) \\ \times \zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)\sum_{\alpha\geq 0}\frac{\mu(p^\alpha)}{p^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}}\sum_{A,B=0}^{2} C(A,B)(p^A)^{2s} \\ \times \sum_{\beta\geq \nu-A+1}\frac{(p^B,p^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)}p^{\beta(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)}}.\end{aligned}$$

The sum over β is given by

$$\begin{aligned}q^{t-s}\sum_{\beta\geq\nu-A+1}\frac{1}{(p^{1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2})^\beta} \\ = \frac{1}{q}(p^{A-1})^{1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2}\sum_{\beta\geq 0}\frac{1}{(p^{1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2})^\beta}.\end{aligned}$$

Hence the contribution of this case to M^{OOD} is $O_{\epsilon,\boldsymbol{r}}(q^{-1+\epsilon})$. ∎

5.6.2. *Case 2: $\beta \leq \nu - A$.* The condition $\beta \leq \nu - A$ means $\delta = \nu - A$ and

$$\begin{aligned}\Phi(s,t) = \hat{q}^{2i\epsilon_1 r_1+2i\epsilon_2 r_2}\frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)} \\ \times \zeta(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2) \\ \times \zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2)\sum_{\alpha\geq 0}\frac{\mu(p^\alpha)}{p^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} \\ \times \sum_{A,B=0}^{2} C(A,B)(p^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_r}\sum_{0\leq\beta\leq\nu-A}\frac{(p^B,p^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)}p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}}.\end{aligned}$$

The sum over β can be decomposed in the following way:

$$\begin{aligned}\sum_{0\leq\beta\leq\nu-A}\frac{(p^B,p^{\alpha+\beta})^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)}p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} \\ = \sum_{\substack{0\leq\beta\leq\nu-A\\ B\leq\alpha+\beta}}\frac{(p^B)^{i\epsilon_2 r_2}}{p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} + \sum_{\substack{0\leq\beta\leq\nu-A\\ B>\alpha+\beta}}\frac{(p^\alpha)^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)}p^{\beta(2i\epsilon_1 r_1)}}\end{aligned}$$

$$= \sum_{0\le\beta\le\nu-A} \frac{(p^B)^{i\epsilon_2 r_2}}{p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} - \sum_{\substack{0\le\beta\le\nu-A\\ B>\alpha+\beta}} \frac{(p^B)^{i\epsilon_2 r_2}}{p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} + \sum_{\substack{0\le\beta\le\nu-A\\ B>\alpha+\beta}} \frac{(p^\alpha)^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)}p^{\beta(2i\epsilon_1 r_1)}}.$$

The first sum does not contribute to $\Phi(s,t)$ because

$$\sum_{0\le\beta\le\nu-A} \frac{1}{p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} = \left(1-\frac{1}{p^{1+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}\right)^{-1}\left(1+O\left(\frac{1}{q}\right)\right)$$

and

$$\sum_{A,B=0}^{2} C(A,B)(p^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2}(p^B)^{i\epsilon_2 r_2} = 0.$$

Therefore,

$$\begin{aligned}\Phi(s,t) = \hat{q}^{2i\epsilon_1 r_1+2i\epsilon_2 r_2} & \frac{\zeta_q(1+t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)} \\ & \times \zeta(1-t+s+i\epsilon_1 r_1+i\epsilon_2 r_2)\zeta(1-t-s+i\epsilon_1 r_1+i\epsilon_2 r_2) \\ & \times \zeta_q(1+t-s+i\epsilon_1 r_1+i\epsilon_2 r_2) \sum_{A,B=0}^{2} C(A,B)(p^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_r} \\ \times \sum_{\alpha\ge 0} \frac{\mu(p^\alpha)}{p^{\alpha(2+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} & \sum_{\substack{0\le\beta\le\nu-A\\ B>\alpha+\beta}} \left(\frac{-(p^B)^{i\epsilon_2 r_2}}{p^{\beta(1+2i\epsilon_1 r_1+2i\epsilon_2 r_2)}} + \frac{(p^\alpha)^{1+2i\epsilon_2 r_2}}{p^{B(1+i\epsilon_2 r_2)}p^{\beta(2i\epsilon_1 r_1)}}\right).\end{aligned}$$

