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The joint modulus of variation of metric space valued functions and pointwise selection principles

by

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Abstract. Given $T \subset \mathbb{R}$ and a metric space M, we introduce a nondecreasing sequence $\{\nu_n\}$ of pseudometrics on M^T (the set of all functions from T into M), called the joint modulus of variation. We prove that if two sequences $\{f_j\}$ and $\{g_j\}$ of functions from M^T are such that $\{f_j\}$ is pointwise precompact, $\{g_j\}$ is pointwise convergent, and $\limsup_{j\to\infty} \nu_n(f_j, g_j) = o(n)$ as $n \to \infty$, then $\{f_j\}$ admits a pointwise convergent subsequence whose limit is a conditionally regulated function. We illustrate the sharpness of this result by examples (in particular, the assumption on the lim sup is necessary for uniformly convergent sequences $\{f_j\}$ and $\{g_j\}$, and 'almost necessary' when they converge pointwise) and show that most of the known Helly-type pointwise selection theorems are its particular cases.

1. Introduction. The purpose of this paper is to present a new sufficient condition (which is almost necessary) on a pointwise precompact sequence $\{f_j\} \equiv \{f_j\}_{j=1}^{\infty}$ of functions f_j mapping a subset T of the real line $\mathbb{R}$ into a metric space (M, d), under which the sequence admits a pointwise convergent subsequence. The historically first result in this direction is the classical *Helly Selection Principle*, in which the assumptions are as follows: $T = [a, b]$ is a closed interval, $M = \mathbb{R}$, and $\{f_j\}$ is uniformly bounded and consists of monotone functions ([29], [31, II.8.9–10], [40, VIII.4.2], and [10, Theorem 1.3] for $T \subset \mathbb{R}$ arbitrary). Since a real function on T of bounded (Jordan) variation is the difference of two nondecreasing bounded functions, Helly's theorem extends to uniformly bounded sequences of functions whose Jordan variations are uniformly bounded. Further generalizations of the latter pointwise selection principle are concerned

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with replacement of Jordan variation by more general notions of variation [2, 3, 6–10, 15, 16, 22, 25, 26, 33, 34, 39, 42, 44]. In all these papers, the pointwise limit of the extracted subsequence of $\{f_j\}$ is a function of bounded generalized variation (in the corresponding sense), and so it is a regulated function (with finite one-sided limits at all points of the domain). Note that pointwise selection principles (or sequential compactness in the topology of pointwise convergence) and regulated functions are of importance in real analysis [28, 31, 40], stochastic analysis and generalized integration [38], optimization [1, 37], set-valued analysis [2, 10, 19, 20, 30], and other fields.

A unified approach to the diverse selection principles mentioned above was proposed in [11, 12]. It is based on the notion of *modulus of variation* of a function introduced in [4, 5] (see also [28, 11.3.7]) and does not refer to the uniform boundedness of variations of any kind, and so can be applied to sequences of non-regulated functions. However, the pointwise limit of the extracted subsequence of $\{f_j\}$ is *again* a regulated function. In order to clarify this situation and expand the amount of sequences having pointwise convergent subsequences, we define the notion of the *joint modulus of variation* for metric space valued functions: this is a certain sequence $\{\nu_n\}$ of pseudometrics on the product set M^T (of all functions from T into M). Making use of $\{\nu_n\}$, we obtain a powerful pointwise selection principle (Theorem 1 in Section 2). Setting $g_j = c$ for all $j \in \mathbb{N}$, where $c : T \to M$ is a constant function, we get the selection principle from [11], which already contains all selection principles alluded to above as particular cases. In contrast to results from [11, 12], the pointwise limit f from Theorem 1 may not be regulated in general—this depends on the limit function g, namely, since $\nu_n(f, g) = o(n)$, the function f is only conditionally regulated with respect to g (for short, *g-regulated*). In particular, if $g = c$, then f is regulated in the usual sense.

Finally, we point out that by following the ideas of [13], Theorem 1 may be extended to sequences of functions with values in a *uniform space M*.

The paper is organized as follows. In Section 2, we present necessary definitions and our main result, Theorem 1. In Section 3, we establish essential properties of the joint modulus of variation, which are needed in the proof of Theorem 1 in Section 5. Section 4 is devoted to the study of g-regulated (and in particular regulated) functions. In the final Section 6, we extend the Helly-type selection theorems of [25] and [22, 33] by exploiting Theorem 1.

2. Main result. Let $\emptyset \neq T \subset \mathbb{R}$, let (M, d) be a metric space with metric d, and let M^T denote the set of all functions $f : T \to M$. The letter c stands for a *constant* function $c : T \to M$.

