STUDIA MATHEMATICA 235 (2) (2016)

Small Valdivia compacta and trees

by

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Abstract. We present a characterization of Valdivia compact spaces of small weight in terms of path spaces of trees and we use it to obtain (under $\diamondsuit$) a counterexample to a conjecture related to an open problem concerning twisted sums of $C(K)$ spaces.

1. Introduction. The purpose of this article is twofold: first, we use the description given by Kubiś and Michalewski [11] of the class of small Valdivia compacta involving inverse limits of compact metric spaces to obtain a new characterization of this class in terms of path spaces of certain types of trees, endowed with appropriate compatible topologies. This characterization allows one to fine-tune the structure of a Valdivia compactum by manipulating the properties of the corresponding tree. We then use this technique to construct, under $\diamondsuit$, a counterexample to the following conjecture stated in [6, Section 4].

Conjecture. *If K is a nonempty Valdivia compact space satisfying the countable chain condition (ccc), then either K has a G_δ point, or K admits a nontrivial convergent sequence in the complement of a dense Σ-subset.*

Recall that a compact Hausdorff space K is called a *Valdivia compactum* if it admits a dense *Σ-subset*, i.e., a subset of K of the form $\varphi^{-1}[\Sigma(\Gamma)]$, where $\varphi : K \to \mathbb{R}^\Gamma$ is a continuous injection and $\Sigma(\Gamma)$ denotes the set of points $x \in \mathbb{R}^\Gamma$ such that $\{\gamma \in \Gamma : x_\gamma \neq 0\}$ is countable; here Γ is an arbitrary index set and $\mathbb{R}^\Gamma$ is endowed with the product topology. We call a Valdivia compactum *small* if its weight is not greater than ω_1. Valdivia compact spaces constitute a large superclass of Corson compact spaces, closed under arbitrary products, and they were introduced by Argyros, Mercourakis, and Negrepontis [2]. This class and its relation to the theory of Banach spaces

2010 *Mathematics Subject Classification*: Primary 54D30, 54G20, 06A06; Secondary 54A35.
Key words and phrases: Valdivia compacta, inverse limits, trees.
Received 8 December 2015; revised 24 July 2016.
Published online 30 September 2016.

DOI: 10.4064/sm8477-7-2016

have since then been studied by several authors [1, 7, 8, 10, 11, 13, 14] (see [9] for a survey).

We observe that finding examples of nonempty Valdivia compact spaces with no G_δ points and no nontrivial convergent sequences in the complement of a dense Σ-subset is not a trivial task, since the absence of G_δ points tends to make the complement of dense Σ-subsets "large" (see, for instance, [9, Theorem 3.3] for a more precise statement). In [6, Proposition 4.7], it was shown that the path space of a certain tree T, endowed with the product topology of 2^T, provides such an example. However, using this topology it is not possible to have a nonempty path space with no G_δ points and ccc.

The techniques presented in this article allow us to handle more complicated topologies on path spaces, and after a technically elaborate construction a counterexample to the Conjecture is obtained under $\diamondsuit$. Recall that $\diamondsuit$ is a combinatorial principle stronger than the continuum hypothesis (CH) and consistent with ZFC (see [12]).

As shown in [6], the Conjecture implies, under CH, that for every nonmetrizable Valdivia compact space K there exists a *nontrivial twisted sum* of c_0 and $C(K)$, i.e., a Banach space X containing a noncomplemented isomorphic copy Y of c_0 such that X/Y is isomorphic to $C(K)$. As usual, $C(K)$ denotes the Banach space of continuous real-valued functions on K endowed with the supremum norm. The existence of a nontrivial twisted sum of c_0 and $C(K)$, for every nonmetrizable compact Hausdorff space K, is an open problem discussed in many articles [3, 4, 5]. This problem remains open even in the context of Valdivia compact spaces [5] and was only recently settled, under CH, for Corson compacta [6, Theorem 3.1]. In fact, a nontrivial twisted sum of c_0 and $C(K)$ for every nonmetrizable Corson compact space K is known to exist also under Martin's Axiom [6, Remark 3.5], but it is not known whether this can be proven in ZFC.

Even though the Conjecture turned out to be false (under $\diamondsuit$), a deeper understanding of the class of counterexamples to the Conjecture should shed light upon the problem of existence of nontrivial twisted sums of c_0 and $C(K)$ for K a nonmetrizable Valdivia compact space. The authors do not know whether such a nontrivial twisted sum exists if K is the counterexample to the Conjecture constructed in this article. We note, in addition, that to prove the existence of a nontrivial twisted sum of c_0 and $C(K)$ for an arbitrary nonmetrizable Valdivia compact space K, one can restrict attention to the case when K is small, since every nonmetrizable Valdivia compact space contains a nonmetrizable small Valdivia subspace as a retract [2, Lemma 1.3].

Here is an overview of this article. In Section 2, we review the relevant material concerning inverse limits and we recall the characterization of small

Valdivia compacta from [11]. In Section 3, we develop the theory relating inverse limits to trees endowed with some additional structure and we study the relevant topologies on the path space. Finally, Section 4 is devoted to the construction of the counterexample to the Conjecture.

2. Inverse limits and Valdivia compacta. Throughout the article, we denote by $|\mathcal{X}|$ the cardinality of a set $\mathcal{X}$, by $w(\mathcal{X})$ the weight of a topological space $\mathcal{X}$ and by $\mathbb{S}$ the class of all successor ordinals. We start by recalling some standard definitions and facts concerning inverse limits.

Definition 2.1. Let $(I, \leq)$ be a partially ordered directed set. An *inverse system of sets* $\mathcal{K} = ((K_i)_{i\in I}, (r_{ij})_{i\leq j\in I})$ indexed by I consists of a family $(K_i)_{i\in I}$ of sets and a family $(r_{ij} : K_j \to K_i)_{i\leq j\in I}$ of maps such that r_{ii} is the identity of K_i for all $i \in I$, and $r_{ij} \circ r_{jk} = r_{ik}$ for all $i, j, k \in I$ with $i \leq j \leq k$. We call $\big(K, (r_i)_{i\in I}\big)$ a *cone* over $\mathcal{K}$ if K is a set and $(r_i : K \to K_i)_{i\in I}$ is a family of maps with $r_{ij} \circ r_j = r_i$ for all $i, j \in I$ with $i \leq j$. An *inverse limit* of $\mathcal{K}$ is a cone $\big(K, (r_i)_{i\in I}\big)$ over $\mathcal{K}$ such that, for every cone $(K', (r'_i)_{i\in I})$ over $\mathcal{K}$, there exists a unique map $f : K' \to K$ such that $r_i \circ f = r'_i$ for all $i \in I$.

A concrete description of an inverse limit of $\mathcal{K}$ is obtained by considering the set

$$\Big\{(x_i)_{i\in I} \in \prod_{i\in I} K_i : r_{ij}(x_j) = x_i \text{ for all } i, j \in I \text{ with } i \leq j\Big\} \tag{1}$$

together with the restrictions of the projections. Note that $(K, (r_i)_{i\in I})$ is a cone over $\mathcal{K}$ if and only if the image of the map $(r_i)_{i\in I} : K \to \prod_{i\in I} K_i$ is contained in (1), and that $(K, (r_i)_{i\in I})$ is an inverse limit of $\mathcal{K}$ if and only if the map $(r_i)_{i\in I}$ is a bijection between K and (1). When the sets K_i are endowed with compact Hausdorff topologies such that the maps r_{ij} are continuous, we call $\mathcal{K}$ an *inverse system of compact Hausdorff spaces*. Cones and inverse limits are then defined by replacing "set" with "compact Hausdorff space" and "map" with "continuous map" in Definition 2.1. In this context, the set (1) should be endowed with the product topology.

The following simple results give information on the closed G_δ subsets of an inverse limit.