For each fixed B the sum over A can be evaluated using Table 1:

$$\sum_{A=0}^{2} C(A,0)(p^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2} = (1-p^{t+s+1-i\epsilon_1 r_1-i\epsilon_2 r_2})(1-p^{t-s+1-i\epsilon_1 r_1-i\epsilon_2 r_2}),$$

$$\sum_{A=0}^{2} C(A,1)(p^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2} = -(p^{ir_2}+p^{-ir_2})(1-p^{t+s+1-i\epsilon_1 r_1-i\epsilon_2 r_2})(1-p^{t-s+1-i\epsilon_1 r_1-i\epsilon_2 r_2}),$$

$$\sum_{A=0}^{2} C(A,2)(p^A)^{t+s-i\epsilon_1 r_1-i\epsilon_2 r_2} = (1-p^{t+s+1-i\epsilon_1 r_1-i\epsilon_2 r_2})(1-p^{t-s+1-i\epsilon_1 r_1-i\epsilon_2 r_2}).$$

Since $B = 0, 1, 2$, the requirement $B > \alpha + \beta$ is satisfied in four cases:

$$(B, \alpha, \beta) \in \{(1,0,0), (2,0,0), (2,1,0), (2,0,1)\}.$$

Thus,

$$\begin{aligned}\Phi(s,t) &= \hat{q}^{2i\epsilon_1 r_1 + 2i\epsilon_2 r_2} \\ &\quad\times \frac{\zeta_q(1+t+s+i\epsilon_1 r_1 + i\epsilon_2 r_2)\zeta_q(1+t-s+i\epsilon_1 r_1 + i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)} \\ &\quad\times \zeta_q(1-t+s+i\epsilon_1 r_1 + i\epsilon_2 r_2)\zeta_q(1-t-s+i\epsilon_1 r_1 + i\epsilon_2 r_2) \\ &\quad\times \bigg[\bigg(p^{ir_2} + \frac{1}{p^{ir_2}}\bigg)\bigg(p^{i\epsilon_2 r_2} - \frac{1}{p^{1+i\epsilon_2 r_2}}\bigg) - p^{2i\epsilon_2 r_2} + \frac{1}{p^{2+2i\epsilon_2 r_2}} \\ &\qquad + \frac{1}{p^{2+2i\epsilon_1 r_1}} - \frac{1}{p^{3+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}} - \frac{1}{p^{1+2i\epsilon_1 r_1}} + \frac{1}{p^{2+2i\epsilon_1 r_1+2i\epsilon_2 r_2}}\bigg].\end{aligned}$$

Simplifying, we have

$$\begin{aligned}\Phi(s,t) &= \frac{\phi(q)}{q}\hat{q}^{2i\epsilon_1 r_1 + 2i\epsilon_2 r_2}\bigg(1 - \frac{1}{p^{1+2i\epsilon_1 r_1}}\bigg)\bigg(1 - \frac{1}{p^{1+2i\epsilon_2 r_2}}\bigg) \\ &\quad\times \frac{\zeta_q(1+t+s+i\epsilon_1 r_1 + i\epsilon_2 r_2)\zeta_q(1+t-s+i\epsilon_1 r_1 + i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)} \\ &\quad\times \zeta_q(1-t+s+i\epsilon_1 r_1 + i\epsilon_2 r_2)\zeta_q(1-t-s+i\epsilon_1 r_1 + i\epsilon_2 r_2).\end{aligned}$$

Substituting this result in (5.15), we obtain Theorem 5.1.

6. Off-off-diagonal term: asymptotic evaluation

THEOREM 6.1. *Up to a negligible error term, we have*

$$\begin{aligned}M^{\mathrm{OOD}} &= \frac{\phi(q)}{q}\sum_{\epsilon_1,\epsilon_2=\pm 1}\frac{\zeta_q(1+2i\epsilon_1 r_1)\zeta_q(1+2i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)} \\ &\quad\times \prod_{\epsilon_3,\epsilon_4=\pm 1}\zeta_q(1+\epsilon_3 t_1 + \epsilon_4 t_2 + i\epsilon_1 r_1 + i\epsilon_2 r_2)\hat{q}^{-2t_1-2t_2+2i\epsilon_1 r_1+2i\epsilon_2 r_2} \\ &\qquad\qquad\times \frac{\Gamma(k/2 - t_1 + i\epsilon_1 r_1)\Gamma(k/2 - t_2 + i\epsilon_2 r_2)}{\Gamma(k/2 + t_1 - i\epsilon_1 r_1)\Gamma(k/2 + t_2 - i\epsilon_2 r_2)}.\end{aligned}\tag{6.1}$$

