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NUMBER THEORY

Zeros of the Derivatives of L-Functions Attached to Cusp Forms

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Summary. Let f be a holomorphic cusp form of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$ which is a normalized Hecke eigenform, and $L_f(s)$ the L-function attached to f. We shall give a relation between the number of zeros of $L_f(s)$ and of the derivatives of $L_f(s)$ using Berndt's method, and an estimate of zero-density of the derivatives of $L_f(s)$ based on Littlewood's method.

1. Introduction. Let f be a cusp form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ which is a normalized Hecke eigenform. Let $a_f(n)$ be the nth Fourier coefficient of f and set $\lambda_f(n) = a_f(n)/n^{(k-1)/2}$. Rankin showed that $\sum_{n\le x} |\lambda_f(n)|^2 = C_f x + O(x^{3/5})$ for $x \in \mathbb{R}_{>0}$, where C_f is a positive constant depending on f (see [8, (4.2.3), p. 364]). The L-function attached to f is defined by

$$L_f(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_{p\,\mathrm{prime}} \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1} \quad (\mathrm{Re}\, s > 1), \tag{1.1}$$

where $\alpha_f(p)$ and $\beta_f(p)$ satisfy $\alpha_f(p)+\beta_f(p) = \lambda_f(p)$ and $\alpha_f(p)\beta_f(p) = 1$. By Hecke's work [4], the function $L_f(s)$ is analytically continued to the whole s-plane by

$$(2\pi)^{-s-(k-1)/2}\Gamma\left(s + \frac{k-1}{2}\right)L_f(s) = \int_0^{\infty} f(iy) y^{s+(k-1)/2-1}\, dy, \tag{1.2}$$

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and has a functional equation

$$L_f(s) = \chi_f(s) L_f(1-s)$$

where $\chi_f(s)$ is given by

$$\begin{aligned}(1.3)\qquad \chi_f(s) &= (-1)^{-k/2}(2\pi)^{2s-1}\frac{\Gamma\big(1-s+\frac{k-1}{2}\big)}{\Gamma\big(s+\frac{k-1}{2}\big)}\\ &= 2(2\pi)^{-2(1-s)}\Gamma\left(s+\frac{k-1}{2}\right)\Gamma\left(s-\frac{k-1}{2}\right)\cos(\pi(1-s)).\end{aligned}$$

The second equality is deduced from the fact that $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ and $\sin(\pi(s+(k-1)/2)) = (-1)^{k/2}\cos(\pi(1-s))$. Similarly to the case of the Riemann zeta function $\zeta(s)$, it is conjectured that all complex zeros of $L_f(s)$ lie on the critical line $\operatorname{Re} s = 1/2$, which is the Generalized Riemann Hypothesis (GRH). In order to support the truth of the GRH, the distribution and the density of complex zeros of $L_f(s)$ are studied without assuming the GRH.

Lekkerkerker [6] proved an approximate formula for the number of complex zeros of $L_f(s)$:

$$(1.4)\qquad N_f(T) = \frac{T}{\pi}\log\frac{T}{2\pi e} + O(\log T),$$

where $T > 0$ is sufficiently large, and $N_f(T)$ denotes the number of complex zeros of $L_f(s)$ in $0 < \operatorname{Im} s \le T$. The formula (1.4) is analogous to the formula for $N(T)$, the number of complex zeros of $\zeta(s)$ in $0 < \operatorname{Im} s \le T$, given by Riemann [9]:

$$(1.5)\qquad N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T).$$

(Later von Mangoldt [14] proved (1.5) rigorously.) In the Riemann zeta function, the zeros of the derivative of $\zeta(s)$ have a connection with the Riemann Hypothesis (RH). Speiser [10] showed that RH is equivalent to the non-existence of a complex zero of $\zeta'(s)$ in $\operatorname{Re} s < 1/2$. Levinson and Montgomery [7] proved that if RH is true, then $\zeta^{(m)}(s)$ has at most finitely many complex zeros in $0 < \operatorname{Re} s < 1/2$ for any $m \in \mathbb{Z}_{\ge 0}$.

There are many studies of the zeros of $\zeta^{(m)}(s)$ without assuming RH. Spira [11], [12] showed that there exist $\sigma_m \ge (7m+8)/4$ and $\alpha_m < 0$ such that $\zeta^{(m)}(s)$ has no zero with $\operatorname{Re} s \le \sigma_m$ or $\operatorname{Re} s \le \alpha_m$, and exactly one real zero in each open interval $(-1-2n, 1-2n)$ for $1-2n \le \alpha_m$. Later, Yıldırım [16] showed that $\zeta''(s)$ and $\zeta'''(s)$ have no zeros in $0 \le \operatorname{Re} s < 1/2$. Berndt [2] gave a relation between the numbers of complex zeros of $\zeta(s)$ and of $\zeta^{(m)}(s)$:

$$(1.6)\qquad N_m(T) = N(T) - \frac{T\log 2}{2\pi} + O(\log T),$$

where $m \in \mathbb{Z}_{\geq 1}$ is fixed and $N_m(T)$ denotes the number of complex zeros of $\zeta^{(m)}(s)$ in $0 < \operatorname{Im} s \leq T$. Recently, Aoki and Minamide [1] studied the density of zeros of $\zeta^{(m)}(s)$ in the right hand side of the critical line $\operatorname{Re} s = 1/2$ by using Littlewood's method. Let $N_m(\sigma, T)$ be the number of zeros of $\zeta^{(m)}(s)$ in $\operatorname{Re} s \geq \sigma$ and $0 < \operatorname{Im} s \leq T$. They showed that

$$N_m(\sigma, T) = O\left(\frac{T}{\sigma - 1/2} \log \frac{1}{\sigma - 1/2}\right), \tag{1.7}$$

uniformly for $\sigma > 1/2$. From (1.6) and (1.7), we see that almost all complex zeros of $\zeta^{(m)}(s)$ lie in the neighbourhood of the critical line.

The purpose of this paper is to study the counterparts of the results of Berndt, Aoki and Minamide for the derivatives of $L_f(s)$, namely, the relation between the number of complex zeros of $L_f(s)$ and that of $L_f^{(m)}(s)$, and the density of zeros of $L_f^{(m)}(s)$ in the right half-plane $\operatorname{Re} s > 1/2$. Let n_f be the smallest integer greater than 1 such that $\lambda_f(n_f) \neq 0$. The mth derivative of $L_f(s)$ is given by

$$L_f^{(m)}(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^m}{n^s} = \sum_{n=n_f}^{\infty} \frac{\lambda_f(n)(-\log n)^m}{n^s} \qquad (\operatorname{Re} s > 1),$$

Differentiating both sides of (1.2), we find that $L_f^{(m)}(s)$ is holomorphic in the whole s-plane and has the functional equation

$$L_f^{(m)}(s) = \sum_{r=0}^{m} \binom{m}{r} (-1)^r \chi_f^{(m-r)}(s) L_f^{(r)}(1-s). \tag{1.8}$$

First, we shall exhibit zero free regions for $L_f^{(m)}(s)$ by following Berndt's method (see [2]) and Spira's method (see [11], [12]).