Lemma 2.2. *Let $\mathcal{K} = ((K_i)_{i\in I}, (r_{ij})_{i\leq j\in I})$ be an inverse system of compact Hausdorff spaces and $(K, (r_i)_{i\in I})$ be an inverse limit of $\mathcal{K}$. Given a closed subset F of K and an open subset U of K containing F, there exists $i \in I$ with $r_i^{-1}[r_i[F]] \subset U$. In particular, if every countable subset of I is bounded and F is a closed G_δ subset of K, then there exists $i \in I$ with $F = r_i^{-1}[r_i[F]]$.*

Proof. Note that $\{(x,y)\in K\times K : r_i(x)=r_i(y)\}$, $i\in I$, is a downward directed family of closed subsets of the compact space $K\times K$ whose intersection is the diagonal. This intersection is contained in the complement of the closed set $F\times(K\setminus U)$, and therefore $\{(x,y)\in K\times K : r_i(x)=r_i(y)\}$ is contained in the complement of $F\times(K\setminus U)$ for some $i\in I$. ∎

COROLLARY 2.3. *Let $\mathcal{K}=((K_i)_{i\in I},(r_{ij})_{i\le j\in I})$ be an inverse system of compact metric spaces and assume that every countable subset of I is bounded. If $(K,(r_i)_{i\in I})$ is an inverse limit of $\mathcal{K}$, then the set*

$$\{r_i^{-1}[F] : i\in I,\ F \text{ closed in } K_i\}$$

coincides with the collection of closed G_δ subsets of K. In particular, K has a G_δ point if and only if there exist $i\in I$ and $x\in K_i$ with $|r_i^{-1}(x)|=1$. □

DEFINITION 2.4. Let $\mathcal{K}=((K_i)_{i\in I},(r_{ij})_{i\le j\in I})$ be an inverse system of sets (resp., of compact Hausdorff spaces). A *right inverse* of $\mathcal{K}$ is a family of maps (resp., of continuous maps) $(\sigma_{ij}:K_i\to K_j)_{i\le j\in I}$ such that σ_{ij} is a right inverse of r_{ij} and $\sigma_{jk}\circ\sigma_{ij}=\sigma_{ik}$ for all $i,j,k\in I$ with $i\le j\le k$.

If $(\sigma_{ij})_{i\le j\in I}$ is a right inverse of $\mathcal{K}$ and $(K,(r_i)_{i\in I})$ is an inverse limit of $\mathcal{K}$, then there exists a unique family $(\sigma_i:K_i\to K)_{i\in I}$ of maps such that σ_i is a right inverse of r_i and $\sigma_j\circ\sigma_{ij}=\sigma_i$ for all $i,j\in I$ with $i\le j$; these maps are automatically continuous if $\mathcal{K}$ is an inverse system of compact Hausdorff spaces [11, Lemma 3.1]. We call the maps σ_i *induced* by the right inverse $(\sigma_{ij})_{i\le j\in I}$, and they will always be denoted by σ_i whenever $(\sigma_{ij})_{i\le j\in I}$ denotes a right inverse of an inverse system. Note that, for $i,j\in I$, we have $r_j\circ\sigma_i=\sigma_{ij}$ if $i\le j$, and $r_j\circ\sigma_i=r_{ji}$ if $j\le i$.

DEFINITION 2.5. An inverse system $\mathcal{K}=((K_\alpha)_{\alpha\in\theta},(r_{\alpha\beta})_{\alpha\le\beta\in\theta})$ whose index set θ is a nonzero ordinal (endowed with the natural order) is called an *inverse θ-sequence* or, more simply, an *inverse sequence*. It is called *continuous* if, for every limit ordinal $\alpha\in\theta$, $(K_\alpha,(r_{\beta\alpha})_{\beta\in\alpha})$ is an inverse limit of $((K_\beta)_{\beta\in\alpha},(r_{\beta\gamma})_{\beta\le\gamma\in\alpha})$.

The next lemma gives a useful description of the image of the maps σ_α induced by a right inverse of a continuous inverse sequence.

LEMMA 2.6. *If $\mathcal{K}=((K_\alpha)_{\alpha\in\theta},(r_{\alpha\beta})_{\alpha\le\beta\in\theta})$ is a continuous inverse sequence with a right inverse $(\sigma_{\alpha\beta})_{\alpha\le\beta\in\theta}$, and $(K,(r_\alpha)_{\alpha\in\theta})$ is an inverse limit of $\mathcal{K}$, then for all $\alpha\in\theta$, the image of σ_α is equal to*

$$\{x\in K : r_{\beta+1}(x)\in\sigma_{\beta,\beta+1}[K_\beta] \text{ for all } \beta\ge\alpha \text{ with } \beta+1<\theta\}. \tag{2}$$

Proof. Obviously $\sigma_\alpha[K_\alpha]$ is contained in (2). To prove equality, take x in (2) and show by induction on β that $r_\beta((\sigma_\alpha\circ r_\alpha)(x))=r_\beta(x)$ for all $\beta\ge\alpha$. Since the maps r_β, $\beta\ge\alpha$, separate the points of K, it follows that $(\sigma_\alpha\circ r_\alpha)(x)=x$. ∎

A collection $\mathcal{D}$ of nonempty open subsets of a topological space $\mathcal{X}$ is called *dense* in the topology of $\mathcal{X}$ if every nonempty open subset of $\mathcal{X}$ contains an element of $\mathcal{D}$. Obviously, if $\mathcal{D}$ is dense, then $\mathcal{X}$ has ccc if and only if every family of pairwise disjoint elements of $\mathcal{D}$ is countable. We are interested in determining conditions under which an inverse limit has ccc, and to this end we describe in the next lemma a convenient dense subset of its topology.

LEMMA 2.7. *Let $\mathcal{K} = ((K_\alpha)_{\alpha\in\theta}, (r_{\alpha\beta})_{\alpha\le\beta\in\theta})$ be a continuous inverse sequence of compact Hausdorff spaces with a right inverse $(\sigma_{\alpha\beta})_{\alpha\le\beta\in\theta}$ and let $(K, (r_\alpha)_{\alpha\in\theta})$ be an inverse limit of $\mathcal{K}$. Consider the collection $\mathcal{D}$ of open subsets of K of the form $r_{\alpha+1}^{-1}[U]$ with U a nonempty open subset of $K_{\alpha+1} \setminus \sigma_{\alpha,\alpha+1}[K_\alpha]$ and $\alpha + 1 < \theta$. If $\bigcup_{\alpha\in\theta} \sigma_\alpha[K_\alpha]$ has empty interior in K, then $\mathcal{D}$ is dense in the topology of K.*

Proof. Let V be a nonempty open subset of K and pick $x \in V$ not in $\bigcup_{\alpha\in\theta} \sigma_\alpha[K_\alpha]$. By Lemma 2.2 there exists $\beta \in \theta$ with $r_\beta^{-1}(r_\beta(x)) \subset V$, and by Lemma 2.6 there exists $\alpha \ge \beta$ with $\alpha+1 < \theta$ and $r_{\alpha+1}(x)$ not in $\sigma_{\alpha,\alpha+1}[K_\alpha]$. Then $r_{\alpha+1}^{-1}(r_{\alpha+1}(x)) \subset r_\beta^{-1}(r_\beta(x)) \subset V$, and setting

$$U = K_{\alpha+1} \setminus (r_{\alpha+1}[K \setminus V] \cup \sigma_{\alpha,\alpha+1}[K_\alpha])$$

we find that $r_{\alpha+1}(x) \in U$ and $r_{\alpha+1}^{-1}[U] \subset V$. ∎

COROLLARY 2.8. *Let $\mathcal{K} = ((K_\alpha)_{\alpha\in\omega_1}, (r_{\alpha\beta})_{\alpha\le\beta\in\omega_1})$ be a continuous inverse sequence of compact metric spaces with a right inverse $(\sigma_{\alpha\beta})_{\alpha\le\beta\in\omega_1}$ and let $(K, (r_\alpha)_{\alpha\in\omega_1})$ be an inverse limit of $\mathcal{K}$. Assume that $\bigcup_{\alpha\in\omega_1} \sigma_\alpha[K_\alpha]$ has empty interior in K. Then K does not have ccc if and only if there exist an uncountable subset Λ of ω_1 and a family $(U_\alpha)_{\alpha\in\Lambda}$ with each U_α a nonempty open subset of $K_{\alpha+1} \setminus \sigma_{\alpha,\alpha+1}[K_\alpha]$ such that the sets $r_{\alpha+1}^{-1}[U_\alpha]$, $\alpha \in \Lambda$, are pairwise disjoint.*

Proof. Follows directly from the lemma, if we keep in mind that the spaces K_α have ccc. ∎

We recall the following characterization of small Valdivia compacta in terms of inverse limits.