Proof. Consider

$$\begin{aligned}M^{\mathrm{OOD}} &= \frac{\phi(q)}{q}\sum_{\epsilon_1,\epsilon_2=\pm 1}\frac{\zeta_q(1+2i\epsilon_1 r_1)\zeta_q(1+2i\epsilon_2 r_2)}{\zeta_q(2+2i\epsilon_1 r_1 + 2i\epsilon_2 r_2)}\hat{q}^{-2t_1-2t_2+2i\epsilon_1 r_1+2i\epsilon_2 r_2} \\ &\quad\times \frac{1}{(2\pi i)^2}\int\limits_{\Re t = k/2+0.7}\int\limits_{\Re s = k/2-0.4} I_{\epsilon_1,\epsilon_2}(s,t)\,\frac{2s\,ds}{s^2 - t_1^2}\,\frac{2t\,dt}{t^2 - t_2^2},\end{aligned}$$

where

$$
\begin{aligned}
&I_{\epsilon_1,\epsilon_2}(s,t)\\
&\quad= \frac{P_r(s)P_r(t)}{P_r(t_1)P_r(t_2)}\zeta_q(1+t+s+i\epsilon_1r_1+i\epsilon_2r_2)\zeta_q(1+t-s+i\epsilon_1r_1+i\epsilon_2r_2)\\
&\qquad\times \zeta_q(1-t+s+i\epsilon_1r_1+i\epsilon_2r_2)\zeta_q(1-t-s+i\epsilon_1r_1+i\epsilon_2r_2)\\
&\times \frac{\Gamma(k/2+s+i\epsilon_1r_1)\Gamma(k/2+t+i\epsilon_2r_2)\Gamma(k/2-s+i\epsilon_1r_1)\Gamma(k/2-t+i\epsilon_2r_2)}{\Gamma(k/2+t_1+ir_1)\Gamma(k/2+t_1-ir_1)\Gamma(k/2+t_2+ir_2)\Gamma(k/2+t_2-ir_2)}.
\end{aligned}
$$

The function $I_{\epsilon_1,\epsilon_2}(s,t)$ is even in both s and t. Therefore,

$$
\begin{aligned}
&4\frac{1}{(2\pi i)^2}\int\limits_{\Re t=k/2+0.7}\int\limits_{\Re s=k/2-0.4} I_{\epsilon_1,\epsilon_2}(s,t)\frac{2s\,ds}{s^2-t_1^2}\,\frac{2t\,dt}{t^2-t_2^2}\\
&\quad= \operatorname{Res}_{\substack{s=t_1\\t=t_2}} I(s,t)\frac{2s}{s^2-t_1^2}\,\frac{2t}{t^2-t_2^2}+\operatorname{Res}_{\substack{s=t_1\\t=-t_2}} I(s,t)\frac{2s}{s^2-t_1^2}\,\frac{2t}{t^2-t_2^2}\\
&\qquad+\operatorname{Res}_{\substack{s=-t_1\\t=t_2}} I(s,t)\frac{2s}{s^2-t_1^2}\,\frac{2t}{t^2-t_2^2}+\operatorname{Res}_{\substack{s=-t_1\\t=-t_2}} I(s,t)\frac{2s}{s^2-t_1^2}\,\frac{2t}{t^2-t_2^2}.
\end{aligned}
$$

Each of the four residues has the same value. Computing it yields the assertion. ■

Theorem 6.2. *Up to a negligible error, we have*

$$
\lim_{(\boldsymbol{t},\boldsymbol{r})\to(0,0)} M^{\mathrm{OOD}} = \frac{1}{(2\pi i)^2}\int\limits_{\Re t=k/2+0.7}\int\limits_{\Re s=k/2-0.4} g(s,t)\frac{2ds}{s}\,\frac{2dt}{t}, \tag{6.2}
$$