THEOREM 1.1. *The following assertions hold for any $m \in \mathbb{Z}_{\geq 0}$.*

(i) *There exists $\sigma_{f,m} \in \mathbb{R}_{>1}$ such that $L_f^{(m)}(s)$ has no zero in $\operatorname{Re} s \geq \sigma_{f,m}$.*
(ii) *For any $\varepsilon \in \mathbb{R}_{>0}$, there exists $\delta_{f,m,\varepsilon} \in \mathbb{R}_{>(k-1)/2+1}$ such that $L_f^{(m)}(s)$ has no zero with $|s| \geq \delta_{f,m,\varepsilon}$ satisfying $\operatorname{Re} s \leq -\varepsilon$ and $|\operatorname{Im} s| \geq \varepsilon$.*
(iii) *There exists $\alpha_{f,m} \in \mathbb{R}_{<-(k-1)/2-1}$ such that $L_f^{(m)}(s)$ has only real zeros in $\operatorname{Re} s \leq \alpha_{f,m}$, and one real zero in each interval $(n-1, n)$ for $n \in \mathbb{Z}_{\leq \alpha_{f,m}}$.*

Next, based on Berndt's proof, we can obtain the following formula for the number of complex zeros of $L_f^{(m)}(s)$:

THEOREM 1.2. *For any fixed $m \in \mathbb{Z}_{\geq 1}$, let $N_{f,m}(T)$ be the number of complex zeros of $L_f^{(m)}(s)$ in $0 < \operatorname{Im} s \leq T$. Then for any large $T > 0$, we* *have*

$$N_{f,m}(T) = \frac{T}{\pi}\log\frac{T}{2\pi e} - \frac{T}{2\pi}\log n_f + O(\log T).$$

Moreover the relation between $N_f(T)$ *and* $N_{f,m}(T)$ *is given by*

$$N_{f,m}(T) = N_f(T) - \frac{T}{2\pi}\log n_f + O(\log T).$$

Finally, using the mean value formula for $L_f^{(m)}(s)$ obtained in [15] and Littlewood's method, we obtain the estimate of the density of zeros:

Theorem 1.3. *For any* $m \in \mathbb{Z}_{\geq 0}$, *let* $N_{f,m}(\sigma, T)$ *be the number of complex zeros of* $L_f^{(m)}(s)$ *with* $\operatorname{Re} s \geq \sigma$ *and* $0 < \operatorname{Im} s \leq T$. *For any large* $T > 0$, *we have*

$$N_{f,m}(\sigma, T) = O\left(\frac{T}{\sigma - 1/2}\log\frac{1}{\sigma - 1/2}\right) \tag{1.9}$$

uniformly for $1/2 < \sigma \leq 1$. *More precisely,*

$$\begin{aligned}
&N_{f,m}(\sigma, T) \\
&\leq \frac{2m+1}{2\pi}\frac{T}{\sigma - 1/2}\log\frac{1}{\sigma - 1/2} + \frac{1}{2\pi}\frac{T}{\sigma - 1/2}\log\frac{(2m)! n_f C_f}{|\lambda_f(n_f)|^2(\log n_f)^{2m}} \\
&\quad + O\left(\frac{\log T}{\sigma - 1/2}\right) \\
&\quad + \frac{1}{2\pi}\frac{T}{\sigma - 1/2}\begin{cases}
\log\left(1 + O\left(\dfrac{(2\sigma-1)^{2m+1}(\log T)^{2m}}{T^{2\sigma-1}}\right)\right), & 1/2 < \sigma < 1, \\
\log\left(1 + O\left(\dfrac{(2\sigma-1)^{2m+1}(\log T)^{2m+2}}{T}\right)\right), & \sigma = 1, \\
\log\left(1 + O\left(\dfrac{(2\sigma-1)^{2m+1}}{T}\right)\right), & 1 < \sigma < \sigma_{f,m},
\end{cases}
\end{aligned} \tag{1.10}$$

where $\sigma_{f,m}$ *is given by Theorem* 1.1(i).

2. Proof of Theorem 1.1. To show (i), we write

$$L_f^{(m)}(s) = \lambda_f(n_f)(-\log n_f)^m F(s) n_f^{-s}$$

where

$$F(s) = 1 + \sum_{n=n_f+1}^{\infty}\frac{\lambda_f(n)}{\lambda_f(n_f)}\left(\frac{\log n}{\log n_f}\right)^m\left(\frac{n_f}{n}\right)^s \quad (\operatorname{Re} s > 1). \tag{2.1}$$

Deligne's result $|\lambda_f(n)| \le d(n) \ll n^{\varepsilon}$ implies that there exist $c_f \in \mathbb{R}_{>0}$ and $\sigma_{f,m} \in \mathbb{R}_{>1}$ depending on f and m such that

$$
\begin{aligned}
(2.2) \qquad |F(\sigma+it)-1| &\le \sum_{n=n_f+1}^{\infty} \left|\frac{\lambda_f(n)}{\lambda_f(n_f)}\right| \left(\frac{\log n}{\log n_f}\right)^m \left(\frac{n_f}{n}\right)^{\sigma} \\
&\le c_f \sum_{n=n_f+1}^{\infty} \frac{(\log n/\log n_f)^m}{(n/n_f)^{\sigma-\varepsilon}} \le \frac{1}{2}
\end{aligned}
$$

for $\sigma \in \mathbb{R}_{\ge \sigma_{f,m}}$ and $t \in \mathbb{R}$, where ε is an arbitrary positive number. Hence $L_f^{(m)}(s)$ has no zeros with $\operatorname{Re} s \ge \sigma_{f,m}$, that is, (i) is proved.

Next we shall show (ii) and (iii). Replacing s with $1-s$ in (1.3) and (1.8), we have

$$
\begin{aligned}
(2.3) \qquad & (-1)^m L_f^{(m)}(1-s) \\
&= \sum_{r=0}^{m} \binom{m}{r} L_f^{(m-r)}(s) \left(2(2\pi)^{-2s}\cos(\pi s)\Gamma\left(s-\frac{k-1}{2}\right)\Gamma\left(s+\frac{k-1}{2}\right)\right)^{(r)}.
\end{aligned}
$$