THEOREM 2.9. *Let K be a Valdivia compact space with $w(K) \le \omega_1$ and let S be a dense Σ-subset of K. Then there exist a continuous inverse sequence $\mathcal{K} = ((K_\alpha)_{\alpha\in\omega_1}, (r_{\alpha\beta})_{\alpha\le\beta\in\omega_1})$ of compact metric spaces with a right inverse $(\sigma_{\alpha\beta})_{\alpha\le\beta\in\omega_1}$ and a family $(r_\alpha : K \to K_\alpha)_{\alpha\in\omega_1}$ of continuous maps such that $(K, (r_\alpha)_{\alpha\in\omega_1})$ is an inverse limit of $\mathcal{K}$ and*

$$S = \bigcup_{\alpha\in\omega_1} \sigma_\alpha[K_\alpha]. \tag{3}$$

Conversely, if $(K, (r_\alpha)_{\alpha\in\omega_1})$ is an inverse limit of a continuous inverse sequence $\mathcal{K} = ((K_\alpha)_{\alpha\in\omega_1}, (r_{\alpha\beta})_{\alpha\le\beta\in\omega_1})$ of compact metric spaces with a right *inverse* $(\sigma_{\alpha\beta})_{\alpha\le\beta\in\omega_1}$, *then* K *is a Valdivia compact space with* $w(K) \le \omega_1$ *and* (3) *is a dense* Σ*-subset of* K.

Proof. The first part of the statement follows from [2, Lemma 1.3], and the fact that (3) is a dense Σ-subset of K follows from the proof of [11, Theorem 4.2]. ∎

In view of Theorem 2.9, when a small Valdivia compact space K is represented as an inverse limit of a continuous ω_1-sequence of compact metric spaces with a right inverse, the assumption of Corollary 2.8 states that a certain Σ-subset of K has empty interior. It turns out that this condition is satisfied for a Valdivia compact space without G_δ points, as we now show. Recall that a compact Hausdorff space is said to be *Corson* if it is a Σ-subset of itself.

Lemma 2.10. *If* K *is a Valdivia compact space without* G_δ *points, then every* Σ*-subset of* K *has empty interior.*

Proof. If S is a Σ-subset of K with nonempty interior, then S contains a nonempty closed G_δ subset F of K. It follows that F is a nonempty Corson compact space and therefore has a G_δ point [9, Theorem 3.3]. ∎

3. Inverse limits and trees. Recall that a *tree* is a partially ordered set $(T, \le)$ such that, for all $t \in T$, the set $]\cdot, t[\, = \{s \in T : s < t\}$ is well-ordered. A subset X of T is called an *initial part* of T if $]\cdot, t[\, \subset X$ for all $t \in X$; a *chain* if it is totally ordered; an *antichain* if any two distinct elements of X are incomparable; a *path* if it is both a chain and an initial part of T. We say that $X \subset T$ satisfies the *countable chain condition* (ccc) if every antichain contained in X is countable. We denote by $P(T)$ the set of all paths of T and by $P^*(T)$ the set of nonempty paths of T. We have a canonical embedding $\mathfrak{p}$ of $(T, \le)$ into $(P^*(T), \subset)$ defined by

$$\mathfrak{p}(t) = \{s \in T : s \le t\}$$

for all $t \in T$. Any subset of T endowed with the restriction of $\le$ is itself a tree and is called a *subtree* of T; an initial part of T endowed with the restriction of $\le$ is called an *initial subtree* of T. If Z is an initial subtree of T, then the paths of Z are precisely the paths of T that are contained in Z. Given a path $A \in P(T)$, we set

$$N_A = \{t \in T : \,]\cdot, t[\, = A\}.$$

We now introduce the structure that makes the connection between trees and inverse systems.

Definition 3.1. A *graded tree* (T, δ) consists of a tree T and a mapping $\delta : T \to \mathbb{S} \cup \{0\}$ such that $\delta(t) = 0$ for every minimal element t of T, and $\delta(t) < \delta(s)$ for all $t, s \in T$ with $t < s$. The map δ is called a *grading function* for T. For every ordinal α, we denote by T_α the initial subtree of T defined as

$$T_\alpha = \{t \in T : \delta(t) \le \alpha\},$$

and by $\rho_\alpha : P^*(T) \to P^*(T_\alpha)$ the map given by $\rho_\alpha(A) = A \cap T_\alpha$ for all $A \in P^*(T)$. By a *compatible topology* on $P^*(T)$ we mean a compact Hausdorff topology on $P^*(T)$ such that the maps $\rho_\alpha : P^*(T) \to P^*(T)$ are continuous for every ordinal α. Given a nonzero ordinal θ, we say that the graded tree (T, δ) is *θ-graded* if $\delta(t) < \theta$ for all $t \in T$.

Note that if (T, δ) is a graded tree, then

$$P^*(T_0) = \mathfrak{p}[\delta^{-1}(0)] \tag{4}$$

and, for every ordinal λ,

$$P^*(T_{\lambda+1}) = P^*(T_\lambda) \cup \mathfrak{p}[\delta^{-1}(\lambda+1)], \tag{5}$$

with the union in (5) being disjoint.

A θ-graded tree (T, δ) is associated in a natural way with a continuous inverse θ-sequence of sets $\mathcal{K}_\theta(T, \delta)$ with a right inverse (Proposition 3.2); moreover, a compatible topology on $P^*(T)$ makes $\mathcal{K}_\theta(T, \delta)$ an inverse sequence of compact Hausdorff spaces. It turns out (Proposition 3.3) that every continuous inverse θ-sequence with a right inverse is of the form $\mathcal{K}_\theta(T, \delta)$, up to isomorphism.

Proposition 3.2. *Let (T, δ) be a θ-graded tree and let*

$$\mathcal{K}_\theta(T, \delta) = \big((P^*(T_\alpha))_{\alpha \in \theta}, (\rho_{\alpha\beta})_{\alpha \le \beta \in \theta}\big),$$

where $\rho_{\alpha\beta} = \rho_\alpha|_{P^*(T_\beta)} : P^*(T_\beta) \to P^*(T_\alpha)$. Then:

- (a) *$\mathcal{K}_\theta(T, \delta)$ is a continuous inverse sequence of sets;*
- (b) *$(P^*(T), (\rho_\alpha)_{\alpha \in \theta})$ is an inverse limit of $\mathcal{K}_\theta(T, \delta)$;*
- (c) *the inclusion maps $P^*(T_\alpha) \to P^*(T_\beta)$, $\alpha \le \beta \in \theta$, constitute a right inverse of $\mathcal{K}_\theta(T, \delta)$ and the induced maps $P^*(T_\alpha) \to P^*(T)$ are inclusion maps as well.*

Moreover, if $P^*(T)$ is endowed with a compatible topology and each $P^*(T_\alpha)$ has the subspace topology, then (a)–(c) *hold when "set" is replaced with "compact Hausdorff space".*

Proof. The proof of (a)–(c) is straightforward. For the last statement, note that $P^*(T_\alpha)$ is closed in $P^*(T)$, being the set of fixed points of the continuous map ρ_α. ∎

Proposition 3.3. *If $\mathcal{K} = ((K_\alpha)_{\alpha \in \theta}, (r_{\alpha\beta})_{\alpha \le \beta \in \theta})$ is a continuous inverse sequence of sets and $(\sigma_{\alpha\beta})_{\alpha \le \beta \in \theta}$ is a right inverse of $\mathcal{K}$, then there exist a θ-graded tree (T, δ) and a family of bijections $\varphi_\alpha : K_\alpha \to P^*(T_\alpha)$ such that,* *for all $\alpha, \beta \in \theta$ with $\alpha \le \beta$, the diagrams*

$$\text{(6)}\qquad \begin{tikzcd} K_\beta \arrow[r, "\varphi_\beta"] \arrow[d, "r_{\alpha\beta}"'] & P^*(T_\beta) \arrow[d, "\rho_{\alpha\beta}"] \\ K_\alpha \arrow[r, "\varphi_\alpha"] & P^*(T_\alpha) \end{tikzcd} \qquad \begin{tikzcd} K_\beta \arrow[r, "\varphi_\beta"] & P^*(T_\beta) \\ K_\alpha \arrow[r, "\varphi_\alpha"] \arrow[u, "\sigma_{\alpha\beta}"] & P^*(T_\alpha) \arrow[u, "inclusion"'] \end{tikzcd}$$

commute. In particular, if $(K, (r_\alpha)_{\alpha\in\theta})$ is an inverse limit of $\mathcal{K}$, then there exists a bijection $\varphi : K \to P^*(T)$ such that the diagrams

