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On totally smooth subspaces of Banach spaces: the Vlasov theorem revisited

by

EVE OJA (Tartu and Tallinn), MÄRT PÕLDVERE (Tartu) and
TAURI VIIL (Tartu)

Abstract. Let X be a Banach space and let Y be a closed subspace of X. We establish new geometric characterizations for Y to be totally smooth in X, meaning that every closed subspace of Y has Phelps' property U in X. In particular, this gives a new self-contained proof for a recent theorem of Liao and Wong, and an improved proof for a theorem of Vlasov.

1. Introduction. Let X be a Banach space and let Y be a closed subspace of X. By the Hahn–Banach theorem, every continuous linear functional $g \in Y^*$ has a norm-preserving extension $f \in X^*$. In general, such an extension is not unique. Following Phelps [Ph1], we say that Y has *property* U in X if every $g \in Y^*$ has a unique norm-preserving extension $f \in X^*$.

By the Taylor–Foguel theorem (see [T] and [F] or, e.g., [C, p. 265, Theorem 5.9-2]), *every subspace of* X *has property* U *if and only if the dual space* X^* *is strictly convex*. A theorem of Vlasov [V], in turn, says that X^* *is strictly convex if and only if the union of every unbounded nested sequence of open balls in* X *is either the whole space* X *or an open half-space.*

DEFINITION 1.1 (see [BFLM]). A sequence $B_n = B(x_n, r_n)$ of open balls in X is *nested* and *unbounded* if $B_n \subset B_{n+1}$ for all $n \in \mathbb{N}$ and $r_n \to \infty$ as $n \to \infty$.

Recently, Liao and Wong [LW] introduced the notion of a totally smooth subspace, which was essentially considered already in 1977 by Sullivan [Su] (see Remark 2.7).

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Definition 1.2 (see [LW]). A closed subspace Y of X is *totally smooth* if every closed subspace of Y has property U in X.

They also described totally smooth subspaces as follows, generalizing the Vlasov theorem.

Theorem 1.3 (see [LW]). *Let X be a real Banach space and let Y be a closed subspace of X. The following assertions are equivalent:*

(a) *Y is totally smooth.*
(b) *The union of every unbounded nested sequence of open balls in X with centers in Y is either the whole space X or an open half-space.*

Clearly, the special case of Theorem 1.3 when $Y = X$ reduces to the Vlasov theorem. The proof of Theorem 1.3 in [LW] is not self-contained: it heavily relies on the Vlasov theorem.

The objective of this paper is to prove an omnibus theorem, Theorem 2.1 below, giving equivalent conditions for a subspace Y of a Banach space X to be totally smooth. These include conditions in terms of sequences of balls which are formally weaker than condition (b) of Theorem 1.3. Among other things, our Theorem 2.1 gives Theorem 1.3 a new self-contained proof which does not use the Vlasov theorem (and which is valid for both real and complex spaces). Our proof refines some ideas from the paper [OP1] by Oja and Põldvere which, in [OP1], led to a new proof of the Vlasov theorem. The special case when $Y = X$ of Theorem 2.1, in its turn, provides a new proof to the Vlasov theorem which represents a qualitative improvement of the argument in [OP1, proof of Theorem 2]. (Concerning different proofs of the Vlasov theorem, see Remark 2.6.)

Our notation is standard. We consider Banach spaces over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In a Banach space X, we denote the open ball with center x and radius r by $B(x, r)$, the unit sphere by S_X, and for a subset A of X, its norm closure by $\overline{A}$.

2. Omnibus characterization of total smoothness. Theorem 1.3 is precisely (the real case of) the equivalence (a)⇔(d) below.

Theorem 2.1. *Let Y be a closed subspace of a Banach space X. The following assertions are equivalent:*

(a) *Y is totally smooth.*
(b) *For every sequence (y_n) in S_Y, there exists at most one functional $f \in S_{X^*}$ satisfying the condition*

$$\operatorname{Re} f(y_n) \xrightarrow[n\to\infty]{} \|f\| = 1. \tag{2.1}$$

(c) *For every unbounded nested sequence of open balls B_n, $n \in \mathbb{N}$, in X with centers in Y, there exists at most one functional $f \in S_{X^*}$ such that $\operatorname{Re} f$ is bounded from above on $\bigcup_{n=1}^{\infty} B_n$.*

(d) *For every unbounded nested sequence of open balls B_n, $n \in \mathbb{N}$, in X with centers in Y, the union $\bigcup_{n=1}^{\infty} B_n$ is either the whole space X or an open half-space.*

(e) *For every unbounded nested sequence of open balls $B_n = B(y_n, r_n)$, $n \in \mathbb{N}$, in X with $y_n \in Y$ satisfying, for some $\delta > 0$,*

$$\|y_n\| \geq r_n - \delta \quad \text{for all } n \in \mathbb{N},$$

the union $\bigcup_{n=1}^{\infty} B_n$ is an open half-space in X.