Since $(\cos(\pi s))^{(r)} = \pi^r(a_r\cos(\pi s)+b_r\sin(\pi s))$ where $a_r, b_r \in \{0,\pm 1\}$, and $((2\pi)^{-2s})^{(r)} = (-2\log 2\pi)^r(2\pi)^{-2s}$ for $r \in \mathbb{Z}_{\ge 0}$, the formula (2.3) can be written as

$$
\begin{aligned}
(2.4) \qquad & (-1)^m L_f^{(m)}(1-s) \\
&= 2(2\pi)^{-2s} \sum_{r=0}^{m} R_{m-r}(s) \left(\Gamma\left(s-\frac{k-1}{2}\right)\Gamma\left(s+\frac{k-1}{2}\right)\right)^{(r)},
\end{aligned}
$$

where

$$
R_{m-r}(s) = \cos(\pi s)\sum_{j=0}^{m-r} a'_j L_f^{(j)}(s) + \sin(\pi s)\sum_{j=0}^{m-r} b'_j L_f^{(j)}(s)
$$

and $a'_r, b'_r \in \mathbb{R}$. It is clear that $a'_0 = 1$, $b'_0 = 0$ and $R_0(s) = L_f(s)\cos(\pi s)$. Moreover we write (2.4) as

$$
(2.5) \qquad \frac{(-1)^m L_f^{(m)}(1-s)}{2(2\pi)^{-2s}} = f(s) + g(s)
$$

where

$$
\begin{aligned}
f(s) &= R_0(s)\left(\Gamma\left(s-\frac{k-1}{2}\right)\Gamma\left(s+\frac{k-1}{2}\right)\right)^{(m)}, \\
g(s) &= \sum_{r=0}^{m-1} R_{m-r}(s)\left(\Gamma\left(s-\frac{k-1}{2}\right)\Gamma\left(s+\frac{k-1}{2}\right)\right)^{(r)}.
\end{aligned}
$$

The formula (2.5) implies that if $|f(s)| > |g(s)|$ in some region then $L_f^{(m)}(s)$ has no zero in that region.

In order to investigate the behavior of $f(s)$ and $g(s)$, we shall consider the approximate formula for $\big(\Gamma\big(s-\frac{k-1}{2}\big)\Gamma\big(s+\frac{k-1}{2}\big)\big)^{(r)}/\big(\Gamma\big(s-\frac{k-1}{2}\big)\Gamma\big(s+\frac{k-1}{2}\big)\big)$. By Stirling's formula, it is known that

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + \int_0^\infty \frac{\{u\}-1/2}{(u+s)^2}\,du \tag{2.6}$$

for $s \in \mathbb{C}$ such that $|\arg s| \le \pi - \delta$ where $\delta \in \mathbb{R}_{>0}$ is fixed (see [5, Theorem A.5 b)]). Let $D := \mathbb{C} \setminus \{s \in \mathbb{C} \mid \operatorname{Re} s < \varepsilon, |\operatorname{Im} s| < \varepsilon\}$, where ε is any fixed positive number. Setting $G^{(j)}(s) := (d^{j-1}/ds^{j-1})G^{(1)}(s)$ for $j \in \mathbb{Z}_{\ge 1}$ and $s \in D$ where $G^{(1)}(s)$ is the right-hand side of (2.6), we shall use the following lemma:

Lemma 2.1 ([15, Lemma 2.3]). *Let F and G be holomorphic functions such that $F(s) \ne 0$ and $\log F(s) = G(s)$ for $s \in D$, where $D \subset \mathbb{C}$ is a domain. Then for any fixed $r \in \mathbb{Z}_{\ge 1}$, there exist $l_1, \dots, l_r \in \mathbb{Z}_{\ge 0}$ and $C_{(l_1,\dots,l_r)} \in \mathbb{Z}_{\ge 0}$ such that*

$$\frac{F^{(r)}}{F}(s) = \sum_{1l_1+\cdots+rl_r=r} C_{(l_1,\dots,l_r)}(G^{(1)}(s))^{l_1}\cdots(G^{(r)}(s))^{l_r}$$

for $s \in D$. In particular, $C_{(r,0,\dots,0)} = 1$.

The estimates

$$\begin{aligned} |u+s|^2 &= u^2 + |s|^2 + 2u|s|\cos\arg s \\ &\ge \begin{cases} |s|^2, & u \le |s|,\ |\arg s| \le \pi/2, \\ |s|^2(\sin\arg s)^2, & u \le |s|,\ \pi/2 \le |\arg s| \le \pi-\delta, \\ u^2, & u \ge |s|,\ |\arg s| \le \pi/2, \\ 4(1+\cos\arg s)u^2, & u \ge |s|,\ \pi/2 \le |\arg s| \le \pi-\delta \end{cases} \end{aligned}$$

give

$$\int_0^\infty \frac{\{u\}-1/2}{(u+s)^{j+1}}\,du \ll \int_0^{|s|} \frac{du}{|s|^{j+1}} + \int_{|s|}^\infty \frac{du}{u^{j+1}} \ll \frac{1}{|s|^j}$$

for $j \in \mathbb{Z}_{\ge 1}$ and $s \in D$. Then

$$\begin{aligned} G^{(1)}(s) &= \log s + O\bigg(\frac{1}{|s|}\bigg), \\ G^{(j)}(s) &= \frac{(-1)^{j-1}(j-2)!}{s^{j-1}} + \frac{(-1)^j(j-1)!}{2s^j} + (-1)^{j+1}j!\int_0^\infty \frac{\{u\}-1/2}{(u+s)^{j+1}}\,du \\ &= O\bigg(\frac{1}{|s|^{j-1}}\bigg) \end{aligned}$$

for $j \in \mathbb{Z}_{\geq 2}$. Hence the approximate formula for $(\Gamma^{(r)}/\Gamma)(s)$ is

$$\frac{\Gamma^{(r)}}{\Gamma}(s) = (G^{(1)}(s))^r + O\Bigl(\sum_{1q_1+\cdots+rq_r=r,\, q_1\neq r} \prod_{j=1}^{r} |G^{(j)}(s)|^{q_j}\Bigr) \tag{2.7}$$

$$= \biggl(\log s + O\Bigl(\frac{1}{|s|}\Bigr)\biggr)^r + O\Bigl(\frac{|\log s|^{r-1}}{|s|}\Bigr) = (\log s)^r \sum_{j=0}^{r} \frac{M_j(s)}{(\log s)^j}$$

for $s \in D$ and $r \in \mathbb{Z}_{\geq 0}$, where $M_j(s) = O(1/|s|^j)$ for $j \in \mathbb{Z}_{\geq 1}$ and $M_0(s) = 1$. Using (2.7) and the approximate formula

$$\log\Bigl(s \pm \frac{k-1}{2}\Bigr) = \log s + \log\Bigl(1 \pm \frac{k-1}{2s}\Bigr) = \log|s| + i\arg s + O(1/|s|) \tag{2.8}$$

for $|s| > (k-1)/2$, we can write

$$\begin{aligned} &\frac{\bigl(\Gamma\bigl(s-\frac{k-1}{2}\bigr)\Gamma\bigl(s+\frac{k-1}{2}\bigr)\bigr)^{(l)}}{\Gamma\bigl(s-\frac{k-1}{2}\bigr)\Gamma\bigl(s+\frac{k-1}{2}\bigr)} \\ &= \sum_{j=0}^{l} \binom{l}{j} \frac{\Gamma^{(j)}}{\Gamma}\Bigl(s-\frac{k-1}{2}\Bigr)\frac{\Gamma^{(l-j)}}{\Gamma}\Bigl(s+\frac{k-1}{2}\Bigr) \\ &= \sum_{j=0}^{l} \binom{l}{j} \Bigl(\log\Bigl(s-\frac{k-1}{2}\Bigr)\Bigr)^j \Bigl(\log\Bigl(s+\frac{k-1}{2}\Bigr)\Bigr)^{l-j} \\ &\quad\times \sum_{\substack{0\leq j_1+j_2\leq l \\ 0\leq j_1\leq j,\, 0\leq j_2\leq l-j}} \frac{M_{j_1}\bigl(s-\frac{k-1}{2}\bigr)}{\bigl(\log\bigl(s-\frac{k-1}{2}\bigr)\bigr)^{j_1}} \frac{M_{j_2}\bigl(s+\frac{k-1}{2}\bigr)}{\bigl(\log\bigl(s+\frac{k-1}{2}\bigr)\bigr)^{j_2}} \\ &= S_l(s) + T_l(s) \end{aligned} \tag{2.9}$$