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "r_{\alpha}"'] & P^*(T) \arrow[d, "\rho_{\alpha}"] \\ K_\alpha \arrow[r, "\varphi_\alpha"] & P^*(T_\alpha) \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "\varphi"] & P^*(T) \\ K_\alpha \arrow[r, "\varphi_\alpha"] \arrow[u, "\sigma_{\alpha}"] & P^*(T_\alpha) \arrow[u, "inclusion"'] \end{tikzcd}$$

commute, for all $\alpha \in \theta$. Moreover, if $\mathcal{K}$ is a continuous inverse sequence of compact Hausdorff spaces, then the topology on $P^*(T)$ that makes φ a homeomorphism is a compatible topology, and each map φ_α is a homeomorphism if $P^*(T_\alpha)$ is endowed with the subspace topology.

Proof. Consider the set $\overline{T} = \bigcup_{\alpha\in\theta}(K_\alpha \times \{\alpha\})$ endowed with the partial order defined by

$$(x, \alpha) \le (y, \beta) \ \Leftrightarrow\ \alpha \le \beta \text{ and } r_{\alpha\beta}(y) = x.$$

For every $t \in \overline{T}$, the projection $\pi_2 : \overline{T} \to \theta$ restricts to an order isomorphism between $]\cdot, t[$ and a subset of θ, so that $\overline{T}$ is a tree. Let T be the subtree of $\overline{T}$ defined by

$$T = (K_0 \times \{0\}) \cup \bigcup\{(K_{\alpha+1} \setminus \sigma_{\alpha,\alpha+1}[K_\alpha]) \times \{\alpha+1\} : \alpha \in \theta \text{ with } \alpha+1 \in \theta\}$$

and let $\delta : T \to \mathbb{S} \cup \{0\}$ be the θ-grading function given by the restriction of π_2. For each $\alpha \in \theta$, define $\varphi_\alpha : K_\alpha \to P^*(T_\alpha)$ by setting

$$\varphi_\alpha(x) = \{(z, \gamma) \in T : (z, \gamma) \le (x, \alpha)\}$$

for all $x \in K_\alpha$. It is clear that the left diagram in (6) commutes. To see that the right diagram commutes, pick $x \in K_\alpha$ and observe that no $t \in \overline{T}$ with $(x, \alpha) < t \le (\sigma_{\alpha\beta}(x), \beta)$ is in T: if it were, it would be of the form $(z, \gamma + 1)$ with $\alpha \le \gamma < \beta$ and

$$z = r_{\gamma+1,\beta}(\sigma_{\alpha\beta}(x)) = \sigma_{\alpha,\gamma+1}(x) = \sigma_{\gamma,\gamma+1}(\sigma_{\alpha\gamma}(x)),$$

which contradicts $z \notin \sigma_{\gamma,\gamma+1}[K_\gamma]$. The fact that the maps φ_α are bijective is proven by induction on α using the commutativity of the diagrams in (6), keeping in mind (5) and the continuity of the inverse sequences $\mathcal{K}$ and $\mathcal{K}_\theta(T, \delta)$. The proof of the remaining parts of the proposition is straightforward. ∎

Using the relation between graded trees and continuous inverse sequences with a right inverse presented above, we now obtain a characterization of small Valdivia compacta in terms of graded trees.

THEOREM 3.4. *Let K be a Valdivia compact space with $w(K) \le \omega_1$ and let S be a dense Σ-subset of K. Then there exist an ω_1-graded tree (T,δ), a compatible topology on $P^*(T)$ such that $P^*(T_\alpha)$ is metrizable for all $\alpha \in \omega_1$, and a homeomorphism between K and $P^*(T)$ that carries S to the set*

$$\bigcup_{\alpha\in\omega_1} P^*(T_\alpha) = \{A \in P^*(T) : |A| \le \omega\}. \tag{7}$$

Conversely, given an ω_1-graded tree (T,δ) and a compatible topology on $P^*(T)$ such that $P^*(T_\alpha)$ is metrizable for all $\alpha \in \omega_1$, then $K = P^*(T)$ is a Valdivia compact space with $w(K) \le \omega_1$ and (7) is a dense Σ-subset of K.

Proof. Follows from Theorem 2.9 using Propositions 3.2 and 3.3. ∎

Let (T,δ) be a graded tree and let $P^*(T)$ be endowed with a compatible topology. It follows from (5) that, for every $\alpha \in \mathbb{S}$, the set $\mathfrak{p}[\delta^{-1}(\alpha)]$ is open in $P^*(T_\alpha)$. Thus, the topology on $\delta^{-1}(\alpha)$ induced by $\mathfrak{p}$ is locally compact Hausdorff and, by (4), the topology on $\delta^{-1}(0)$ induced by $\mathfrak{p}$ is compact Hausdorff. Our goal is to construct a compatible topology on $P^*(T)$ from given locally compact Hausdorff topologies on the sets $\delta^{-1}(\alpha)$ satisfying certain compatibility conditions. To this end, we make the following definition.

DEFINITION 3.5. Let (T,δ) be a graded tree. For each $\alpha \in \mathbb{S}\cup\{0\}$, we denote by $g_\alpha : P^*(T) \to \delta^{-1}(\alpha)\cup\{\infty\}$ the map defined by $g_\alpha(A) = t$ if t is the (automatically unique) element of $A\cap\delta^{-1}(\alpha)$, and $g_\alpha(A) = \infty$ if $A\cap\delta^{-1}(\alpha)$ is empty; here ∞ denotes any point not in $\delta^{-1}(\alpha)$. Given $\alpha,\beta \in \mathbb{S}\cup\{0\}$ with $\alpha \le \beta$, we set $g_{\alpha\beta} = g_\alpha \circ \mathfrak{p}|_{\delta^{-1}(\beta)} : \delta^{-1}(\beta) \to \delta^{-1}(\alpha)\cup\{\infty\}$. We call $g_{\alpha\beta}$ the *connecting maps* of the graded tree (T,δ).

When a locally compact Hausdorff topology is given on a set $\mathcal{X}$, we always endow the disjoint union $\mathcal{X}\cup\{\infty\}$ with the unique compact Hausdorff topology that induces the given topology on $\mathcal{X}$, i.e., $\mathcal{X}\cup\{\infty\}$ is the one-point compactification of $\mathcal{X}$ if $\mathcal{X}$ is not compact, and the point ∞ is isolated in $\mathcal{X}\cup\{\infty\}$ otherwise.

PROPOSITION 3.6. *Let (T,δ) be a graded tree. If $P^*(T)$ is endowed with a compatible topology and the sets $\delta^{-1}(\alpha)$, $\alpha \in \mathbb{S}\cup\{0\}$, are endowed with the topologies induced by $\mathfrak{p}$, then the maps g_α and the connecting maps $g_{\alpha\beta}$ are continuous for all $\alpha,\beta \in \mathbb{S}\cup\{0\}$ with $\alpha \le \beta$; moreover, the topology of $P^*(T)$ coincides with the topology induced by the maps g_α, $\alpha \in \mathbb{S}\cup\{0\}$. Conversely, let the set $\delta^{-1}(\alpha)$ be endowed with a locally compact Hausdorff topology τ_α for each $\alpha \in \mathbb{S}$, and let $\delta^{-1}(0)$ be endowed with a compact Hausdorff topology τ_0. If the connecting maps of (T,δ) are continuous, then there exists a unique* *compatible topology on $P^*(T)$ such that τ_α is equal to the topology on $\delta^{-1}(\alpha)$ induced by $\mathfrak{p}$ for all $\alpha \in \mathbb{S} \cup \{0\}$.*