The *joint oscillation* of two functions $f, g \in M^T$ is the quantity

$$|(f, g)(T)| = \sup\big\{|(f, g)(\{s, t\})| : s, t \in T\big\} \in [0, \infty],$$

where

$$|(f,g)(\{s,t\})| = \sup_{z\in M}\bigl|d(f(s),z)+d(g(t),z)-d(f(t),z)-d(g(s),z)\bigr| \tag{2.1}$$

is the *joint increment* of f and g on the two-point set $\{s,t\}\subset T$, for which the following two inequalities hold:

$$|(f,g)(\{s,t\})| \le d(f(s),f(t)) + d(g(s),g(t)), \tag{2.2}$$

$$|(f,g)(\{s,t\})| \le d(f(s),g(s)) + d(f(t),g(t)). \tag{2.3}$$

Since $|(f,c)(\{s,t\})| = d(f(s),f(t))$ (= the increment of f on $\{s,t\}\subset T$) is independent of c, the quantity $|f(T)| = |(f,c)(T)|$ is the usual *oscillation* of f on T, also known as the *diameter of the image* $f(T) = \{f(t) : t\in T\}\subset M$. Clearly, by (2.2), $|(f,g)(T)| \le |f(T)| + |g(T)|$.

We denote by $\mathrm{B}(T;M) = \{f\in M^T : |f(T)|<\infty\}$ the family of all *bounded* functions on T equipped with the *uniform metric* d_∞ given by

$$d_\infty(f,g) = \sup_{t\in T} d(f(t),g(t)) \quad \text{for } f,g\in \mathrm{B}(T;M)$$

(d_∞ is an *extended* metric on M^T, i.e., may assume the value ∞). We have

$$d_\infty(f,g) \le d(f(s),g(s)) + |f(T)| + |g(T)| \quad \text{for all } s\in T$$

and, by (2.3), $|(f,g)(T)| \le 2d_\infty(f,g)$.

If $n\in\mathbb{N}$, we write $\{I_i\}_{i=1}^n \prec T$ to denote a collection of n two-point subsets $I_i = \{s_i,t_i\}$ of T ($i=1,\dots,n$) such that $s_1 < t_1 \le s_2 < t_2 \le \dots \le s_{n-1} < t_{n-1} \le s_n < t_n$ (so that the intervals $[s_1,t_1],\dots,[s_n,t_n]$ with end-points in T are non-overlapping). We say that a collection $\{I_i\}_{i=1}^n\prec T$ with $I_i=\{s_i,t_i\}$ is a *partition* of T if (setting $t_0 = s_1$) $s_i = t_{i-1}$ for all $i=1,\dots,n$, which is written as $\{t_i\}_{i=0}^n \prec T$.

The *joint modulus of variation* of $f,g\in M^T$ is the sequence $\{\nu_n(f,g)\}_{n=1}^\infty \subset [0,\infty]$ defined by

$$\nu_n(f,g) = \sup\Bigl\{\sum_{i=1}^n |(f,g)(I_i)| : \{I_i\}_{i=1}^n \prec T\Bigr\} \quad \text{for all } n\in\mathbb{N}, \tag{2.4}$$

where $|(f,g)(I_i)| = |(f,g)(\{s_i,t_i\})|$ is the quantity from (2.1) if $I_i = \{s_i,t_i\}$ (for finite T with $\#(T)\ge 2$, we make use of (2.4) for $n\le \#(T)-1$, and set $\nu_n(f,g) = \nu_{\#(T)-1}(f,g)$ for all $n > \#(T)-1$).

Note that, given $f,g\in M^T$, we have $\nu_1(f,g) = |(f,g)(T)|$ and

$$\nu_1(f,g) \le \nu_n(f,g) \le n\nu_1(f,g) \quad \text{for all } n\in\mathbb{N}. \tag{2.5}$$

Further properties of the joint modulus of variation are presented in Section 3.

For a sequence $\{f_j\}\subset M^T$ and $f\in M^T$, we write: (a) $f_j\to f$ on T to denote the *pointwise* (or *everywhere*) *convergence* of $\{f_j\}$ to f (that is,

$\lim_{j\to\infty} d(f_j(t), f(t)) = 0$ for all $t \in T$); (b) $f_j \rightrightarrows f$ on T to denote the *uniform convergence* of $\{f_j\}$ to f, meaning, as usual, that $\lim_{j\to\infty} d_\infty(f_j, f) = 0$. Uniform convergence implies pointwise convergence, but not vice versa. Recall that a sequence $\{f_j\} \subset M^T$ is said to be *pointwise precompact* on T if the closure in M of the set $\{f_j(t) : j \in \mathbb{N}\}$ is compact for all $t \in T$.

Making use of E. Landau's notation, given a sequence $\{\mu_n\}_{n=1}^\infty \subset \mathbb{R}$, we write $\mu_n = o(n)$ when $\lim_{n\to\infty} \mu_n/n = 0$.