for $l \in \mathbb{Z}_{\geq 0}$, $s \in D'$ and $|s| > (k-1)/2$, where $D' := \mathbb{C} \setminus \{s \in \mathbb{C} \mid \operatorname{Re} s < (k-1)/2 + \varepsilon,\ |\operatorname{Im} s| < \varepsilon\}$ and

$$S_l(s) = \biggl(\log\Bigl(s-\frac{k-1}{2}\Bigr) + \log\Bigl(s+\frac{k-1}{2}\Bigr)\biggr)^l,$$

$$T_l(s) = O\Bigl(\frac{1}{|s|\log|s|}\sum_{j=0}^{l}(\log|s|)^j(\log|s|)^{l-j}\Bigr) = O\Bigl(\frac{(\log|s|)^{l-1}}{|s|}\Bigr) \tag{2.10}$$

for $l \in \mathbb{Z}_{\geq 1}$, in particular $S_0(s) = 1$ and $T_0(s) = 0$.

Next using $R_r(s)$, $S_r(s)$ and $T_r(s)$, we shall give a condition which implies $|f(s)| > |g(s)|$ for some region. From (2.5) and (2.9), the inequality $|f(s)| > |g(s)|$ is equivalent to

$$|S_m(s) + T_m(s)| > \Bigl|\sum_{r=0}^{m-1} \frac{R_r}{R_0}(s)(S_r(s) + T_r(s))\Bigr|.$$

Dividing both sides by $S_{m-1}(s)$ and applying the triangle inequality, we see that if

$$(2.11)\qquad |S_1(s)| > \left|\frac{T_m}{S_{m-1}}(s)\right| + \left|\sum_{r=0}^{m-1}\frac{R_r}{R_0}(s)\left(\frac{1}{S_{m-1-r}}(s)+\frac{T_r}{S_{m-1}}(s)\right)\right|,$$

then $|f(s)| > |g(s)|$ for $s \in D'$ and $|s| > (k-1)/2$. To prove (2.11), we shall obtain upper bounds of $(1/S_r)(s)$ and $(R_r/R_0)(s)$. The formula (2.8) gives

$$(2.12)\qquad \left|\frac{1}{S_r}(s)\right| \le \frac{C_1}{(\log|s|)^r}$$

for the above s and $r \in \mathbb{Z}_{\ge 0}$; here and later, $C_1, C_2, \dots$ denote positive constants depending on f, r and ε. Since $L_f^{(j)}(s)$ and $(1/L_f)(s)$ are absolutely convergent for $\operatorname{Re} s > 1$, it follows that

$$(2.13)\qquad \left|\frac{R_r}{R_0}(s)\right| = \left|\sum_{j=0}^{r} a'_j\frac{L_f^{(j)}}{L_f}(s)+\tan(\pi s)\sum_{j=0}^{r} b'_j\frac{L_f^{(j)}}{L_f}(s)\right| \le C_2+C_3|\tan(\pi s)|$$

for $\operatorname{Re} s \ge 1+\varepsilon$. Here $\tan(\pi s)$ is estimated as

$$(2.14)\qquad |\tan\pi(\sigma+it)| = \left|\frac{e^{-2t}e^{2\pi i\sigma}-1}{e^{-2t}e^{2\pi i\sigma}+1}\right| \le \begin{cases} 2/(1-e^{-2\varepsilon}) & \text{if } |t|\ge\varepsilon,\\ 3 & \text{if } \sigma\in\mathbb{Z}.\end{cases}$$

Combining (2.10) and (2.12)–(2.14), we see that the right-hand side of (2.11) is estimated as

$$(2.15)\qquad \begin{aligned}&\left|\frac{T_m}{S_{m-1}}(s)\right| + \left|\sum_{r=0}^{m-1}\frac{R_r}{R_0}(s)\left(\frac{1}{S_{m-1-r}}(s)+\frac{T_r}{S_{m-1}}(s)\right)\right|\\ &\quad\le \frac{C_4}{|s|}+C_5|\tan(\pi s)|\sum_{r=0}^{m-1}\left(\frac{1}{(\log|s|)^{m-1-r}}+\frac{1}{|s|(\log|s|)^{m-r}}\right) \le C_{f,m,\varepsilon}\end{aligned}$$

for $|s| > (k-1)/2$ and $\operatorname{Re} s \ge 1+\varepsilon$ provided $|\operatorname{Im} s| \ge \varepsilon$ or $\operatorname{Re} s \in \mathbb{Z}$, where $C_{f,m,\varepsilon}$ is a positive constant depending on f, m and ε. Choose $r_{f,m,\varepsilon} \in \mathbb{R}_{>(k-1)/2}$ such that $C_{f,m,\varepsilon} < (\log r_{f,m,\varepsilon})/C_1$. The inequalities (2.12) and (2.15) imply that (2.11) is true, that is, $L_f^{(m)}(1-s)$ has no zero for $s \in \mathbb{C}$ such that $|s| \ge r_{f,m,\varepsilon}$, $\operatorname{Re} s \ge 1+\varepsilon$ and $|\operatorname{Im} s| \ge \varepsilon$. Therefore, we conclude that for any $\varepsilon \in \mathbb{R}_{>0}$ there exists $\delta_{f,m,\varepsilon} \in \mathbb{R}_{>(k-1)/2+1}$ such that $L_f^{(m)}(s)$ has no zero with $|s| \ge \delta_{f,m,\varepsilon}$, $\operatorname{Re} s \le -\varepsilon$ and $|\operatorname{Im} s| \ge \varepsilon$, that is, the proof of (ii) is complete.