Proof. Let $P^*(T)$ be endowed with a compatible topology and let each set $\delta^{-1}(\alpha)$ be endowed with the topology induced by $\mathfrak{p}$. To see that g_α is continuous, note that $g_\alpha = q \circ \rho_\alpha$, where $q : P^*(T_\alpha) \to \delta^{-1}(\alpha) \mathbin{\dot\cup} \{\infty\}$ is defined by $q(A) = \mathfrak{p}^{-1}(A)$ for $A \in \mathfrak{p}[\delta^{-1}(\alpha)]$, and $q(A) = \infty$ otherwise. The continuity of the connecting maps then follows. The fact that the topology of $P^*(T)$ is induced by the maps g_α is a consequence of the observation that the map

$$(g_\alpha)_{\alpha\in\delta[T]} : P^*(T) \to \prod_{\alpha\in\delta[T]} (\delta^{-1}(\alpha) \mathbin{\dot\cup} \{\infty\}) \tag{8}$$

is continuous and injective. To prove the converse, let $P^*(T)$ be endowed with the topology induced by (8) and let us show that this topology satisfies the required properties. It is easy to see that the image of (8) is

$$F = \Big\{(t_\alpha)_{\alpha\in\delta[T]} \in \prod_{\alpha\in\delta[T]} (\delta^{-1}(\alpha) \mathbin{\dot\cup} \{\infty\}) : t_0 \neq \infty \text{ and, for all } \alpha, \beta \in \delta[T] \\ \text{with } \alpha \le \beta, \text{ if } t_\beta \neq \infty, \text{ then } g_{\alpha\beta}(t_\beta) = t_\alpha\Big\}$$

and that the continuity of the connecting maps implies that F is closed; hence $P^*(T)$ is compact. The continuity of ρ_α follows from the fact that $g_\beta \circ \rho_\alpha = g_\beta$, for $\beta \le \alpha$, and $g_\beta \circ \rho_\alpha \equiv \infty$, for $\beta > \alpha$. Finally, the topology induced on $\delta^{-1}(\alpha)$ by $\mathfrak{p}$ is equal to the topology induced by the maps $g_{\beta\alpha}$, with $\beta \in \delta[T]$ and $\beta \le \alpha$. That this topology is equal to τ_α follows from the continuity of the connecting maps and from the fact that $g_{\alpha\alpha}$ is the inclusion of $\delta^{-1}(\alpha)$ in $\delta^{-1}(\alpha) \mathbin{\dot\cup} \{\infty\}$. ■

Corollary 3.7. *Let (T, δ) be a θ-graded tree, $P^*(T)$ be endowed with a compatible topology and the sets $\delta^{-1}(\alpha)$ be endowed with the topology induced by $\mathfrak{p}$. Then*

$$w(P^*(T)) \le \max\Big\{|\theta|, \sup_{\alpha\in\theta} w(\delta^{-1}(\alpha))\Big\}.$$

In particular, if $\delta^{-1}(\alpha)$ is second countable for all $\alpha \in \omega_1$, then $P^*(T_\alpha)$ is second countable for all $\alpha \in \omega_1$. □

4. The counterexample to the Conjecture. Combining Theorem 3.4 with Proposition 3.6 and Corollary 3.7, we obtain the following strategy for constructing a small Valdivia compact space: Take an ω_1-graded tree (T, δ) and locally compact Hausdorff second countable topologies on the sets $\delta^{-1}(\alpha)$ such that $\delta^{-1}(0)$ is compact and the connecting maps of (T, δ) are continuous. Then combine the topologies into a compatible topology on $P^*(T)$, which is a small Valdivia compact space. Moreover, every small Valdivia compact space is of this form.

The purpose of this section is to use this strategy to prove the following result.

THEOREM 4.1. *Assume $\diamondsuit$. There exists a Valdivia compact space K such that:*

(a) $w(K) = \omega_1$;
(b) *K has ccc;*
(c) *K has no G_δ points;*
(d) *K does not have a nontrivial convergent sequence in the complement of a dense Σ-subset.*

REMARK 4.2. We observe that property (d) in the statement of Theorem 4.1 is independent of the choice of the dense Σ-subset: more precisely, if K is a Valdivia compact space and if $K \setminus S$ contains a nontrivial convergent sequence for some dense Σ-subset S of K, then $K \setminus S$ contains a nontrivial convergent sequence for *any* dense Σ-subset S of K (see [6, Remark 4.5]).

We start by investigating conditions on the ω_1-graded tree (T, δ) and on the topologies of the sets $\delta^{-1}(\alpha)$ that imply conditions (a)–(d) of Theorem 4.1.

LEMMA 4.3. *Let (T, δ) be an ω_1-graded tree and let $P^*(T)$ be endowed with a compatible topology such that $P^*(T_\alpha)$ is metrizable for all $\alpha \in \omega_1$. Then $P^*(T)$ has no G_δ points if and only if $\delta[N_A]$ is uncountable for every countable path $A \in P^*(T)$.*

Proof. Follows from Corollary 2.3 and Proposition 3.2, if we keep in mind that, for $\alpha \in \omega_1$ and $A \in P^*(T_\alpha)$, we have $|\rho_\alpha^{-1}(A)| = 1$ if and only if $\delta[N_A]$ is contained in $[0, \alpha]$. ∎

In what follows, whenever (T, δ) is a graded tree and $P^*(T)$ is endowed with a compatible topology, we will consider the sets $\delta^{-1}(\alpha)$ endowed with the topology induced by $\mathfrak{p}$.

LEMMA 4.4. *Let (T, δ) be an ω_1-graded tree and let $P^*(T)$ be endowed with a compatible topology such that $P^*(T_\alpha)$ is metrizable for all $\alpha \in \omega_1$. Assume that* (7) *has empty interior in $P^*(T)$. Then $P^*(T)$ has ccc if and only if for every antichain $X \subset T$ the set*

$$\{\alpha \in \mathbb{S} : X \cap \delta^{-1}(\alpha) \text{ has nonempty interior in } \delta^{-1}(\alpha)\} \tag{9}$$

is countable.

Proof. By Corollary 2.8 and Proposition 3.2, $P^*(T)$ does not have ccc if and only if there exist an uncountable subset Λ of $\mathbb{S} \cap \omega_1$ and a family $(U_\alpha)_{\alpha \in \Lambda}$ with each U_α a nonempty open subset of $\delta^{-1}(\alpha)$ such that the sets $\rho_\alpha^{-1}[\mathfrak{p}[U_\alpha]]$,

$\alpha \in \Lambda$, are pairwise disjoint. The conclusion follows by observing that these sets are pairwise disjoint if and only if $X = \bigcup_{\alpha\in\Lambda} U_\alpha$ is an antichain of T. ∎

Corollary 4.5. *Let (T, δ) be an ω_1-graded tree and let $P^*(T)$ be endowed with a compatible topology such that $P^*(T_\alpha)$ is metrizable for all $\alpha \in \omega_1$. Assume that* (7) *has empty interior in $P^*(T)$. Set*

$$Z = \{t \in T : t \text{ is an isolated point of } \delta^{-1}(\delta(t))\}. \tag{10}$$

If $Z \cap \delta^{-1}(\alpha)$ is dense in $\delta^{-1}(\alpha)$ for all $\alpha \in \mathbb{S} \cup \{0\}$, then $P^*(T)$ has ccc if and only if Z has ccc.