where

$$
\begin{aligned}
g(s,t) &= \left(\frac{\phi(q)}{q}\right)^3\frac{1}{\zeta_q(2)}\,\frac{P_r(s)P_r(t)}{P_r(0)^2}\prod_{\epsilon_1,\epsilon_2=\pm1}\zeta_q(1+\epsilon_1t+\epsilon_2s)\\
&\quad\times\frac{\Gamma(k/2+\epsilon_1t)\Gamma(k/2+\epsilon_2s)}{\Gamma(k/2)^4}\Bigg[(2\log\hat q+\gamma)^2+\sum_{\epsilon_1,\epsilon_2=\pm1}\frac{\zeta_q''}{\zeta_q}(1+\epsilon_1t+\epsilon_2s)\\
&\qquad+2\sum_{\substack{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4=\pm1\\(\epsilon_1,\epsilon_2)\neq(\epsilon_3,\epsilon_4)}}\frac{\zeta_q'}{\zeta_q}(1+\epsilon_1t+\epsilon_2s)\frac{\zeta_q'}{\zeta_q}(1+\epsilon_3t+\epsilon_4s)+(2\log\hat q+\gamma)\\
&\qquad\times\left(4\frac{\zeta_q'}{\zeta_q}(2)-2\sum_{\epsilon_1,\epsilon_2=\pm1}\frac{\zeta_q'}{\zeta_q}(1+\epsilon_1t+\epsilon_2s)-\sum_{\epsilon=\pm1}\frac{\Gamma'}{\Gamma}(k/2+\epsilon s)\right.\\
&\qquad\left.-\sum_{\epsilon=\pm1}\frac{\Gamma'}{\Gamma}(k/2+\epsilon t)\right)+\sum_{\epsilon_1,\epsilon_2=\pm1}\frac{\zeta_q'}{\zeta_q}(1+\epsilon_1t+\epsilon_2s)\\
&\qquad\times\left(-4\frac{\zeta_q'}{\zeta_q}(2)+\sum_{\epsilon=\pm1}\frac{\Gamma'}{\Gamma}(k/2+\epsilon t)+\sum_{\epsilon=\pm1}\frac{\Gamma'}{\Gamma}(k/2+\epsilon s)\right)
\end{aligned}
$$

$$+\sum_{\epsilon=\pm1}\frac{\Gamma'}{\Gamma}(k/2+\epsilon t)\sum_{\epsilon=\pm1}\frac{\Gamma'}{\Gamma}(k/2+\epsilon s)-4\frac{\zeta_q''}{\zeta_q}(2)+8\biggl(\frac{\zeta_q'}{\zeta_q}(2)\biggr)^2$$
$$-2\frac{\zeta_q'}{\zeta_q}(2)\biggl(\sum_{\epsilon=\pm1}\frac{\Gamma'}{\Gamma}(k/2+\epsilon t)\sum_{\epsilon=\pm1}\frac{\Gamma'}{\Gamma}(k/2+\epsilon s)\biggr)\biggr].$$

COROLLARY 6.3. *The off-off-diagonal term at the critical point is a polynomial in* $\log q$ *of order* 2.

Proof. First, we let $t_1,t_2\to 0$. Then

$$\lim_{\boldsymbol{t}\to 0}M^{\mathrm{OOD}}=\frac{\phi(q)}{q}\sum_{\epsilon_1,\epsilon_2=\pm1}\frac{\zeta_q(1+2i\epsilon_1r_1)\zeta_q(1+2i\epsilon_2r_2)}{\zeta_q(2+2i\epsilon_1r_1+2i\epsilon_2r_2)}\hat{q}^{2i\epsilon_1r_1+2i\epsilon_2r_2}$$
$$\times\frac{1}{(2\pi i)^2}\int\limits_{\Re t=k/2+0.7}\ \int\limits_{\Re s=k/2-0.4}I_{\epsilon_1,\epsilon_2}(s,t)\frac{2\,ds}{s}\,\frac{2\,dt}{t},$$

where

$$I_{\epsilon_1,\epsilon_2}(s,t)=\frac{P_r(s)P_r(t)}{P_r(0)^2}\zeta_q(1+t+s+i\epsilon_1r_1+i\epsilon_2r_2)$$
$$\times\zeta_q(1+t-s+i\epsilon_1r_1+i\epsilon_2r_2)\zeta_q(1-t+s+i\epsilon_1r_1+i\epsilon_2r_2)\zeta_q(1-t-s+i\epsilon_1r_1+i\epsilon_2r_2)$$
$$\times\frac{\Gamma(k/2+s+i\epsilon_1r_1)\Gamma(k/2+t+i\epsilon_2r_2)\Gamma(k/2-s+i\epsilon_1r_1)\Gamma(k/2-t+i\epsilon_2r_2)}{\Gamma(k/2+ir_1)\Gamma(k/2-ir_1)\Gamma(k/2+ir_2)\Gamma(k/2-ir_2)}.$$