(f) *For every sequence (y_n) in Y satisfying, for some $\delta > 0$,*

$$\|y_1\| = \|y_{n+1} - y_n\| = 1 \quad \text{and} \quad \|y_n\| \geq n - \delta \quad \text{for all } n \in \mathbb{N}, \tag{2.2}$$

the union $\bigcup_{n=1}^{\infty} B(y_n, n)$ is an open half-space in X.

(g) *There is a constant $\delta > 0$ such that, for every sequence (y_n) in Y satisfying (2.2), the union $\bigcup_{n=1}^{\infty} B(y_n, n)$ is an open half-space in X.*

The following lemma collects some simple facts about half-spaces and unbounded nested sequences of balls used in the proof of Theorem 2.1.

LEMMA 2.2. *Let $B_n = B(y_n, r_n)$, $n \in \mathbb{N}$, be an unbounded nested sequence of open balls in a Banach space X, and let $f \in X^* \setminus \{0\}$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$. Set*

$$B := \bigcup_{n=1}^{\infty} B_n \quad \text{and} \quad A := \{x \in X : \operatorname{Re} f(x) < \alpha\}.$$

(a) *If $B \subset A$, then*

$$\operatorname{Re} f\left(\frac{y_1 - y_n}{r_n}\right) \xrightarrow[n\to\infty]{} \|f\|. \tag{2.3}$$

(b) *$B \neq X$ if and only if there are $\delta > 0$ and $N \in \mathbb{N}$ such that*

$$\|y_n\| \geq r_n - \delta \quad \text{for all } n \geq N. \tag{2.4}$$

(c) *$\bigcup_{n=1}^{\infty} B(y_n, r_n + \varepsilon) = \{x \in X : d(x, B) < \varepsilon\}$.*

(d) *$\{x \in X : d(x, A) < \varepsilon\} = \{x \in X : \operatorname{Re} f(x) < \alpha + \varepsilon\|f\|\}$.*

Proof. (a) Assume that $B \subset A$. Then, for every $n \in \mathbb{N}$,

$$\operatorname{Re} f(y_n) + r_n\|f\| = \sup_{x \in B_n} \operatorname{Re} f(x) \leq \sup_{x \in B} \operatorname{Re} f(x) \leq \alpha.$$

Therefore, since $y_1 \in B_n$,

$$\|f\| - \frac{\alpha}{r_n} + \frac{\operatorname{Re} f(y_1)}{r_n} \leq \operatorname{Re} f\left(\frac{y_1 - y_n}{r_n}\right) \leq \|f\|.$$

This implies (2.3), because $r_n \to \infty$ as $n \to \infty$.

(b) Suppose $B \neq X$. Then there exists an $x \in X$ such that $\|x - y_n\| \geq r_n$ for every $n \in \mathbb{N}$, and therefore

$$\|y_n\| = \|(y_n - x) + x\| \geq \|y_n - x\| - \|x\| \geq r_n - \|x\|.$$

Conversely, suppose that there are $\delta > 0$ and $N \in \mathbb{N}$ satisfying (2.4). Since $r_n \to \infty$ as $n \to \infty$, one has $r_m \geq 2\delta$ for some $m \in \mathbb{N}$. Letting $n \geq \max\{N, m\}$, it suffices to show that $-y_m \notin B_n$, i.e., $\|y_n + y_m\| \geq r_n$. Since $B_m \subset B_n$ and thus $\|y_n - y_m\| \leq r_n - r_m$, it follows that

$$\begin{aligned} \|y_n + y_m\| &= \|2y_n - (y_n - y_m)\| \geq 2\|y_n\| - \|y_n - y_m\| \\ &\geq 2r_n - 2\delta - (r_n - r_m) = r_n + r_m - 2\delta \geq r_n. \end{aligned}$$

(c) Let $x \in \bigcup_{n=1}^{\infty} B(y_n, r_n + \varepsilon)$, i.e., $\|x - y_n\| < r_n + \varepsilon$ for some $n \in \mathbb{N}$. If $x = y_n$, then $d(x, B) = 0$. Assume that $x \neq y_n$. Since

$$\theta := \frac{\|x - y_n\|}{r_n + \varepsilon} < 1,$$

one has

$$z := y_n + \theta r_n \frac{x - y_n}{\|x - y_n\|} \in B(y_n, r_n) \subset B.$$

And since

$$x = y_n + \theta(r_n + \varepsilon) \frac{x - y_n}{\|x - y_n\|},$$

it follows that $\|x - z\| = \theta\varepsilon < \varepsilon$; hence $d(x, B) < \varepsilon$.