Our main result, a *pointwise selection principle* for metric space valued functions in terms of the joint modulus of variation, is as follows.

THEOREM 1. *Let $\emptyset \neq T \subset \mathbb{R}$ and (M, d) be a metric space. Suppose $\{f_j\}, \{g_j\} \subset M^T$ are two sequences of functions such that*

(a) *$\{f_j\}$ is pointwise precompact on T,*
(b) *$\{g_j\}$ is pointwise convergent on T to a function $g \in M^T$,*

and

$$\mu_n \equiv \limsup_{j\to\infty} \nu_n(f_j, g_j) = o(n). \tag{2.6}$$

Then there is a subsequence of $\{f_j\}$ which converges pointwise on T to a function $f \in M^T$ such that $\nu_n(f, g) \le \mu_n$ for all $n \in \mathbb{N}$.

This theorem will be proved in Section 5. Now, a few remarks are in order. Given $f \in M^T$ and a constant function $c : T \to M$, the quantity

$$\nu_n(f) \equiv \nu_n(f, c) = \sup\Big\{\sum_{i=1}^n d(f(s_i), f(t_i)) : \{I_i\}_{i=1}^n \prec T\Big\} \tag{2.7}$$

(with $I_i = \{s_i, t_i\}$) is independent of c, and the sequence $\{\nu_n(f)\}_{n=1}^\infty \subset [0, \infty]$ is known as the *modulus of variation* of f in the sense of Chanturiya [4, 5, 11–13, 28]. It characterizes regulated (or proper) functions on $T = [a, b]$ as follows. We say that $f : [a, b] \to M$ is *regulated* and write $f \in \mathrm{Reg}([a, b]; M)$ if $d(f(s), f(t)) \to 0$ as $s, t \to \tau - 0$ for every $a < \tau \le b$, and $d(f(s), f(t)) \to 0$ as $s, t \to \tau' + 0$ for every $a \le \tau' < b$ (and so, by Cauchy's criterion, the one-sided limits $f(\tau - 0), f(\tau' + 0) \in M$ exist provided M is complete). We have

$$\mathrm{Reg}([a, b]; M) = \{f \in M^{[a,b]} : \nu_n(f) = o(n)\} \tag{2.8}$$

(more general characterizations for dense subsets T of $[a, b]$ can be found in [12, 13]). A certain relationship between characterizations of regulated functions and pointwise selection principles is exhibited in [18].

3. The joint modulus of variation. We begin by studying the joint increment (2.1), whose properties are gathered in the following lemma.

LEMMA 1. *Given $f, g, h \in M^T$ and $s, t \in T$, we have:*

(a) $|(f,f)(\{s,t\})| = 0$;
(b) $|(f,g)(\{s,t\})| = |(g,f)(\{s,t\})|$;
(c) $|(f,g)(\{s,t\})| \le |(f,h)(\{s,t\})| + |(h,g)(\{s,t\})|$;
(d) $d(f(s),f(t)) \le d(g(s),g(t)) + |(f,g)(\{s,t\})|$;
(e) $d(f(t),g(t)) \le d(f(s),g(s)) + |(f,g)(\{s,t\})|$.

Proof. Properties (a)–(c), showing that $(f,g) \mapsto |(f,g)(\{s,t\})|$ is a *pseudometric* on M^T, are straightforward. To establish (d) and (e), take into account the equality $d(x,y) = \max_{z\in M} |d(x,z) - d(y,z)|$. ∎

REMARK 1. (a) If $|(f,g)(\{s,t\})| = 0$, then (d), (e), and (b) imply $d(f(s),f(t)) = d(g(s),g(t))$ and $d(f(t),g(t)) = d(f(s),g(s))$. In addition to Lemma 1, the function $(s,t) \mapsto |(f,g)(\{s,t\})|$ is a pseudometric on T.

(b) If $F(z)$ denotes the absolute value under the supremum sign in (2.1), then $F : M \to [0,\infty)$ and $|F(z) - F(z_0)| \le 4d(z,z_0)$ for all $z, z_0 \in M$.