Finally, we show (iii) by applying Rouché's theorem to $f(s)$ and $g(s)$. For $n \in \mathbb{Z}_{\ge 1}$ let D_n be the region where $n \le \operatorname{Re} s \le n+1$ and $|\operatorname{Im} s| \le 1/2$. By (2.14) and (2.15), we see that there exists $\delta_{f,m,1/2} \in \mathbb{R}_{>(k-1)/2}$ such that $|f(s)| > |g(s)|$ on the boundary of D_n and in the region where $|s| > \delta_{f,m,1/2}$ and $\operatorname{Re} s \ge 1+1/2$. Then the number of zeros of $f(s)$ is equal to that of

$f(s)+g(s)$ in the interior of D_n. From (2.4), (2.5) and (2.9),

$$f(s)=L_f(s)\cos(\pi s)\Gamma\left(s-\frac{k-1}{2}\right)\Gamma\left(s+\frac{k-1}{2}\right)(S_m(s)+T_m(s)). \tag{2.16}$$

When $R_{f,m}$ is chosen such that $R_{f,m}\ge\delta_{f,m,1/2}$ and $C_6/(R_{f,m}\log R_{f,m})<1$, (2.12) gives

$$\left|\frac{T_m}{S_m}(s)\right|\le\frac{C_6}{|s|\log|s|}<1, \tag{2.17}$$

that is, $S_m(s)+T_m(s)$ has no zero with $|s|\ge R_{f,m}$. Hence $f(s)$ has the only real zero $s=n+1/2$ in D_n. It is clear that $\overline{f(s)}=f(\overline{s})$ and $\overline{g(s)}=g(\overline{s})$ for $s\in\mathbb{C}$, which implies that $L_f^{(m)}(1-s)$ can only have a real zero in the interior of D_n. Replacing $1-s$ with s, we conclude that there exists $\alpha_{f,m}\in\mathbb{R}_{<-(k-1)/2-1}$ such that $L_f^{(m)}(s)$ has no complex zero in $\operatorname{Re}s<\alpha_{f,m}$ and one real zero in each open interval $(n-1,n)$ for $n\in\mathbb{Z}_{\le\alpha_{f,m}}$. The proof of (iii) is complete.

3. Proof of Theorem 1.2. Using Theorem 1.1, we can choose $\alpha_{f,m}\in\mathbb{R}_{<-(k-1)/2}$ and $\sigma_{f,m}\in\mathbb{R}_{>1}$ such that $L_f^{(m)}(s)$ has no zeros with $\operatorname{Re}s\le\alpha_{f,m}$ or $\operatorname{Re}s\ge\sigma_{f,m}$. Moreover, choose $\tau_{f,m}\in\mathbb{R}_{>2}$ and $T\in\mathbb{R}_{>0}$ such that $L_f^{(m)}(s)$ has no zeros with $0<\operatorname{Im}s\le\tau_{f,m}$ or $\operatorname{Im}s=T$. Using the residue theorem in the region where $\alpha_{f,m}\le\operatorname{Re}s\le\sigma_{f,m}$ and $\tau_{f,m}\le\operatorname{Im}s\le T$, we get

$$\begin{aligned}N_{f,m}(T)=\frac{1}{2\pi i}\Bigg(&\int_{\alpha_{f,m}+i\tau_{f,m}}^{\sigma_{f,m}+i\tau_{f,m}}+\int_{\sigma_{f,m}+i\tau_{f,m}}^{\sigma_{f,m}+iT}+\int_{\sigma_{f,m}+iT}^{\alpha_{f,m}+iT}\\&+\int_{\alpha_{f,m}+iT}^{\alpha_{f,m}+i\tau_{f,m}}\Bigg)(\log L_f^{(m)}(s))'ds=:I_1+I_2+I_3+I_4.\end{aligned} \tag{3.1}$$

First, it is clear that

$$I_1=\frac{\log L_f^{(m)}(\sigma_{f,m}+i\tau_{f,m})-\log L_f^{(m)}(\alpha_{f,m}+i\tau_{f,m})}{2\pi i}=O(1). \tag{3.2}$$

To approximate I_2, we write $L_f^{(m)}(s)=\lambda_f(n_f)(-\log n_f)^mF(s)n_f^{-s}$ where $F(s)$ is given by (2.1). Using (2.2) we find that $1/2\le|F(s)|\le 3/2$, $\operatorname{Re}F(s)\ge 1/2$ and $|\arg F(s)|<\pi/2$ for $s=\sigma_{f,m}+it$ ($t\in\mathbb{R}$). Hence

$$\begin{aligned}I_2&=\frac{1}{2\pi i}\left[\log\frac{\lambda_f(n_f)(-\log n_f)^m}{n_f^s}+\log F(s)\right]_{\sigma_{f,m}+i\tau_{f,m}}^{\sigma_{f,m}+iT}\\&=\frac{-(\sigma_{f,m}+iT)\log n_f}{2\pi i}+O(1)=-\frac{T}{2\pi}\log n_f+O(1).\end{aligned} \tag{3.3}$$

Next we shall estimate I_3. By (1.8), the approximate functional equation for $L_f^{(m)}(s)$ (see [15, Theorem 1.2]) and Rankin's result, there exists $A \in \mathbb{R}_{\geq 0}$ such that $L_f^{(m)}(\sigma+it) = O(|t|^A)$ uniformly for $\sigma \in [\alpha_{f,m}, \sigma_{f,m}]$. This implies that

$$\begin{aligned} I_3 &= \frac{\log L_f^{(m)}(\alpha_{f,m}+iT) - \log L_f^{(m)}(\sigma_{f,m}+iT)}{2\pi i} \\ &= \frac{\arg L_f^{(m)}(\alpha_{f,m}+iT) - \arg L_f^{(m)}(\sigma_{f,m}+iT)}{2\pi} + O(\log T). \end{aligned} \tag{3.4}$$

To estimate the first term of the right-hand side, we write $L_f^{(m)}(\sigma+iT) = (-1)^m e^{-iT\log n_f}\lambda_f(n_f)G(\sigma+iT)$ where

$$G(\sigma+iT) = \frac{(\log n_f)^m}{n_f^\sigma} + \frac{1}{\lambda_f(n_f)} \sum_{n=n_f+1}^{\infty} \frac{\lambda_f(n)(\log n)^m}{n^\sigma} e^{iT\log n_f/n}$$

for $\sigma \in \mathbb{R}_{>1}$. Let Q be the number of zeros of $\operatorname{Re} G(s)$ on the line segment $(\alpha_{f,m}+iT, \sigma_{f,m}+iT)$. Divide this line into $Q+1$ subintervals by these zeros. Then on each subinterval, the sign of $\operatorname{Re} G(s)$ is constant, and the variation of $\arg G(s)$ is at most π. Hence, there exists a constant C such that $\arg G(s) = \arg L_f^{(m)}(s) + C$ on the divided line, and so

$$|\arg L_f^{(m)}(\alpha_{f,m}+iT) - \arg L_f^{(m)}(\sigma_{f,m}+iT)| \leq (Q+1)\pi. \tag{3.5}$$

In order to estimate Q, let $H(z) = (G(z+iT) + \overline{G(\overline{z}+iT)})/2$. Then we find that

$$\begin{aligned} H(\sigma) &= \operatorname{Re} G(\sigma+iT) \\ &= \frac{(\log n_f)^m}{n_f^\sigma}\left(1 + \sum_{n=n_f+1}^{\infty} \frac{\lambda_f(n)}{\lambda_f(n_f)}\left(\frac{\log n}{\log n_f}\right)^m \left(\frac{n_f}{n}\right)^\sigma \cos\left(T\log\frac{n_f}{n}\right)\right) \end{aligned} \tag{3.6}$$

for $\sigma \in \mathbb{R}_{>1}$. Now (2.2) and (3.6) give

$$\frac{1}{2}\,\frac{(\log n_f)^m}{{n_f}^{\sigma_{f,m}}} \leq H(\sigma_{f,m}) \leq \frac{3}{2}\,\frac{(\log n_f)^m}{{n_f}^{\sigma_{f,m}}}. \tag{3.7}$$