Let

$$f(r_1,r_2):=\frac{\phi(q)}{q}\,\frac{P_r(s)P_r(t)}{P_r(0)^2}\,\frac{\zeta_q(1+t+s+ir_1+ir_2)}{\zeta_q(2+2ir_1+2ir_2)}$$
$$\times\zeta_q(1+t-s+ir_1+ir_2)\zeta_q(1-t+s+ir_1+ir_2)\zeta_q(1-t-s+ir_1+ir_2)$$
$$\times\frac{\Gamma(k/2+s+ir_1)\Gamma(k/2+t+ir_2)\Gamma(k/2-s+ir_1)\Gamma(k/2-t+ir_2)}{\Gamma(k/2+ir_1)\Gamma(k/2-ir_1)\Gamma(k/2+ir_2)\Gamma(k/2-ir_2)}.$$

Consider

$$g(s,t)=\lim_{r_1\to0}\lim_{r_2\to0}\sum_{\epsilon_1,\epsilon_2=\pm1}\zeta_q(1+2i\epsilon_1r_1)\zeta_q(1+2i\epsilon_2r_2)\hat{q}^{2i\epsilon_1r_1+2i\epsilon_2r_2}f(\epsilon_1r_1,\epsilon_2r_2)$$
$$=\biggl(\frac{\phi(q)}{q}\biggr)^2\biggl[(2\log\hat{q}+\gamma)^2f(0,0)$$
$$+i(2\log\hat{q}+\gamma)\biggl(\frac{\partial f}{\partial r_1}(0,0)+\frac{\partial f}{\partial r_2}(0,0)\biggr)-\frac{\partial^2 f}{\partial r_1\partial r_2}(0,0)\biggr].$$

Here

$$\frac{\partial f}{\partial r_1}(0,0)=-if(0,0)\biggl(2\frac{\zeta_q'}{\zeta_q}(2)-\sum\frac{\zeta_q'}{\zeta_q}(1\pm t\pm s)-\sum\frac{\Gamma'}{\Gamma}(k/2\pm s)\biggr),$$

$$\frac{\partial f}{\partial r_2}(0,0) = -if(0,0)\Big(2\frac{\zeta_q'}{\zeta_q}(2) - \sum \frac{\zeta_q'}{\zeta_q}(1\pm t\pm s) - \sum \frac{\Gamma'}{\Gamma}(k/2\pm t)\Big),$$

$$\begin{aligned}\frac{\partial^2 f}{\partial r_1\partial r_2}(0,0) = -f(0,0)\Big[&\sum \frac{\zeta_q''}{\zeta_q}(1\pm t\pm s)\\
&+2\sum\nolimits^* \frac{\zeta_q'}{\zeta_q}(1\pm t\pm s)\frac{\zeta_q'}{\zeta_q}(1\pm t\pm s) + \sum \frac{\zeta_q'}{\zeta_q}(1\pm t\pm s)\\
&\times\Big(-4\frac{\zeta_q'}{\zeta_q}(2) + \sum\frac{\Gamma'}{\Gamma}(k/2\pm t) + \sum\frac{\Gamma'}{\Gamma}(k/2\pm s)\Big)\\
&+\sum\frac{\Gamma'}{\Gamma}(k/2\pm t)\sum\frac{\Gamma'}{\Gamma}(k/2\pm s) - 4\frac{\zeta_q''}{\zeta_q}(2) + 8\Big(\frac{\zeta_q'}{\zeta_q}(2)\Big)^2\\
&-2\frac{\zeta_q'}{\zeta_q}(2)\Big(\sum\frac{\Gamma'}{\Gamma}(k/2\pm t)\sum\frac{\Gamma'}{\Gamma}(k/2\pm s)\Big)\Big],\end{aligned}$$

where $\sum^*$ means that the sum does not include squared terms. Then

$$\lim_{(\boldsymbol{t},\boldsymbol{r})\to(0,0)} M^{\mathrm{OOD}} = \frac{1}{(2\pi i)^2}\int\limits_{\Re t=k/2+0.7}\int\limits_{\Re s=k/2-0.4} g(s,t)\,\frac{2\,ds}{s}\,\frac{2\,dt}{t}.$$