(c) By Lemma 1(d), $|f(T)| \le |g(T)| + |(f,g)(T)| = |g(T)| + \nu_1(f,g)$. So,

$$\big||f(T)| - |g(T)|\big| \le |(f,g)(T)| \le |f(T)| + |g(T)|, \qquad f, g \in \mathrm{B}(T;M).$$

Moreover, it follows from Lemma 1(e) that

$$d_\infty(f,g) \le d(f(s),g(s)) + |(f,g)(T)| \le 3d_\infty(f,g) \quad \text{for all } s \in T.$$

(d) Suppose the triple $(M,d,+)$ is a *metric semigroup* [10, Section 4], i.e., (M,d) is a metric space, $(M,+)$ is an Abelian semigroup with addition $+$, and $d(x,y) = d(x+z, y+z)$ for all $x,y,z \in M$. Then the joint increment (2.1) may be alternatively replaced by

$$|(f,g)(\{s,t\})| = d(f(s)+g(t), f(t)+g(s)). \tag{3.1}$$

The joint modulus of variation (2.4) involving (3.1) was employed in [17]. Furthermore, if $(M,\|\cdot\|)$ is a normed linear space (over $\mathbb{R}$ or $\mathbb{C}$), we may set

$$|(f,g)(\{s,t\})| = \|f(s)+g(t)-f(t)-g(s)\| = \|(f-g)(s)-(f-g)(t)\|. \tag{3.2}$$

Quantities (3.1) and (3.2) have the same properties as (2.1): see (2.2), (2.3), Lemma 1 and Remark 1(a). In the following, we make use of the more general quantity (2.1).

If $f, g \in M^T$, $n \in \mathbb{N}$ and $\emptyset \ne E \subset T$, we set $\nu_n(f,g;E) = \nu_n(f|_E, g|_E)$, where $f|_E \in M^E$ is the restriction of f to E. Accordingly, $\nu_n(f,g) = \nu_n(f,g;T)$.

The following properties of the joint modulus of variation are immediate. The sequence $\{\nu_n(f,g)\}_{n=1}^\infty$ is nondecreasing, $\nu_{n+m}(f,g) \le \nu_n(f,g) + \nu_m(f,g)$ for all $n, m \in \mathbb{N}$, and $\nu_n(f,g;E) \le \nu_n(f,g;T)$ provided $n \in \mathbb{N}$ and $E \subset T$. It follows from (2.4) and Lemma 1(a)–(c) that, for every $n \in \mathbb{N}$, the function $(f,g) \mapsto \nu_n(f,g)$ is a *pseudometric* on M^T (possibly assuming infinite values), and in particular (cf. (2.4) and (2.7))

$$\nu_n(f,g) \le \nu_n(f) + \nu_n(g) \quad \text{and} \quad \nu_n(f) \le \nu_n(g) + \nu_n(f,g) \tag{3.3}$$

for all $n \in \mathbb{N}$ and $f, g \in M^T$. Furthermore, if $f, g \in \mathrm{B}(T; M)$, then, by (2.5), the sequence $\{\nu_n(f,g)/n\}_{n=1}^{\infty}$ is bounded in $[0, \infty)$.

Essential properties of the joint modulus of variation are presented in

Lemma 2. *Given $n \in \mathbb{N}$, $f, g \in M^T$, and $\emptyset \ne E \subset T$, we have*:

(a) $|(f,g)(\{s,t\})| + \nu_n(f,g;E_s^-) \le \nu_{n+1}(f,g;E_t^-)$ *for all $s, t \in E$ with $s \le t$, where $E_\tau^- = (-\infty, \tau] \cap E$ for $\tau \in E$*;

(b) $\nu_{n+1}(f,g;E) \le \nu_n(f,g;E) + \nu_{n+1}(f,g;E)/(n+1)$;

(c) *if $\{f_j\}, \{g_j\} \subset M^T$ are such that $f_j \to f$ and $g_j \to g$ on E, then* $\nu_n(f,g;E) \le \liminf_{j\to\infty} \nu_n(f_j,g_j;E)$;

(d) *if $\{f_j\}, \{g_j\} \subset M^T$ are such that $f_j \rightrightarrows f$ and $g_j \rightrightarrows g$ on E, then* $\nu_n(f,g;E) = \lim_{j\to\infty} \nu_n(f_j,g_j;E)$.

Proof. (a) We may assume that $s < t$. Let $\{I_i\}_{i=1}^n \prec E_s^-$. Setting $I_0 = \{s,t\}$, we find $\{I_i\}_{i=0}^n \prec E_t^-$, and so

$$|(f,g)(I_0)| + \sum_{i=1}^n |(f,g)(I_i)| \le \nu_{n+1}(f,g;E_t^-).$$

The inequality in (a) follows by taking the supremum over all $\{I_i\}_{i=1}^n \prec E_s^-$.

(b) We may assume that $\nu_{n+1}(f,g;E)$ is finite, and apply the idea from [5, Lemma]. Given $\varepsilon > 0$, there is $\{I_i\}_{i=1}^{n+1} \prec E$ (depending on ε) such that

$$\sum_{i=1}^{n+1} |(f,g)(I_i)| \le \nu_{n+1}(f,g;E) \le \sum_{i=1}^{n+1} |(f,g)(I_i)| + \varepsilon.$$

If we set $a_0 = \min_{1\le i\le n+1} |(f,g)(I_i)|$, the left-hand inequality implies $(n+1)a_0 \le \nu_{n+1}(f,g;E)$. The right-hand inequality gives

$$\nu_{n+1}(f,g;E) \le \nu_n(f,g;E) + a_0 + \varepsilon,$$

from which our inequality follows due to the arbitrariness of $\varepsilon > 0$.