Proof. Note that the set (9) is equal to $\delta[X\cap Z]\setminus\{0\}$ and that $Z\cap\delta^{-1}(\alpha)$ is countable for all α. ∎

Lemma 4.6. *Let (T, δ) be an ω_1-graded tree and let $P^*(T)$ be endowed with a compatible topology such that $P^*(T_\alpha)$ is metrizable for all $\alpha \in \omega_1$. Assume that $\delta^{-1}(\alpha)$ has at most one nonisolated point for all $\alpha \in \mathbb{S} \cup \{0\}$, and that the set Z defined in* (10) *is an initial part of T. If $P^*(T)$ has a nontrivial convergent sequence outside a dense Σ-subset, then Z contains an uncountable path.*

Proof. By Remark 4.2, we can assume that the complement of (7) has a nontrivial convergent sequence. Let then $A \in P^*(T)$ be an uncountable path which is the limit of a sequence $(A_n)_{n\in\omega}$ in $P^*(T)\setminus\{A\}$ and let $\alpha_0 \in \omega_1$ be such that $\rho_{\alpha_0}(A_n) \neq \rho_{\alpha_0}(A)$ for all $n \in \omega$. We claim that, for $\alpha \in \delta[A]$ with $\alpha \geq \alpha_0$, there exists $n(\alpha) \in \omega$ such that $A_n \cap Z \cap \delta^{-1}(\alpha) \neq \emptyset$ for all $n \geq n(\alpha)$. Namely, for such α we have $g_\alpha(A) \neq \infty$, so that, by the continuity of g_α, there exists $n(\alpha) \in \omega$ with $g_\alpha(A_n) \neq \infty$ for all $n \geq n(\alpha)$. Since $g_\alpha(A_n) \neq g_\alpha(A)$ for all n, we see that $g_\alpha(A)$ is not isolated in $\delta^{-1}(\alpha)$, and thus $g_\alpha(A_n) \in Z \cap \delta^{-1}(\alpha)$ for all $n \geq n(\alpha)$. This proves the claim. To conclude the proof, pick $n \in \omega$ such that $n = n(\alpha)$ for uncountably many $\alpha \in \delta[A] \setminus \alpha_0$ and note that A_n is an uncountable path contained in Z. ∎

A tree T is called *ever-branching* if, for every $t \in T$, the set

$$\{s \in T : s > t\}$$

is not a chain. We recall (see [12, Lemma 7.4]) that if T is an ever-branching tree with ccc such that $]\cdot, t[$ is countable for all $t \in T$, then every path of T is countable.

Lemma 4.7. *Let (T, δ) be a nonempty ω_1-graded tree and let $P^*(T)$ be endowed with a compatible topology such that $P^*(T_\alpha)$ is metrizable for all $\alpha \in \omega_1$. Assume that:*

(i) *N_A is uncountable for every countable path $A \in P^*(T)$;*
(ii) *$\delta^{-1}(\alpha)$ has at most one nonisolated point for all $\alpha \in \mathbb{S} \cup \{0\}$;*
(iii) *the set Z defined in* (10) *is an initial part of T with ccc.*

Then $K = P^*(T)$ *is a Valdivia compact space satisfying conditions* (a)–(d) *in Theorem* 4.1.

Proof. By Theorem 3.4, K is a Valdivia compact space, (7) is a dense Σ-subset of K and $w(K) \le \omega_1$. Since $\delta^{-1}(\alpha)$ is countable for all $\alpha \in \mathbb{S} \cup \{0\}$, we find that $\delta[N_A]$ is uncountable for every countable path $A \in P^*(T)$, so Lemma 4.3 implies that K has no G_δ points. It then follows from Lemma 2.10 that (7) has empty interior, and from Corollary 4.5 that K has ccc. By Lemma 4.6, to conclude the proof, it suffices to show that Z does not contain an uncountable path. To this end, we show that Z must be everbranching. Let $z \in Z$ and note that, since $N_{\mathfrak{p}(z)}$ is an uncountable antichain, there exists $t \in T \setminus Z$ with $t \in N_{\mathfrak{p}(z)}$. Setting $\alpha = \delta(z)$ and $\beta = \delta(t)$, we have $g_{\alpha\beta}(t) = z$. Since z is isolated in $\delta^{-1}(\alpha)$ and t is not isolated in $\delta^{-1}(\beta)$, it follows from the continuity of $g_{\alpha\beta}$ that $g_{\alpha\beta}^{-1}(z)$ is infinite; moreover, by (ii), $g_{\alpha\beta}^{-1}(z) \setminus \{t\}$ is contained in Z. Hence, $g_{\alpha\beta}^{-1}(z) \setminus \{t\}$ is an infinite antichain in Z consisting of elements greater than z. ∎

Our goal now is to construct an ω_1-graded tree (T, δ) and a compatible topology on $P^*(T)$ such that the assumptions of Lemma 4.7 hold. Observe that the initial subtree Z of T will be a *Suslin tree*, i.e., $|Z| = \omega_1$, every path of Z is countable and Z has ccc. Our construction is similar to the standard construction of a Suslin tree using $\diamondsuit$ (see [12, Theorem 7.8]), but technically much more involved, since we have to ensure the continuity of the connecting maps. The first step is to develop the technique that will be later used to prove that Z has ccc. The construction itself is the content of Subsection 4.1.

Definition 4.8. Let (T, δ) be a graded tree. An antichain X of T is said to be *special* in T if, given a finite subset $\mathcal{F}$ of $\mathbb{S} \cup \{0\}$ and an element $t \in T$ with $\mathfrak{p}(t) \cap X = \emptyset$, there exists $s \in X$ with $s > t$ such that $\delta\big[]t, s]\big] \cap \mathcal{F} = \emptyset$, where $]t, s] = \{u \in T : t < u \le s\}$.

Obviously a special antichain is also a maximal antichain.

Lemma 4.9. *Let* (T, δ) *be an* ω_1*-graded tree and let* X *be a special antichain in* T. *If* T_α *is countable for all* $\alpha \in \omega_1$, *then the set*

$$\{\alpha \in \omega_1 : X \cap T_\alpha \text{ is a special antichain in } T_\alpha\} \tag{11}$$

is closed and unbounded (*club*).

Proof. The above set is clearly closed. To see that it is unbounded note that, given $\alpha \in \omega_1$, the fact that T_α is countable implies that there exists $f(\alpha)$ in ω_1 with $f(\alpha) > \alpha$ having the following property: for every $t \in T_\alpha$ with $\mathfrak{p}(t) \cap X = \emptyset$ and every finite subset $\mathcal{F}$ of $[0, \alpha]$, there exists $s \in X \cap T_{f(\alpha)}$ with $s > t$ and $\delta\big[]t, s]\big] \cap \mathcal{F} = \emptyset$. Hence, denoting by f^n the nth iterate of f, we conclude that $\sup_{n \in \omega} f^n(\alpha)$ is in (11) for all $\alpha \in \omega_1$. ∎

Given a set Γ with $|\Gamma| = \omega_1$, by a *continuous filtration of* Γ *by countable sets* we mean an increasing family $(\Gamma_\alpha)_{\alpha\in\omega_1}$ of countable subsets of Γ such that $\Gamma = \bigcup_{\alpha\in\omega_1} \Gamma_\alpha$ and $\Gamma_\alpha = \bigcup_{\beta\in\alpha} \Gamma_\beta$ for every limit ordinal $\alpha \in \omega_1$. A $\diamondsuit$*-sequence* for this filtration is a family $(\Gamma_\alpha^\diamondsuit)_{\alpha\in\omega_1}$ such that each $\Gamma_\alpha^\diamondsuit$ is a subset of Γ_α and, for any $X \subset \Gamma$, the set $\{\alpha \in \omega_1 : X \cap \Gamma_\alpha = \Gamma_\alpha^\diamondsuit\}$ is stationary. The combinatorial principle $\diamondsuit$ states that there exists a $\diamondsuit$-sequence for the filtration $\omega_1 = \bigcup_{\alpha\in\omega_1} \alpha$. It is easy to prove that $\diamondsuit$ implies the existence of a $\diamondsuit$-sequence for any continuous filtration $\Gamma = \bigcup_{\alpha\in\omega_1} \Gamma_\alpha$ by countable sets.

COROLLARY 4.10. *Let (T, δ) be an ω_1-graded tree such that T_α is countable for all $\alpha \in \omega_1$. Let $(T_\alpha^\diamondsuit)_{\alpha\in\omega_1}$ be a $\diamondsuit$-sequence for the continuous filtration $T = \bigcup_{\alpha\in\omega_1} T_\alpha$. Assume that:*

(i) *for all $\alpha \in \omega_1$, if $T_\alpha^\diamondsuit$ is a special antichain in T_α, then $\mathfrak{p}(t)$ intersects $T_\alpha^\diamondsuit$ for every $t \in T \setminus T_\alpha$;*
(ii) *every maximal antichain in T is special in T.*

Then T has ccc.