Conversely, let $d(x, B) < \varepsilon$. Then $\|x - z\| < \varepsilon$ for some $z \in B$. Letting $m \in \mathbb{N}$ be such that $z \in B(y_m, r_m)$, one has

$$\|x - y_m\| \leq \|x - z\| + \|z - y_m\| < r_m + \varepsilon;$$

therefore $x \in B(y_m, r_m + \varepsilon) \subset \bigcup_{n=1}^{\infty} B(y_n, r_n + \varepsilon)$.

(d) If $d(x, A) < \varepsilon$, i.e., $\|x - z\| < \varepsilon$ for some $z \in A$, then

$$\operatorname{Re} f(x) = \operatorname{Re} f(z) + \operatorname{Re} f(x - z) \leq \operatorname{Re} f(z) + \|f\|\,\|x - z\| < \alpha + \varepsilon\,\|f\|.$$

Conversely, let $x \in X$ be such that $\operatorname{Re} f(x) < \alpha + \varepsilon\|f\|$. Now, if $\operatorname{Re} f(x) \leq \alpha$, then $x \in \overline{A}$ and thus $d(x, A) = 0$.

Suppose that $\alpha < \operatorname{Re} f(x) < \alpha + \varepsilon\,\|f\|$; then

$$\operatorname{Re} f(x) = \alpha + \varepsilon_0\,\|f\|$$

for some $\varepsilon_0 \in (0, \varepsilon)$. It suffices to find a $u \in X$ such that $\operatorname{Re} f(u) = \alpha$ and $\|x - u\| < \varepsilon$, as in this case $u \in \overline{A}$, and so $d(x, A) = d(x, \overline{A}) \leq \|x - u\| < \varepsilon$.

Choose $x_0 \in S_X$ such that

$$\beta := \operatorname{Re} f(x_0) > \frac{\varepsilon_0}{\varepsilon}\,\|f\|,$$

and set

$$u := x + \frac{\alpha - \operatorname{Re} f(x)}{\beta}\, x_0.$$

Then $\operatorname{Re} f(u) = \alpha$, and

$$\|x-u\| = \frac{\operatorname{Re} f(x) - \alpha}{\beta} = \frac{\varepsilon_0 \|f\|}{\beta} < \varepsilon. \qquad \blacksquare$$

The proof of the implication (g)⇒(a) of Theorem 2.1 uses the following general lemma, inspired by an argument of Vlasov [V, pp. 37–38].

LEMMA 2.3 (see [OP2, Lemma 1.3]). *Let $Z \neq \{0\}$ be a Banach space and let $h \in Z^*$. For every $\varepsilon > 0$, there is a sequence (y_n) in Z such that $\|y_1\| = \|y_{n+1} - y_n\| = 1$ and $h(y_n) \geq (n-\varepsilon)\|h\|$ for all $n \in \mathbb{N}$.*

Proof of Theorem 2.1. (a)⇒(b). As Y is totally smooth, Y has property U and, by the Taylor–Foguel theorem, Y^* is strictly convex.

Let a sequence (y_n) in S_Y and functionals $f, g \in S_{X^*}$ be such that $\operatorname{Re} f(y_n), \operatorname{Re} g(y_n) \to 1$ as $n \to \infty$. We need to show that $f = g$.

Since $\operatorname{Re} \frac{1}{2}(f+g)(y_n) \to 1$ as $n \to \infty$, one has $\frac{1}{2} f\big|_Y + \frac{1}{2} g\big|_Y \in S_{Y^*}$, and thus $f|_Y = g|_Y$ (because Y^* is strictly convex). Since f and g are norm-preserving extensions of the functional $h := f|_Y = g|_Y \in S_{Y^*}$, one has $f = g$ (because Y has property U), as desired.

(b)⇒(c). Let $B_n := B(y_n, r_n)$, where $y_n \in Y$, $n \in \mathbb{N}$, be an unbounded nested sequence of open balls in X. Let $f, g \in S_{X^*}$ be such that the real parts $\operatorname{Re} f$ and $\operatorname{Re} g$ are bounded from above on $B := \bigcup_{n=1}^{\infty} B_n$, i.e.,

$$B \subset \{x \in X : \operatorname{Re} f(x) < \alpha\} \quad \text{and} \quad B \subset \{x \in X : \operatorname{Re} g(x) < \beta\}$$

for some $\alpha, \beta \in \mathbb{R}$. We need to show that $f = g$.