(c) First, we note that, given $j \in \mathbb{N}$ and $s, t \in T$, we have

$$\begin{aligned} \big||(f_j,g_j)(\{s,t\})| - |(f,g)(\{s,t\})|\big| &\le d(f_j(s), f(s)) + d(f_j(t), f(t)) \\ &\quad + d(g_j(s), g(s)) + d(g_j(t), g(t)). \end{aligned} \tag{3.4}$$

In fact, Lemma 1(c) and inequality (2.3) imply

$$\begin{aligned} |(f_j,g_j)(\{s,t\})| &\le |(f_j,f)(\{s,t\})| + |(f,g)(\{s,t\})| + |(g,g_j)(\{s,t\})| \\ &\le d(f_j(s), f(s)) + d(f_j(t), f(t)) + |(f,g)(\{s,t\})| \\ &\quad + d(g(s), g_j(s)) + d(g(t), g_j(t)). \end{aligned} \tag{3.5}$$

Exchanging f_j and f as well as g_j and g, we obtain (3.4).

From the pointwise convergence of $\{f_j\}$ and $\{g_j\}$ and (3.4), we find

$$\lim_{j\to\infty} |(f_j, g_j)(\{s,t\})| = |(f,g)(\{s,t\})| \quad \text{for all } s,t \in E.$$

By definition (2.4), given $\{I_i\}_{i=1}^n \prec E$, we have

$$\sum_{i=1}^n |(f_j, g_j)(I_i)| \le \nu_n(f_j, g_j; E) \quad \text{for all } j \in \mathbb{N}.$$

Hence, by taking lim inf as $j \to \infty$,

$$\sum_{i=1}^n |(f,g)(I_i)| \le \liminf_{j\to\infty} \nu_n(f_j, g_j; E). \tag{3.6}$$

Since $\{I_i\}_{i=1}^n \prec E$ is arbitrary, it remains to take into account (2.4).

(d) It follows from (3.5) that, for any $s,t \in E$ and $j \in \mathbb{N}$,

$$|(f_j, g_j)(\{s,t\})| \le 2 \sup_{\tau\in E} d(f_j(\tau), f(\tau)) + |(f,g)(\{s,t\}) + 2 \sup_{\tau\in E} d(g_j(\tau), g(\tau)),$$

and so definition (2.4) implies

$$\begin{aligned}\nu_n(f_j, g_j; E) \le 2n \sup_{\tau\in E} d(f_j(\tau), f(\tau)) &+ \nu_n(f,g;E) \\ &+ 2n \sup_{\tau\in E} d(g_j(\tau), g(\tau))\end{aligned} \tag{3.7}$$

for all $j \in \mathbb{N}$. Hence

$$\limsup_{j\to\infty} \nu_n(f_j, g_j; E) \le \nu_n(f,g;E).$$

Now, (d) is a consequence of Lemma 2(c). ∎

Remark 2. If the value $\nu_1(f,g;E) = |(f,g)(E)|$ (see (2.5)) is finite for an $E \subset T$ (e.g., when $f,g \in \mathrm{B}(E;M)$), inequality in Lemma 2(b) is equivalent to

$$\frac{\nu_{n+1}(f,g;E)}{n+1} \le \frac{\nu_n(f,g;E)}{n}.$$

Thus, the limit $\lim_{n\to\infty} \nu_n(f,g;E)/n$ always exists in $[0,\infty)$. This also follows from the subadditivity property $\nu_{n+m}(f,g;E) \le \nu_n(f,g;E) + \nu_m(f,g;E)$ mentioned on p. 41.

4. Conditionally regulated functions. Since $\nu_n = \nu_n(\cdot,\cdot)$ is an (extended) pseudometric on M^T, we may introduce an equivalence relation $\sim$ on M^T as follows: given $f, g \in M^T$, we set

$$f \sim g \quad \text{if and only if} \quad \nu_n(f,g) = o(n).$$

The equivalence class $\mathrm{R}(g) = \{f \in M^T : f \sim g\}$ of a function $g \in M^T$ is called the *regularity class* of g, and any representative $f \in \mathrm{R}(g)$ is called a *conditionally regulated* or, more precisely, a *g-regulated* function. This terminology is justified by (2.8): in the framework of the product set $M^{[a,b]}$, we have $\mathrm{Reg}([a,b];M)=\mathrm{R}(c)$ for any constant function $c:[a,b]\to M$.