Take T so large that $T - \tau_{f,m} > 2(\sigma_{f,m} - \alpha_{f,m})$. Since $\operatorname{Im}(z+iT) \geq T - (T-\tau_{f,m}) > 0$ for $z \in \mathbb{C}$ such that $|z-\sigma_{f,m}| < T - \tau_{f,m}$, it follows that $H(z)$ is analytic in the disc $|z-\sigma_{f,m}| < T-\tau_{f,m}$. Note that there exists a positive constant B such that $H(z) = O(T^B)$ in that disc because $L_f(\sigma+it) = O(|t|^A)$. For $u \in \mathbb{R}_{\geq 0}$, let $P(u)$ be the number of zeros of $H(z)$ in $|z-\sigma_{f,m}| \leq u$. Then using the trivial estimate

$$P(\sigma_{f,m}-\alpha_{f,m}) \le \frac{1}{\log 2}\int_{\sigma_{f,m}-\alpha_{f,m}}^{2(\sigma_{f,m}-\alpha_{f,m})} \frac{P(u)}{u}\,du \le \frac{1}{\log 2}\int_{0}^{2(\sigma_{f,m}-\alpha_{f,m})} \frac{P(u)}{u}\,du,$$

Jensen's formula (see [13, Chapter 3.61]), the above remark and (3.7), we have

$$\begin{aligned} P(\sigma_{f,m}-\alpha_{f,m}) \ll & \int_{0}^{2(\sigma_{f,m}-\alpha_{f,m})} \frac{P(u)}{u}\,du \\ = & \frac{1}{2\pi}\int_0^{2\pi} \log|H(\sigma_{f,m}+2(\sigma_{f,m}-\alpha_{f,m})e^{i\theta})|\,d\theta - \log|H(\sigma_{f,m})| \\ \ll & \int_0^{2\pi} \log T^B\,d\theta + 1 \ll \log T, \end{aligned}$$

Therefore

$$Q = \#\{\sigma \in (\alpha_{f,m},\sigma_{f,m}) \mid F(\sigma)=0\} \ll P(\sigma_{f,m}-\alpha_{f,m}) \ll \log T. \tag{3.8}$$

Combining (3.4), (3.5), (3.8), we obtain

$$I_3 = O(\log T). \tag{3.9}$$

Finally, in order to approximate I_4, we shall obtain an approximate formula for $\log L_f^{(m)}(\alpha_{f,m}+iT)$ as $T \to \infty$. By the proof of Theorem 1.1, there exists $\delta_{f,m} \in \mathbb{R}_{>0}$ such that

$$\left|\frac{g}{f}(1-s)\right| < 1, \qquad \left|\frac{T_m}{S_m}(1-s)\right| < 1 \tag{3.10}$$

for $s \in \mathbb{C}$ in the region where $|s-(1-(k-1)/2)| > \delta_{f,m}$, $\operatorname{Re} s < 1-(k-1)/2$ and $|\operatorname{Im} s| > 1/2$. Choose $\alpha_{f,m} \in \mathbb{R}_{<0}$ such that $\alpha_{f,m} < 1-(k-1)/2-\delta_{f,m}$. Then the path of I_4 is contained in the above region. Replacing s with $1-s$ and taking the logarithm of both sides of (2.5), we obtain

$$\begin{aligned} \log L_f^{(m)}(\alpha_{f,m}+iT) & \\ = & -2(1-\alpha_{f,m}-iT)\log 2\pi + \log f(1-\alpha_{f,m}-iT) \\ & + \log\left(1+\frac{g}{f}(1-\alpha_{f,m}-iT)\right) + O(1). \end{aligned} \tag{3.11}$$

The first inequality of (3.10) gives $|\arg(1+(g/f)(1-\alpha_{f,m}-iT))| < \pi/2$ and

$$\begin{aligned} \log\left(1+\frac{g}{f}(1-\alpha_{f,m}-iT)\right) & \\ \ll \sqrt{\left|1+\frac{g}{f}(1-\alpha_{f,m}-iT)\right|^2 + \left(\arg\left(1+\frac{g}{f}(1-\alpha_{f,m}-iT)\right)\right)^2} \ll 1. \end{aligned} \tag{3.12}$$

By (2.16), the second term of the right-hand side of (3.11) can be written as

$$(3.13)\qquad \begin{aligned} &\log f(1-\alpha_{f,m}-iT)\\ &= \log \Gamma\Big(1-\alpha_{f,m}-\frac{k-1}{2}-iT\Big)+\log \Gamma\Big(1-\alpha_{f,m}+\frac{k-1}{2}-iT\Big)\\ &\quad+\log S_m(1-\alpha_{f,m}-iT)+\log\Big(1+\frac{T_m}{S_m}(1-\alpha_{f,m}-iT)\Big)\\ &\quad+\log L_f(1-\alpha_{f,m}-iT)+\log\cos\pi(1-\alpha_{f,m}-iT). \end{aligned}$$

Now it is clear that

$$\cos(\pi(1-\alpha_{f,m}-iT)) = e^{\pi T}e^{i(1-\alpha_{f,m})-\log 2}(1+e^{-2\pi(1-\alpha_{f,m})i}/e^{2\pi T}),$$

$$\log L_f(1-\alpha_{f,m}-iT) = \sum_{n=1}^{\infty}\frac{b_f(n)}{n^{1-\alpha_{f,m}-iT}}$$

where $b_f(n)$ is given by

$$b_f(n) = \begin{cases} (\alpha_f(p)^r+\beta_f(p)^r)/r, & n=p^r,\\ 0, & \text{otherwise,}\end{cases}$$

and $\alpha_f(p)$, $\beta_f(p)$ are given by (1.1). Hence for the last two terms of the right-hand side of (3.13) we have

$$(3.14)\qquad \log L_f(1-\alpha_{f,m}-iT)+\log\cos(\pi(1-\alpha_{f,m}-iT)) = \pi T+O(1).$$

By a similar discussion to (3.12), the fourth term of the right-hand side of (3.13) is estimated as

$$(3.15)\qquad \log\Big(1+\frac{T_m}{S_m}(1-\alpha_{f,m}-iT)\Big)\ll 1.$$

The trivial formula

$$\log\Big(1-\alpha_{f,m}\pm\frac{k-1}{2}-iT\Big) = \log T-(\pi/2)i+O(1/T)$$

shows that the third term of the right-hand side of (3.13) is approximated as

$$(3.16)\qquad \begin{aligned} &\log S_m(1-\alpha_{f,m}-iT)\\ &= m\log\Big(\log\Big(1-\alpha_{f,m}-\frac{k-1}{2}-iT\Big)+\log\Big(1-\alpha_{f,m}+\frac{k-1}{2}-iT\Big)\Big)\\ &= m\log\log T+O(1). \end{aligned}$$