By Lemma 2.2(a), setting $z_n := (y_1 - y_n)/r_n$, $n \in \mathbb{N}$, one has

$$\operatorname{Re} f(z_n) \xrightarrow[n\to\infty]{} \|f\| = 1 \quad \text{and} \quad \operatorname{Re} g(z_n) \xrightarrow[n\to\infty]{} \|g\| = 1.$$

Since $\|z_n\| \leq 1$ for every $n \in \mathbb{N}$ (because $\|y_1 - y_n\| \leq r_n$ by the inclusion $B_1 \subset B_n$), it follows that $\|z_n\| \to 1$ as $n \to \infty$, and thus

$$\operatorname{Re} f\left(\frac{z_n}{\|z_n\|}\right) \xrightarrow[n\to\infty]{} 1 \quad \text{and} \quad \operatorname{Re} g\left(\frac{z_n}{\|z_n\|}\right) \xrightarrow[n\to\infty]{} 1.$$

By assumption (b), one has $f = g$, as desired.

(c)⇒(d). Let $B_n := B(y_n, r_n)$, where $y_n \in Y$, $n \in \mathbb{N}$, be an unbounded nested sequence of open balls in X such that $B := \bigcup_{n=1}^{\infty} B_n \neq X$. Then, since B is open, convex, and non-empty, as pointed out in [BFLM, beginning of Section 2], B is an open half-space, and (d) holds.

(d)⇒(e) is immediate from Lemma 2.2(b).

(e)⇒(f) follows by taking $r_n = n$ in (e).

(f)⇒(g) is obvious.

(g)⇒(a). Let $Z \neq \{0\}$ be a closed subspace of Y, let $h \in S_{Z^*}$, and let H_1 and H_2 be norm-preserving extensions of h to X. We shall prove that if

$x \in S_X$ and $0 < \varepsilon < \delta$, then

$$|\operatorname{Re} H_1(x) - \operatorname{Re} H_2(x)| \le 4\varepsilon. \tag{2.5}$$

This clearly implies that $H_1 = H_2$, and (a) follows.

Thus, let x and ε be fixed as above. Let (y_n) be a sequence from Z as in Lemma 2.3. Then

$$\|y_n\| \ge h(y_n) \ge n - \varepsilon \ge n - \delta,$$

hence (y_n) satisfies (2.2), and $\bigcup_{n=1}^{\infty} B(y_n, n)$ is an open half-space in X. By Lemma 2.2(c),(d), also

$$B_\varepsilon := \bigcup_{n=1}^{\infty} B(y_n, n+\varepsilon) \tag{2.6}$$

is an open half-space, say

$$B_\varepsilon = \{u \in X : \operatorname{Re} f(u) < \alpha\} \tag{2.7}$$

for some $f \in S_{X^*}$ and $\alpha \in \mathbb{R}$. Observe that since $0 \in B_\varepsilon$, one must have $\alpha > 0$. But then

$$\begin{aligned} x + B_\varepsilon &= \{u \in X : \operatorname{Re} f(u) < \operatorname{Re} f(x) + \alpha\}, \\ x - B_\varepsilon &= \{u \in X : \operatorname{Re} f(u) > \operatorname{Re} f(x) - \alpha\}. \end{aligned}$$

From (2.7) and Lemma 2.2(a), we see that $\|f|_Z\| = \|f\|$. Hence $f|_Z \ne 0$, and therefore

$$\{z \in Z : \operatorname{Re} f(x) - \alpha < \operatorname{Re} f(z) < \operatorname{Re} f(x) + \alpha\} \ne \emptyset,$$

meaning that

$$Z \cap (x + B_\varepsilon) \cap (x - B_\varepsilon) \ne \emptyset. \tag{2.8}$$

This allows us to choose $z \in Z$ such that $\pm(z - x) \in B_\varepsilon$. Recalling that $B(y_n, n+\varepsilon)$ is a nested sequence of balls, we have an $n_\varepsilon \in \mathbb{N}$ such that $\pm(z - x) \in B(y_{n_\varepsilon}, n_\varepsilon + \varepsilon)$. Now a rather standard reasoning follows: the functionals H_i, $i = 1, 2$, satisfy

$$\begin{aligned} n_\varepsilon + \varepsilon &> \|y_{n_\varepsilon} \pm (z - x)\| \ge \operatorname{Re} H_i(y_{n_\varepsilon} \pm (z - x)) \\ &= \operatorname{Re} h(y_{n_\varepsilon}) \pm \operatorname{Re}(h(z) - H_i(x)) \ge n_\varepsilon - \varepsilon \pm \operatorname{Re}(h(z) - H_i(x)), \end{aligned}$$

giving $|\operatorname{Re} h(z) - \operatorname{Re} H_i(x)| \le 2\varepsilon$, and (2.5) follows. ∎