Proof. Given a maximal antichain X in T, from (ii) and Lemma 4.9 we get $\alpha \in \omega_1$ such that $X \cap T_\alpha$ is a special antichain in T_α and $X \cap T_\alpha = T_\alpha^\diamondsuit$. It then follows from (i) that $X \subset T_\alpha$. ∎

4.1. Construction of the graded tree. Throughout this subsection, T and Z are defined by

$$T = [0, \omega] \times [(\mathbb{S} \cup \{0\}) \cap \omega_1], \qquad Z = \omega \times [(\mathbb{S} \cup \{0\}) \cap \omega_1],$$

and $\delta : T \to \mathbb{S} \cup \{0\}$ denotes the projection onto the second coordinate. As usual, we write $T_\alpha = \delta^{-1}\big[[0, \alpha]\big]$ and $Z_\alpha = Z \cap T_\alpha$. Moreover, for α in $(\mathbb{S} \cup \{0\}) \cap \omega_1$, we endow $\delta^{-1}(\alpha) = [0, \omega] \times \{\alpha\}$ with the topology that makes the projection onto the first coordinate a homeomorphism, where $[0, \omega]$ has the order topology. Clearly, equality (10) holds. The hard part will be the construction of the partial order of T.

DEFINITION 4.11. Let $(Z_\alpha^\diamondsuit)_{\alpha\in\omega_1}$ be a $\diamondsuit$-sequence for the continuous filtration $Z = \bigcup_{\alpha\in\omega_1} Z_\alpha$. Given $\alpha \in [0, \omega_1]$, we say that a partial order $\le$ in T_α is *admissible* (with respect to the given $\diamondsuit$-sequence) if it satisfies the following properties:

(1) $(T_\alpha, \le)$ is a tree, $\delta|_{T_\alpha}$ is a grading function and the connecting maps of $(T_\alpha, \delta|_{T_\alpha})$ are continuous;
(2) Z_α is an initial part of $(T_\alpha, \le)$;
(3) for all $\beta < \alpha$, if $Z_\beta^\diamondsuit$ is a special antichain in $(Z_\beta, \delta|_{Z_\beta})$, then $\mathfrak{p}(z)$ intersects $Z_\beta^\diamondsuit$ for every $z \in Z_\alpha \setminus Z_\beta$.

Note that (1) and (2) imply that T_β and Z_β are initial parts of $(T_\alpha, \leq)$ for all $\beta \leq \alpha$; in particular, $(T_\beta, \delta|_{T_\beta})$ and $(Z_\beta, \delta|_{Z_\beta})$ are graded trees (endowed with the restriction of $\leq$).

In Corollary 4.13 below, we show that if T is endowed with an admissible partial order satisfying a simple extra property, then the assumptions of Lemma 4.7 are satisfied.

LEMMA 4.12. *Let $\leq$ be an admissible partial order in T and assume that, for every $z \in Z$, the set $N_{\mathfrak{p}(z)} \setminus Z$ is uncountable. Then every maximal antichain in Z is special in Z.*

Proof. Let X be a maximal antichain in Z and pick $z \in Z$ with $\mathfrak{p}(z) \cap X = \emptyset$; write $\alpha = \delta(z)$. If $\mathcal{F}$ is a finite subset of $(\mathbb{S} \cup \{0\}) \cap \omega_1$, then there exists $t \in N_{\mathfrak{p}(z)} \setminus Z$ with $\delta(t) = \beta > \sup \mathcal{F}$. Since $g_{\alpha\beta}(t) = z$ and $g_{\gamma\beta}(t) = \infty$ for all $\gamma \in]\alpha, \beta[\cap \mathbb{S}$, it follows from the continuity of the connecting maps that there exists $w \in Z \cap \delta^{-1}(\beta)$ such that $g_{\alpha\beta}(w) = z$ and $g_{\gamma\beta}(w) = \infty$, for all $\gamma \in \mathcal{F}$ with $\gamma > \alpha$. Note that $w > z$ and $\delta\big[]z, w]\big] \cap \mathcal{F} = \emptyset$. To conclude the proof, use the fact that X is a maximal antichain in Z to obtain $v \in X$ comparable with w and note that $v > z$ and $\delta\big[]z, v]\big] \cap \mathcal{F} = \emptyset$. ∎

COROLLARY 4.13. *Let $\leq$ be an admissible partial order in T and assume that, for every countable path $A \in P^*(T)$, the set $N_A \setminus Z$ is uncountable. If $P^*(T)$ is endowed with the compatible topology given by Proposition* 3.6, *then $K = P^*(T)$ is a Valdivia compact space satisfying conditions* (a)–(d) *in Theorem* 4.1.

Proof. It follows from Corollary 4.10 that Z has ccc, and from Corollary 3.7 that $P^*(T_\alpha)$ is metrizable for all $\alpha \in \omega_1$. The assumptions of Lemma 4.7 are thus satisfied. ∎

We are going to construct, by recursion, an admissible partial order in T satisfying the assumption of Corollary 4.13. Note that, if $\alpha \in [0, \omega_1]$ is a limit ordinal and, for each $\beta < \alpha$, an admissible partial order $\leq_\beta$ is given in T_β such that $(T_\beta, \leq_\beta)$ is a subtree of $(T_\gamma, \leq_\gamma)$ for all $\beta \leq \gamma < \alpha$, then $\leq_\alpha = \bigcup_{\beta<\alpha} \leq_\beta$ is an admissible partial order in T_α. For the recursion step, we will use the lemma below.

LEMMA 4.14. *Given $\alpha \in \omega_1$, an admissible partial order $\leq$ in T_α and a path $A \in P^*(T_\alpha)$, there exists an admissible partial order $\leq'$ in $T_{\alpha+1}$ such that $(T_\alpha, \leq)$ is a subtree of $(T_{\alpha+1}, \leq')$ and $A = \{t \in T_\alpha : t <' (\omega, \alpha + 1)\}$.*

We postpone the proof of Lemma 4.14 for a moment to conclude the proof of Theorem 4.1.

COROLLARY 4.15. *There exists an admissible partial order $\leq$ in T such that $N_A \setminus Z$ is uncountable for every countable path $A \in P^*(T)$.*

Proof. Denote by $\wp_\omega(T)$ the collection of all countable subsets of T and let $\psi : \omega_1 \to \wp_\omega(T)$ be a map with $\psi^{-1}(A)$ uncountable for all $A \in \wp_\omega(T)$. Using Lemma 4.14, construct by recursion a family $(\leq_\alpha)_{\alpha\in\omega_1}$ with $\leq_\alpha$ an admissible partial order in T_α such that $(T_\alpha, \leq_\alpha)$ is a subtree of $(T_\beta, \leq_\beta)$ for $\alpha \leq \beta \in \omega_1$, and such that, for all $\alpha \in \omega_1$,

$$\{t \in T_\alpha : t <_{\alpha+1} (\omega, \alpha + 1)\} = A,$$

where $A = \psi(\alpha)$ if $\psi(\alpha) \in P^*(T_\alpha)$, and $A = \{(\omega, 0)\}$ otherwise. Finally, set $\leq = \bigcup_{\alpha\in\omega_1} \leq_\alpha$. ∎

Proof of Theorem 4.1. Follows directly from Corollaries 4.13 and 4.15. ∎

We now turn to the proof of Lemma 4.14. In order to extend an admissible partial order from T_α to $T_{\alpha+1}$, we need to specify, for each $n \in [0, \omega]$, a path $A_n \in P^*(T_\alpha)$ which will be the set of predecessors of $(n, \alpha + 1)$. We introduce the following definition.

Definition 4.16. Given $\alpha \in \omega_1$, let $\leq$ be a partial order in T_α such that $(T_\alpha, \leq)$ is a tree and let $(A_n)_{n\in[0,\omega]}$ be a sequence in $P(T_\alpha)$. We define a partial order $\leq'$ in $T_{\alpha+1}$ by requiring that, for all $t, s \in T_{\alpha+1}$, we have $t <' s$ if and only if one of the following conditions holds:

- $t, s \in T_\alpha$ and $t < s$;
- $t \in A_n$ and $s = (n, \alpha + 1)$ for some $n \in [0, \omega]$.

We call $\leq'$ the partial order in $T_{\alpha+1}$ *induced* by the sequence $(A_n)_{n\in[0,\omega]}$.