Note that, in Theorem 1, the condition '$\nu_n(f,g)\le\mu_n$ for all $n\in\mathbb{N}$' means that $f\in\mathrm{R}(g)$, and so the class $\mathrm{R}(g)$ is worth studying in more detail.

THEOREM 2. *Given $g\in M^T$, we have:*

(a) $g\in\mathrm{B}(T;M)$ *if and only if* $\mathrm{R}(g)\subset\mathrm{B}(T;M)$*;*

(b) $\mathrm{R}(g)$ *is closed with respect to uniform convergence, but not with respect to pointwise convergence in general;*

(c) *if* (M,d) *is a complete metric space, then so is* $(\mathrm{R}(g),d_\infty)$.

Proof. (a) The sufficiency is clear, because $g\in\mathrm{R}(g)$. Now, suppose that $g\in\mathrm{B}(T;M)$, so that, by (2.7), $\nu_1(g)=|(g,c)(T)|=|g(T)|<\infty$. Given $f\in\mathrm{R}(g)$, $\nu_n(f,g)=o(n)$, and so $\nu_{n_0}(f,g)\le n_0$ for some $n_0\in\mathbb{N}$. It follows from (3.3) and (2.5) that

$$|f(T)|=\nu_1(f)\le\nu_1(g)+\nu_1(f,g)\le|g(T)|+\nu_{n_0}(f,g)\le|g(T)|+n_0<\infty,$$

which implies $f\in\mathrm{B}(T;M)$.

(b) We have to show that if $\{f_j\}\subset\mathrm{R}(g)$ and $f_j\rightrightarrows f$ on T with $f\in M^T$, then $f\in\mathrm{R}(g)$. We will prove a little more: if $\{f_j\},\{g_j\}\subset M^T$, $f_j\in\mathrm{R}(g_j)$ for all $j\in\mathbb{N}$, and $f_j\rightrightarrows f$ and $g_j\rightrightarrows g$ on T with $f,g\in M^T$, then $f\in\mathrm{R}(g)$ (the previous assertion follows if $g_j=g$ for all $j\in\mathbb{N}$). In fact, exchanging f_j and f, and g_j and g, in (3.7), we get

$$\frac{\nu_n(f,g)}{n}\le 2d_\infty(f,f_j)+\frac{\nu_n(f_j,g_j)}{n}+2d_\infty(g,g_j),\qquad n,j\in\mathbb{N}.$$

By the uniform convergence of $\{f_j\}$ and $\{g_j\}$, given $\varepsilon>0$, there is $j_0=j_0(\varepsilon)\in\mathbb{N}$ such that $d_\infty(f,f_{j_0})\le\varepsilon$ and $d_\infty(g,g_{j_0})\le\varepsilon$. Since f_{j_0} is in $\mathrm{R}(g_{j_0})$, we have $\nu_n(f_{j_0},g_{j_0})=o(n)$, and so there exists $n_0=n_0(\varepsilon)\in\mathbb{N}$ such that $\nu_n(f_{j_0},g_{j_0})/n\le\varepsilon$ for all $n\ge n_0$. The estimate above with $j=j_0$ implies $\nu_n(f,g)/n\le 5\varepsilon$, $n\ge n_0$, which means that $\nu_n(f,g)=o(n)$ and $f\in\mathrm{R}(g)$.

As for pointwise convergence, consider a sequence of real step functions converging pointwise to the Dirichlet function (= the characteristic function of the rationals $\mathbb{Q}$) on $T=[0,1]$ (see [11, Examples 4, 5] and Example 2(a) in Section 5).

(c) First, we show that $d_\infty(f,f')<\infty$ for all $f,f'\in\mathrm{R}(g)$. In fact, since $f\sim f'$, we have $\nu_n(f,f')=o(n)$, and so $\nu_{n_0}(f,f')\le n_0$ for some $n_0\in\mathbb{N}$. Given $s\in T$, it follows from Remark 1(c) and (2.5) that

$$d_\infty(f,f')\le d(f(s),f'(s))+\nu_1(f,f')\le d(f(s),f'(s))+\nu_{n_0}(f,f')<\infty.$$

The metric axioms for d_∞ on $\mathrm{R}(g)$ are verified in a standard way.