Using Stirling's formula

$$\log\Gamma(s) = (s-1/2)\log s - s+\log\sqrt{2\pi}+O(1/|s|)$$

and the approximate formula for $\log(1-\alpha_{f,m}\pm(k-1)/2-iT)$, we approximate the sum of the first two terms of the right-hand side of (3.13) as

$$\log\Gamma\Big(1-\alpha_{f,m}+\frac{k-1}{2}-iT\Big)+\log\Gamma\Big(1-\alpha_{f,m}-\frac{k-1}{2}-iT\Big) \tag{3.17}$$
$$=(1-2\alpha_{f,m}-2iT)(\log T-(\pi/2)i+O(1/T))-2(1-\alpha_{f,m}-iT)+O(1)$$
$$=-2iT\log(T/e)-\pi T+(1-2\alpha_{f,m})\log T+O(1).$$

Combining (3.11)–(3.17), we obtain the desired approximate formula

$$\log L_f^{(m)}(\alpha_{f,m}+iT)=-2iT\log\frac{T}{2\pi e}+O(\log T),$$

which implies that

$$I_4=\frac{T}{\pi}\log\frac{T}{2\pi e}+O(\log T). \tag{3.18}$$

By (3.2), (3.3), (3.9) and (3.18), the proof of Theorem 1.2 is complete.

4. Proof of Theorem 1.3. Write $L_f^{(m)}(s)=\lambda_f(n_f)(-\log n_f)^m F(s)/n_f^s$ where $F(s)$ is given by (2.1). By the proof of Theorem 1.1, we can choose $\sigma_{f,m}\in\mathbb{R}_{>1}$ such that $L_f(s)$ has no zero with $\operatorname{Re}s>\sigma_{f,m}$, and

$$\sum_{n=n_f+1}^{\infty}\left|\frac{\lambda_f(n)}{\lambda_f(n_f)}\right|\Big(\frac{\log n}{\log n_f}\Big)^m\Big(\frac{n_f}{n}\Big)^{\sigma_{f,m}/2}\le\frac{1}{2}.$$

Note that (2.2) and the above inequality give

$$\begin{aligned}|F(s)-1|&\le\sum_{n=n_f+1}^{\infty}\left|\frac{\lambda_f(n)}{\lambda_f(n_f)}\right|\Big(\frac{\log n}{\log n_f}\Big)^m\Big(\frac{n_f}{n}\Big)^{\sigma_{f,m}/2+\sigma/2}\\&\le\frac{1}{2}\Big(\frac{n_f}{n_f+1}\Big)^{\sigma/2}\end{aligned} \tag{4.1}$$

for $\operatorname{Re}s\ge\sigma_{f,m}$. Applying Littlewood's formula (see [13, Chapter 3.8]) to $F(s)$, we obtain

$$\begin{aligned}2\pi\sum_{\substack{F(\rho)=0\\ \sigma\le\operatorname{Re}\rho\le\sigma_{f,m}\\ 1\le\operatorname{Im}\rho\le T}}(\operatorname{Re}\rho-\sigma)&=\int_1^T\log|F(\sigma+it)|dt-\int_1^T\log|F(\sigma_{f,m}+it)|\,dt\\&\quad+\int_\sigma^{\sigma_{f,m}}\arg F(u+iT)\,dt-\int_\sigma^{\sigma_{f,m}}\arg F(u+i)\,dt\\&=:I_1+I_2+I_3+I_4\end{aligned} \tag{4.2}$$

for $\sigma \in \mathbb{R}_{>1/2}$. We first estimate I_2. Cauchy's theorem gives

$$(4.3)\quad I_2 = \int_1^T \log|F(v+it)|\,dt + \int_{\sigma_{f,m}}^{v} \log|F(u+i)|\,du - \int_{\sigma_{f,m}}^{v} \log|F(u+iT)|\,du$$

for all $v > \sigma_{f,m}$. The facts that $\log|X| \le |X-1|$ for $X \in \mathbb{C}$ and $-\log|Y| \le 2|Y-1|$ for $Y \in \mathbb{C}$ satisfying $|Y| \ge 1/2$, and (4.1), imply that

$$(4.4)\quad \int_1^T \log|F(v+it)|dt \le \frac{T-1}{2}\left(\frac{n_f}{n_f+1}\right)^{v/2}$$

and

$$(4.5)\quad \int_{\sigma_{f,m}}^{v} \log|F(u+i)|\,du - \int_{\sigma_{f,m}}^{v} \log|F(u+iT)|\,du \ll \int_{\sigma_{f,m}}^{v} \left(\frac{n_f}{n_f+1}\right)^{u/2} du \ll 1.$$

Combining (4.3)–(4.5) we get

$$(4.6)\quad I_2 = O(1).$$

Following the estimation of I_3 in the proof of Theorem 1.2, we obtain

$$(4.7)\quad I_3 + I_4 = O(\log T).$$

To estimate I_1, we calculate

$$(4.8)\quad I_1 = \frac{T-1}{2}\log\frac{n_f^{2\sigma}}{|\lambda_f(n_f)|^2(\log n_f)^{2m}} + \frac{1}{2}\int_1^T \log|L_f^{(m)}(\sigma+it)|^2\,dt.$$

Jensen's inequality gives

$$(4.9)\quad \int_1^T \log|L_f^{(m)}(\sigma+it)|^2 dt \le (T-1)\log\left(\frac{1}{T-1}\int_1^T |L_f^{(m)}(\sigma+it)|^2 dt\right).$$

Combining (4.2) and (4.6)–(4.9), we obtain

$$(4.10)\quad \sum_{\substack{F(\rho)=0\\ \sigma\le \operatorname{Re}\rho\le\sigma_{f,m}\\ 1\le \operatorname{Im}\rho\le T}} (\operatorname{Re}\rho-\sigma) \le \frac{T-1}{4\pi}\log\left(\frac{1}{T-1}\int_1^T |L_f^{(m)}(\sigma+it)|^2\,dt\right) + \frac{T-1}{4\pi}\log\frac{n_f^{2\sigma}}{|\lambda_f(n_f)|^2(\log n_f)^{2m}} + O(\log T).$$

First, we consider the mean square of $L_f^{(m)}(s)$ for $\operatorname{Re} s > 1$. We calculate as follows:

$$\text{(4.11)}\qquad \int_1^T |L_f^{(m)}(\sigma+it)|^2\,dt$$
$$= \sum_{n_1,n_2=1}^{\infty} \frac{\overline{\lambda_f(n_1)}\lambda_f(n_2)(\log n_1)^m(\log n_2)^m}{(n_1n_2)^\sigma} \int_{\max\{n_1,n_2\}}^{T} \left(\frac{n_1}{n_2}\right)^{it} dt$$
$$= (T-1)\sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2(\log n)^{2m}}{n^{2\sigma}}$$
$$+ \frac{1}{i} \sum_{\substack{n_1,n_2=1\\ n_1\neq n_2}}^{\infty} \frac{\overline{\lambda_f(n_1)n_1^{-iT}}\lambda_f(n_2)n_2^{-iT}(\log n_1)^m(\log n_2)^m}{(n_1n_2)^\sigma \log(n_1/n_2)}$$
$$- \frac{1}{i} \sum_{\substack{n_1,n_2=1\\ n_1\neq n_2}}^{\infty} \frac{\overline{\lambda_f(n_1)n_1^{-i\max\{n_1,n_2\}}}\lambda_f(n_2)n_2^{-i\max\{n_1,n_2\}}(\log n_1)^m(\log n_2)^m}{(n_1n_2)^\sigma \log(n_1/n_2)}.$$