Lemma 4.17. *Let $\alpha \in \omega_1$ and fix a partial order $\leq$ in T_α such that $(T_\alpha, \leq)$ is a tree and $\delta|_{T_\alpha}$ is a grading function. Let $(A_n)_{n\in[0,\omega]}$ be a sequence in $P^*(T_\alpha)$ and $\leq'$ be the partial order induced in $T_{\alpha+1}$. Then:*

(1) *$(T_{\alpha+1}, \leq')$ is a tree, $\delta|_{T_{\alpha+1}}$ is a grading function and $(T_\alpha, \leq)$ is a subtree of $(T_{\alpha+1}, \leq')$.*
(2) *If Z_α is an initial part of $(T_\alpha, \leq)$ and A_n is in $P^*(Z_\alpha)$ for all $n \in \omega$, then $Z_{\alpha+1}$ is an initial part of $(T_{\alpha+1}, \leq')$.*
(3) *Assume that the connecting maps of $(T_\alpha, \delta|_{T_\alpha})$ are continuous and let $\beta \leq \alpha$ in $\mathbb{S} \cup \{0\}$ be such that the connecting map $g_{\beta,\alpha+1}$ of $(T_{\alpha+1}, \delta|_{T_{\alpha+1}})$ is continuous. If $g_{\beta,\alpha+1}(\omega, \alpha + 1) \neq \infty$, then the connecting map $g_{\gamma,\alpha+1}$ of $(T_{\alpha+1}, \delta|_{T_{\alpha+1}})$ is continuous for all $\gamma \leq \beta$ in $\mathbb{S} \cup \{0\}$.*

Proof. The proofs of (1) and (2) are straightforward, and to prove (3), note that the equality $g_{\gamma,\alpha+1} = g_{\gamma\beta} \circ g_{\beta,\alpha+1}$ holds in the open subset

$$\{t \in \delta^{-1}(\alpha + 1) : g_{\beta,\alpha+1}(t) \neq \infty\}$$

of $\delta^{-1}(\alpha + 1)$. ∎

Our task now is to find an appropriate sequence $(A_n)_{n\in[0,\omega]}$ to induce the partial order of $T_{\alpha+1}$.

LEMMA 4.18. *Let $\alpha \in \omega_1$ and fix a partial order $\le$ in T_α satisfying conditions* (1) *and* (2) *of Definition* 4.11. *Given $A_\omega \in P^*(T_\alpha)$, there exists a sequence $(z_n)_{n\in\omega}$ in Z_α satisfying the following properties:*

(a) $\sup_{n\in\omega} \delta(z_n) = \sup \delta[A_\omega]$;
(b) *given a sequence $(A_n)_{n\in\omega}$ in $P^*(T_\alpha)$ with $z_n \in A_n$ for all $n \in \omega$, if $T_{\alpha+1}$ is endowed with the partial order induced by $(A_n)_{n\in[0,\omega]}$, then the connecting maps $g_{\beta,\alpha+1}$ of the graded tree $T_{\alpha+1}$ are continuous for all $\beta \in \mathbb{S} \cup \{0\}$ with $\beta \le \sup \delta[A_\omega]$.*

Proof. Assume first that $A_\omega \subset Z_\alpha$. If A_ω has a largest element z, take $z_n = z$ for all $n \in \omega$; otherwise, let $(z_n)_{n\in\omega}$ be an increasing cofinal sequence in A_ω. Assume now that A_ω is not contained in Z_α. If A_ω has a largest element t, then it must be in $T_\alpha \setminus Z_\alpha$, since Z_α is an initial part of T_α. Then t is of the form (ω, β) and we take $z_n = (n, \beta)$ for all $n \in \omega$. Finally, if A_ω does not have a largest element, let $((\omega, \beta_n))_{n\in\omega}$ be an increasing cofinal sequence in A_ω. For each $n \in \omega$, since $g_{\beta_i\beta_n}$ is continuous and $g_{\beta_i\beta_n}(\omega, \beta_n) = (\omega, \beta_i)$ for $i \le n$, we can take $z_n \in \omega \times \{\beta_n\}$ with $g_{\beta_i\beta_n}(z_n) \in [n, \omega] \times \{\beta_i\}$ for all $i \le n$. In all the four cases considered above, it is easy to check that $(z_n)_{n\in\omega}$ satisfies properties (a) and (b), keeping in mind item (3) of Lemma 4.17. ■

LEMMA 4.19. *Let $\alpha \in \omega_1$ and fix an admissible partial order $\le$ in T_α. Given $z \in Z_\alpha$ and a finite subset $\mathcal{F}$ of $\mathbb{S} \cup \{0\}$, there exists $B \in P^*(Z_\alpha)$ with $z \in B$ satisfying the following conditions:*

(a) *for all $\beta \le \alpha$, if $Z_\beta^\diamondsuit$ is a special antichain in Z_β, then B intersects $Z_\beta^\diamondsuit$;*
(b) *for all $w \in B$ with $w > z$, we have $\delta(w) \notin \mathcal{F}$.*

Proof. Note first that if $\beta \le \alpha$ and $Z_\beta^\diamondsuit$ is a special antichain in Z_β, then $Z_\beta^\diamondsuit$ is also a special antichain in Z_α, by property (3) in Definition 4.11. Set

$$\Lambda = \{\beta \in [0, \alpha] : Z_\beta^\diamondsuit \text{ is a special antichain in } Z_\beta \text{ and } \mathfrak{p}(z) \cap Z_\beta^\diamondsuit = \emptyset\}.$$

If $\Lambda = \emptyset$, take $B = \mathfrak{p}(z)$. Otherwise, let $\{\beta_n : n \in \omega\}$ be an enumeration of Λ and define by recursion a chain $\{w_n : n \in \omega\}$ such that $w_n \in Z_{\beta_n}^\diamondsuit$, $w_n > z$, and $\delta\big[]z, w_n]\big] \cap \mathcal{F} = \emptyset$, for all $n \in \omega$. Given w_i, $i \le n$, we obtain w_{n+1} as follows. Setting $w = \max\{w_0, \ldots, w_n\}$, pick $w_{n+1} \in \mathfrak{p}(w) \cap Z_{\beta_{n+1}}^\diamondsuit$ if this intersection is not empty; otherwise, select $w_{n+1} \in Z_{\beta_{n+1}}^\diamondsuit$ with $w_{n+1} > w$ and $\delta\big[]\,w, w_{n+1}]\big] \cap \mathcal{F} = \emptyset$. To conclude the proof, set $B = \bigcup_{n\in\omega} \mathfrak{p}(w_n)$. ■

Proof of Lemma 4.14. Set $A_\omega = A$ and take $(z_n)_{n\in\omega}$ in Z_α as in Lemma 4.18. By recursion, we define $A_n \in P^*(Z_\alpha)$ and injective maps $\phi_n : A_n \to \omega$ as follows: given A_i and ϕ_i for $i < n$, obtain $B \in P^*(Z_\alpha)$ from Lemma 4.19

with $z = z_n$ and

$$\mathcal{F} = \bigcup_{i<n} \delta\big[\phi_i^{-1}[[0,n]]\big]; \tag{12}$$

set $A_n = B$. Let $\leq'$ be the partial order in $T_{\alpha+1}$ induced by the sequence $(A_n)_{n\in[0,\omega]}$. That $\leq'$ satisfies property (3) of Definition 4.11 follows from the fact that each A_n satisfies condition (a) in Lemma 4.19. To conclude the proof, fix $\beta \in \mathbb{S}\cup\{0\}$ with $\sup\delta[A_\omega] < \beta \leq \alpha$ and let us check that $g_{\beta,\alpha+1}$ is continuous at the point $(\omega, \alpha+1)$. To this end, it is sufficient to verify that $\beta \in \delta[A_n]$ for at most a finite number of $n \in \omega$. If $\beta = \delta(v)$ for some $v \in A_i$ with $i \in \omega$, we claim that $\beta \notin \delta[A_n]$ for all $n > \max\{i, \phi_i(v)\}$. Namely, for such n, we know that β is in (12) and then, by the construction of A_n, we get $\delta(w) \neq \beta$ for all $w \in A_n$ with $w > z_n$. Finally, for $w \leq z_n$, we have $\delta(w) \leq \delta(z_n) \leq \sup\delta[A_\omega] < \beta$, proving the claim. ∎

Acknowledgements. The first author is supported by FAPESP (process no. 2014/00848-2).

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