In order to prove that $\mathrm{R}(g)$ is complete, suppose $\{f_j\}\subset\mathrm{R}(g)$ is a Cauchy sequence, i.e., $d_\infty(f_j,f_k)\to 0$ as $j,k\to\infty$. Since $d(f_j(t),f_k(t))\le d_\infty(f_j,f_k)$

for all $t \in T$ and (M,d) is complete, there exists $f \in M^T$ such that $f_j \to f$ on T. Noting that $f_j \to f_j$ and $f_k \to f$ on T as $k \to \infty$ (and arguing as in (3.6)), we get

$$d_\infty(f_j,f) \le \liminf_{k\to\infty} d_\infty(f_j,f_k) = \lim_{k\to\infty} d_\infty(f_j,f_k) < \infty \quad \text{for all } j \in \mathbb{N}.$$

Since $\{f_j\}$ is d_∞-Cauchy, we find

$$\limsup_{j\to\infty} d_\infty(f_j,f) \le \lim_{j\to\infty}\lim_{k\to\infty} d_\infty(f_j,f_k) = 0.$$

Thus, $\lim_{j\to\infty} d_\infty(f_j,f) = 0$, and so $f_j \rightrightarrows f$ on T. Now (b) implies that $f \in \mathrm{R}(g)$. ■

A traditionally important class of regulated functions is the space of functions of bounded Jordan variation, $\mathrm{BV}(T;M)$, which is introduced by means of the joint modulus of variation as follows.

Since the sequence $\{\nu_n(f,g)\}_{n=1}^\infty$ is nondecreasing for all $f,g \in M^T$, the quantity (finite or not) $V(f,g) = \lim_{n\to\infty} \nu_n(f,g) = \sup_{n\in\mathbb{N}} \nu_n(f,g)$ is called the *joint variation* of f and g on T. The value $V(f) \equiv V(f,c)$ is independent of the constant function $c : T \to M$ and is the usual Jordan *variation* of f on T:

$$V(f) = \sup\Big\{\sum_{i=1}^n d(f(t_i), f(t_{i-1})) : n \in \mathbb{N} \text{ and } \{t_i\}_{i=0}^n \prec T\Big\},$$

the supremum being taken over all partitions $\{t_i\}_{i=0}^n$ of T (cf. Section 2). The set $\mathrm{BV}(T;M) = \{f \in M^T : V(f) < \infty\}$ is contained in $\mathrm{B}(T;M) \cap \mathrm{R}(c)$ (in fact, $|f(T)| = \nu_1(f) \le V(f)$ and $\nu_n(f,c)/n \le V(f)/n$ for all $f \in \mathrm{BV}(T;M)$).

The following notion of *ε-variation* $V_\varepsilon(f)$, due to Fraňková [25, Section 3], provides an alternative characterization (cf. (2.8)) of regulated functions: given $f \in M^T$ and $\varepsilon > 0$, set

$$V_\varepsilon(f) = \inf\{V(g) : g \in \mathrm{BV}(T;M) \text{ and } d_\infty(f,g) \le \varepsilon\} \tag{4.1}$$

($\inf\emptyset = \infty$). It was shown in [25, Proposition 3.4] (for $T = [a,b]$) that

$$\mathrm{Reg}([a,b];M) = \{f \in M^{[a,b]} : V_\varepsilon(f) < \infty \text{ for all } \varepsilon > 0\} \tag{4.2}$$

(although in [25] it is assumed that $M = \mathbb{R}^N$, the proof carries over to any metric space M, cf. [11, Lemma 3]).

The notion of ε-variation will be needed in Section 6.

EXAMPLE 1. Given $x,y \in M$ with $x \ne y$, let $f = \mathcal{D}_{x,y} : T = [0,1] \to M$ be the Dirichlet-type function of the form

$$\mathcal{D}_{x,y}(t) = \begin{cases} x & \text{if } t \in [0,1] \text{ is rational,} \\ y & \text{if } t \in [0,1] \text{ is irrational.} \end{cases} \tag{4.3}$$

Clearly, $f \notin \mathrm{Reg}([0,1];M)$. Moreover (cf. (4.2)),

$$V_\varepsilon(f) = \infty \quad \text{if } 0 < \varepsilon < d(x,y)/2, \qquad V_\varepsilon(f) = 0 \quad \text{if } \varepsilon \ge d(x,y). \tag{4.4}$$

To see this, first note that, given $g \in M^{[0,1]}$, the inequality $d_\infty(f,g) \le \varepsilon$ is equivalent to the following two conditions:

$$d(x,g(s)) \le \varepsilon \quad \forall s \in [0,1] \cap \mathbb{Q}, \qquad d(y,g(t)) \le \varepsilon \quad \forall t \in [0,1] \setminus \mathbb{Q}. \tag{4.5}$$