By the same discussion for $U_\sigma(x)$ with

$$(\alpha_{n_1},\beta_{n_2}) = (\lambda_f(n_1)n_1^{-iT}(\log n_1)^m, \lambda_f(n_2)n_2^{-iT}(\log n_2)^m) \quad \text{or}$$
$$(\lambda_f(n_1)n_1^{-i\max\{n_1,n_2\}}(\log n_1)^m, \lambda_f(n_2)n_2^{-i\max\{n_1,n_2\}}(\log n_2)^m)$$

in [3, p. 348, Lemma 6], we find that the second and third terms on the right-hand side of (4.11) are $O(1)$ uniformly for $\sigma > 1$. Hence, for $\sigma > 1$,

$$\text{(4.12)}\qquad \int_1^T |L_f^{(m)}(\sigma+it)|^2\,dt = (T-1)\sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2(\log n)^{2m}}{n^{2\sigma}} + O(1).$$

Next the mean square of $L_f^{(m)}(s)$ for $1/2 < \operatorname{Re} s \le 1$ is obtained as follows:

Lemma 4.1 ([15, Theorem 1.3]). *For any $m \in \mathbb{Z}_{\ge 0}$ and $T > 0$, we have*

$$\text{(4.13)}\qquad \int_1^T |L_f^{(m)}(\sigma+it)|^2\,dt$$
$$= \begin{cases} (T-1)\displaystyle\sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2(\log n)^{2m}}{n^{2\sigma}} + O(T^{2(1-\sigma)}(\log T)^{2m}), & 1/2 < \sigma < 1, \\ (T-1)\displaystyle\sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2(\log n)^{2m}}{n^{2\sigma}} + O((\log T)^{2m+2}), & \sigma = 1. \end{cases}$$

Using Rankin's result mentioned in Introduction and the fact that

$$\int_{n_f}^{\infty} \frac{(\log u)^{2m}}{u^{2\sigma}}\,du = \frac{(2m)!\,n_f^{1-2\sigma}}{(2\sigma-1)^{2m+1}} \sum_{j=0}^{2m} \frac{(\log n_f)^j(2\sigma-1)^j}{j!},$$

which is obtained by induction and integration by parts, we find that the series in the main term of (4.13) is approximated as

$$
\begin{aligned}
(4.14)\qquad & \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2(\log n)^{2m}}{n^{2\sigma}} \\
&= -\int_{n_f}^{\infty} \left(\frac{(\log u)^{2m}}{u^{2\sigma}}\right)' \sum_{n_f<n\le u} |\lambda_f(n)|^2 \, du \\
&= -\frac{C_f(\log n_f)^{2m}}{n_f^{2\sigma-1}} + C_f \int_{n_f}^{\infty} \frac{(\log u)^{2m}}{u^{2\sigma}} \, du + O\biggl(\int_{n_f}^{\infty} \frac{(\log u)^{2m}}{u^{2\sigma+2/5}} \, du\biggr) \\
&= \frac{(2m)! n_f C_f}{n_f^{2\sigma}} \frac{1}{(2\sigma-1)^{2m+1}} + O\biggl(\frac{1}{(2\sigma-1)^{2m}}\biggr)
\end{aligned}
$$

as $\sigma \to 1/2+0$. From (4.10)–(4.14), the following approximate formula is obtained:

$$
\begin{aligned}
(4.15)\qquad & \sum_{\substack{F(\rho)=0,\\ \sigma\le \operatorname{Re}\rho\le\sigma_{f,m}\\ 1\le \operatorname{Im}\rho\le T}} (\operatorname{Re}\rho-\sigma) \\
&\le \frac{(2m+1)(T-1)}{4\pi}\log\frac{1}{2\sigma-1} + \frac{T-1}{4\pi}\log\frac{(2m)! n_f C_f}{|\lambda_f(n_f)|^2(\log n_f)^{2m}} + O(\log T) \\
&\quad + \frac{T-1}{4\pi}\begin{cases}
\log\Bigl(1+O\Bigl(\frac{(2\sigma-1)^{2m+1}(\log T)^{2m}}{T^{2\sigma-1}}\Bigr)\Bigr), & 1/2<\sigma<1, \\
\log\Bigl(1+O\Bigl(\frac{(2\sigma-1)^{2m+1}(\log T)^{2m+2}}{T}\Bigr)\Bigr), & \sigma=1, \\
\log\Bigl(1+O\Bigl(\frac{(2\sigma-1)^{2m+1}}{T}\Bigr)\Bigr), & \sigma>1.
\end{cases}
\end{aligned}
$$

Finally, we shall give an upper bound of $N_{f,m}(\sigma,T)$. Since $N_{f,m}(\sigma,T)$ is decreasing with respect to σ, it follows that

$$
\begin{aligned}
(4.16)\qquad N_{f,m}(\sigma,T) &= N_{f,m}(\sigma,T) - N_{f,m}(\sigma,1) + C \\
&\le \frac{1}{\sigma-\sigma_1}\int_{\sigma_1}^{\sigma_{f,m}} (N_{f,m}(u,T)-N_{f,m}(u,1))\,du + C,
\end{aligned}
$$

where we set $\sigma_1 = 1/2+(\sigma-1/2)/2$. Note that $\sigma-\sigma_1=(\sigma-1/2)/2$, so $2\sigma_1-1=\sigma-1/2$. Since the number of zeros of $F_m(s)$ is equal to that of $L_f^{(m)}(s)$, it follows that

$$(4.17)\quad \int_{\sigma_1}^{\sigma_{f,m}} (N_{f,m}(u,T) - N_{f,m}(u,1))\,du$$
$$= \int_{\sigma_1}^{\sigma_{f,m}} \sum_{\substack{F(\rho)=0\\ u\le \operatorname{Re}\rho\le\sigma_{f,m}\\ 1\le \operatorname{Im}\le T}} 1\,du = \sum_{\substack{F(\rho)=0\\ \sigma_1\le \operatorname{Re}\rho\le\sigma_{f,m}\\ 1\le \operatorname{Im}\rho\le T}} \int_{\sigma_1}^{\operatorname{Re}\rho} 1\,du$$
$$= \sum_{\substack{F(\rho)=0\\ \sigma_1\le \operatorname{Re}\rho\le\sigma_{f,m}\\ 1\le \operatorname{Im}\rho\le T}} (\operatorname{Re}\rho - \sigma_1).$$

Combining (4.15)–(4.17) we obtain (1.10) and (1.9). Hence the proof of Theorem 1.3 is complete.

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