The function $g(s,t)$ is even in both variables s and t. Therefore,

$$\lim_{(\boldsymbol{t},\boldsymbol{r})\to(0,0)} M^{\mathrm{OOD}} = \frac{1}{4}\operatorname{Res}_{s=t=0}\frac{4g(s,t)}{st} = \operatorname{Res}_{t=0}\frac{g(0,t)}{t}.$$

To find the order of the leading term, we replace all $\zeta(1\pm t)$ by $\frac{1}{\pm t}$. Let

$$r(t) := \frac{P_r(t)}{P_r(0)}\,\frac{\Gamma(k/2+t)\Gamma(k/2-t)}{\Gamma(k/2)^2}.$$

Then

$$\begin{aligned}\Big(\frac{\phi(q)}{q}\Big)^7 &\frac{1}{\zeta_q(2)}\operatorname{Res}_{t=0}\frac{r(t)}{t^5}\Big((\log q)^2 + \frac{4}{t^2}\Big)\\
&= \Big(\frac{\phi(q)}{q}\Big)^7\frac{1}{\zeta_q(2)}\,\frac{1}{6!}\big(4r^{(6)}(0) + 30r^{(4)}(0)(\log q)^2\big).\end{aligned}$$

Therefore, $\lim_{(\boldsymbol{t},\boldsymbol{r})\to(0,0)} M^{\mathrm{OOD}}$ is a polynomial in $\log q$ of order 2. ∎

Appendices

A. Bessel functions

Lemma A.1 ([KMV2, Lemma C.1]). *Let $z>0$ and $v\in\mathbb{C}$. Then*

(A.1) $$(z^v J_v(z))' = z^v J_{v-1}(z),$$

(A.2) $$(z^v Y_v(z))' = z^v Y_{v-1}(z),$$

(A.3) $$(z^v K_v(z))' = -z^v K_{v-1}(z).$$

LEMMA A.2 ([KMV2, Lemma C.2]). *For $z > 0$ and $j \geq 0$ we have*

$$\frac{z^j}{(1+z)^j} J_v^{(j)}(z) \ll_{j,v} \frac{z^{\Re v}}{(1+z)^{\Re v + 1/2}}, \tag{A.4}$$

$$\frac{z^j}{(1+z)^j} Y_0^{(j)}(z) \ll_j \frac{(1+|\log z|)}{(1+z)^{1/2}}, \tag{A.5}$$

$$\frac{z^j}{(1+z)^j} K_v^{(j)}(z) \ll_{j,v} \frac{e^{-z}(1+|\log z|)}{(1+z)^{1/2}} \quad \text{if } \Re v = 0. \tag{A.6}$$

LEMMA A.3 ([Wa, p. 149]). *Assume that $\Re(\mu_1 + \mu_2 + 1) > \Re(2s) > 0$. Then*

$$\int_0^\infty \frac{J_{\mu_1}(z) J_{\mu_2}(z)}{z^{2s}}\, dz = \frac{1}{2^{2s}} \frac{\Gamma(2s)}{\Gamma(-\mu_1/2 + \mu_2/2 + s + 1/2)} \times \frac{\Gamma(\mu_1/2 + \mu_2/2 - s + 1/2)}{\Gamma(\mu_1/2 + \mu_2/2 + s + 1/2)\Gamma(\mu_1/2 - \mu_2/2 + s + 1/2)}. \tag{A.7}$$

LEMMA A.4 ([BH, Lemma 3]). *Let $F : (0,\infty) \to \mathbb{C}$ be a smooth function of compact support. For $s \in \mathbb{C}$ let B_s denote one of J_s, Y_s or K_s. Then for $\alpha > 0$ and $j \in \mathbb{N}$ we have*

$$\int_0^\infty F(x) B_s(\alpha\sqrt{x})\, dx = \pm \left(\frac{2}{\alpha}\right)^j \int_0^\infty \frac{\partial^j}{\partial x^j} (F(x) x^{-s/2}) x^{(s+j)/2} B_{s+j}(\alpha\sqrt{x})\, dx. \tag{A.8}$$