Remark 2.4. The following criterion for property U from [OP1] yields a (non-self-contained) geometric proof of (g)⇒(a) of Theorem 2.1.

Theorem 2.5 (see [OP1, Theorem 1]). *Let Y be a closed subspace of a Banach space X. The following assertions are equivalent:*

(a) *Y has property U in X.*

(b) *There is a constant $\delta > 0$ such that whenever $0 < \varepsilon < \delta$, $x \in S_X$, and a sequence (y_n) of elements in Y satisfies* (2.2), *there are $n_\varepsilon \in \mathbb{N}$ and $y \in Y$ such that*

$$\|\pm y_{n_\varepsilon} + x - y\| \le n_\varepsilon + \varepsilon.$$

Indeed, assume that (g) holds. Let Z be a closed subspace of Y, let $0 < \varepsilon < \delta$, let $x \in S_X$, and let a sequence (y_n) of elements in Z satisfy (2.2). By Theorem 2.5, it suffices to show that

$$Z \cap \Big(\bigcup_{n=1}^{\infty} B(x + y_n, n + \varepsilon) \Big) \cap \Big(\bigcup_{n=1}^{\infty} B(x - y_n, n + \varepsilon) \Big) \neq \emptyset,$$

which is equivalent to (2.8) where B_ε is defined by (2.6). Condition (2.8) can be obtained as in the proof of (g)$\Rightarrow$(a) above.

In the proof of Theorem 2.1, we preferred to present a self-contained argument for (g)$\Rightarrow$(a).

Remark 2.6. Vlasov's proof and the proof of a local version of his theorem (the description of rotund points of the closed unit ball of the dual space in terms of sequences of nested balls) by Giles [G] relied, respectively, on the equivalence of the strict convexity of the dual space to the smoothness of every 2-dimensional quotient space of the original space, and on a local version of this equivalence. In [BHLT] (by Bandyopadhyay, Huang, Lin, and Troyanski), in [BHL], and in [BL], the description of rotund points was developed further: among other things, in [BHL, Theorem 3.6] a local version of the Taylor–Foguel theorem in terms of rotund points was presented, and in [BHLT, Theorem 6] and [BL, Theorem 2.1] a local version of the Vlasov theorem was proved without using the Taylor–Foguel theorem (unlike the proof of our Theorem 2.1 and the proof in [OP1]).

Remark 2.7. In [Su], Sullivan introduced a property of Banach spaces in terms of nested sequences of balls which is stronger than the condition from the Vlasov theorem, and which he called property V. Later, e.g., in [BR] and [BL], this property was called the *Vlasov property* to avoid confusion with Pełczyński's property (V). In [Su, Theorem 5.4], it is proved that X *has the Vlasov property if and only if it has property U in its bidual X^{**} and the dual space X^* is strictly convex* (thus, the Vlasov property for X means that, in our terms, X is totally smooth in X^{**}). Also (see [Su, Theorem 5.5] with further references to [S], [Ph2], and [W]), *for a separable space X, its dual X^* is separable if and only if X has an equivalent norm with the Vlasov property.*

Remark 2.8. In [HWW, Theorem III.4.6], it is shown that if X is an M-ideal in its bidual X^{**}, then X has an equivalent norm whose dual norm is strictly convex and under which X is still an M-ideal (we refer to the proof in [HWW] for further references). Thus, thanks to the Taylor–Foguel theorem and the transitivity of property U, the space X is totally smooth in its bidual.

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Eve Oja
Institute of Mathematics and Statistics
University of Tartu
J. Liivi 2
50409 Tartu, Estonia
and
Estonian Academy of Sciences
Kohtu 6
10130 Tallinn, Estonia
E-mail: eve.oja@ut.ee

Märt Põldvere, Tauri Viil
Institute of Mathematics and Statistics
University of Tartu
J. Liivi 2
50409 Tartu, Estonia
E-mail: mart.poldvere@ut.ee
tauriv@ut.ee