LEMMA A.5 ([Ol, 10.6.7 and 10.29.5]). *For $k = 0, 1, 2, \ldots,$*

$$J_s^{(k)}(z) = \frac{1}{2^k} \sum_{n=0}^{k} (-1)^n \binom{k}{n} J_{s-k+2n}(z), \tag{A.9}$$

$$e^{s\pi i} K_s^{(k)}(z) = \frac{1}{2^k} \left(e^{(s-k)\pi i} K_{s-k}(z) + \binom{k}{1} e^{(s-k+2)\pi i} K_{s-k+2}(z) + \binom{k}{2} e^{(s-k+4)\pi i} K_{s-k+4}(z) + \cdots + e^{(s+k)\pi i} K_{s+k}(z) \right). \tag{A.10}$$

B. Mellin transforms

LEMMA B.1 ([Ob, 2.19, p. 15]). *Let $\phi(x) = (b + ax)^{-v}$. Then for $0 < \Re z < v$,*

$$\int_0^\infty \phi(x) x^{z-1}\, dx = (b/a)^z b^{-v} \frac{\Gamma(z)\Gamma(v-z)}{\Gamma(v)}. \tag{B.1}$$

LEMMA B.2 ([Ob, 2.20, p. 16]). *Let* $\Re v > -1$ *and*

$$\phi(x) = \begin{cases} (a-x)^v & \text{if } x < a, \\ 0 & \text{if } x > a. \end{cases}$$

Then for $\Re z > 0$,

$$\int_0^\infty \phi(x) x^{z-1}\, dx = a^{v+z} \frac{\Gamma(v+1)\Gamma(z)}{\Gamma(v+z+1)}. \tag{B.2}$$

LEMMA B.3 ([Ob, 2.21, p. 16]). *Let* $\Re v > -1$ *and*

$$\phi(x) = \begin{cases} (x-a)^v & \text{if } x > a, \\ 0 & \text{if } x < a. \end{cases}$$

Then for $\Re z < -\Re v$,

$$\int_0^\infty \phi(x) x^{z-1}\, dx = a^{v+z} \frac{\Gamma(-v-z)\Gamma(v+1)}{\Gamma(1-z)}. \tag{B.3}$$

LEMMA B.4 ([BE, p. 21]). *For* $x > 0$,

$$J_{k-1}(x) = \frac{1}{2}\frac{1}{2\pi i} \int_{(\sigma)} \left(\frac{x}{2}\right)^{-s} \frac{\Gamma(s/2+k/2-1/2)}{\Gamma(-s/2+k/2+1/2)}\, ds, \tag{B.4}$$

where $-k+1 < \sigma < 1$.

Changing variable $-s := k-1+2z$, we obtain

LEMMA B.5.

$$J_{k-1}(x) = -\frac{1}{2\pi i} \int_{(\sigma)} \frac{\pi}{\Gamma(1+z)\Gamma(k+z)\sin(\pi z)} \left(\frac{x}{2}\right)^{k-1+2z} dz, \tag{B.5}$$

where $-k/2 < \sigma < 0$.

Let

$$\gamma(u,v) := \frac{2^{2u-1}}{\pi}\Gamma(u+v-1/2)\Gamma(u-v+1/2). \tag{B.6}$$

LEMMA B.6 ([K, p. 89]).

$$\int_0^\infty k_0(x,v) x^{w-1}\, dx = \gamma(w/2, v)\cos(\pi w/2), \tag{B.7}$$

$$\int_0^\infty k_1(x,v) x^{w-1}\, dx = \gamma(w/2, v)\sin(\pi v). \tag{B.8}$$

COROLLARY B.7.

$$k_1(x, 1/2+ir) = \frac{\sin(\pi(1/2+ir))}{2\pi i} \int_{(0.7)} x^{-2\beta}\gamma(\beta, 1/2+ir) 2\, d\beta, \tag{B.9}$$

$$k_0(x, 1/2 + ir) = \frac{1}{2\pi i} \int_{(*)} x^{-2\beta} \gamma(\beta, 1/2 + ir) \cos(\pi\beta) 2\, d\beta, \tag{B.10}$$

where the integration contour $(*)$ *can be taken as* $\Re\beta = -1$ *except in the area* $|\Im\beta| < 1$*, where it crosses the real axis at* $\Re\beta > 0$*.*

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