Suppose $0 < \varepsilon < d(x,y)/2$. To show that $d_\infty(f,g) \le \varepsilon$ implies $V(g) = \infty$, we let $n \in \mathbb{N}$, and let $\{t_i\}_{i=0}^{2n} \prec [0,1]$ be a partition of $[0,1]$ such that the points $\{t_{2i}\}_{i=0}^{n}$ are rational and the points $\{t_{2i-1}\}_{i=1}^{n}$ are irrational. By the triangle inequality for d and (4.5), we get

$$\begin{aligned} V(g) &\ge \sum_{i=1}^{2n} d(g(t_i), g(t_{i-1})) \ge \sum_{i=1}^{n} d(g(t_{2i}), g(t_{2i-1})) \\ &\ge \sum_{i=1}^{n} \big(d(x,y) - d(x,g(t_{2i})) - d(g(t_{2i-1}),y)\big) \ge n\big(d(x,y) - 2\varepsilon\big). \end{aligned}$$

If $\varepsilon \ge d(x,y)$, we set $g(t) = x$ (or $g(t) = y$) for all $t \in [0,1]$, so that (4.5) is satisfied and $V(g) = 0$. Thus, $V_\varepsilon(f) = 0$.

The second assertion in (4.4) can be refined, provided

$$d(x,y)/2 = \max\{d(x,z_0), d(y,z_0)\} \quad \text{for some } z_0 \in M. \tag{4.6}$$

In fact, we may set $g(t) = z_0$ for all $t \in [0,1]$, so that (4.5) holds whenever $d(x,y)/2 \le \varepsilon$, and $V(g) = 0$. This implies $V_\varepsilon(f) = 0$ for all $\varepsilon \ge d(x,y)/2$.

A few remarks concerning condition (4.6) are in order. Since

$$d(x,y) \le d(x,z) + d(z,y) \le 2\max\{d(x,z), d(y,z)\} \quad \text{for all } z \in M,$$

condition (4.6) is a certain form of 'convexity' of M (which is not restrictive for our purposes). For instance, if $(M, \|\cdot\|)$ is a normed linear space with $d(x,y) = \|x-y\|$, we may set $z_0 = (x+y)/2$. More generally, by Menger's Theorem ([36], [27, Example 2.7]), if a metric space (M,d) is complete and *metrically convex* (i.e., given $x,y \in M$ with $x \ne y$, there is $z \in M$ such that $x \ne z \ne y$ and $d(x,y) = d(x,z) + d(z,y)$), then, for any $x,y \in M$, there is an isometry $\varphi : [0, d(x,y)] \to M$ such that $\varphi(0) = x$ and $\varphi(d(x,y)) = y$. In this case, we set $z_0 = \varphi(d(x,y)/2)$. More examples of metrically convex metric spaces can be found in [21, 24].

Finally, if $M = \{x,y\}$, then condition (4.6) is not satisfied, and we have $V_\varepsilon(f) = \infty$ for all $0 < \varepsilon < d(x,y)$, which is a consequence of (4.5).

5. Proof of the main result

Proof of Theorem 1. With no loss of generality we may assume that T is uncountable; otherwise, by assumption (a) and the standard Cantor diagonal procedure, we extract a pointwise convergent subsequence of $\{f_j\}$ and apply Lemma 2(c). Note that μ_n is finite for all $n \in \mathbb{N}$: in fact, $\mu_n \le n$ whenever $n \ge n_0$ for some $n_0 \in \mathbb{N}$, and since $n \mapsto \nu_n(f_j, g_j)$ is nondecreasing for all $j \in \mathbb{N}$, we have $\mu_n \le \mu_{n_0} \le n_0$ for all $1 \le n < n_0$.

For clarity we divide the rest of the proof into four steps.

STEP 1. Let us show that there is a subsequence of $\{j\}_{j=1}^{\infty}$, again denoted by $\{j\}$, and a nondecreasing sequence $\{\alpha_n\}_{n=1}^{\infty}\subset[0,\infty)$ such that

$$\lim_{j\to\infty}\nu_n(f_j,g_j)=\alpha_n\le\mu_n\quad\text{for all } n\in\mathbb{N}. \tag{5.1}$$

We set $\alpha_1=\mu_1$. The definition (2.6) of μ_1 implies that there is an increasing sequence $\{J_1(j)\}_{j=1}^{\infty}\subset\mathbb{N}$ (i.e., a subsequence of $\{j\}_{j=1}^{\infty}$) such that $\nu_1(f_{J_1(j)},g_{J_1(j)})\to\alpha_1$ as $j\to\infty$. Setting $\alpha_2=\limsup_{j\to\infty}\nu_2(f_{J_1(j)},g_{J_1(j)})$, we find $\alpha_2\le\mu_2$, and there is a subsequence $\{J_2(j)\}_{j=1}^{\infty}$ of $\{J_1(j)\}_{j=1}^{\infty}$ such that $\nu_2(f_{J_2(j)},g_{J_2(j)})\to\alpha_2$ as $j\to\infty$. Inductively, if $n\ge3$ and a subsequence $\{J_{n-1}(j)\}_{j=1}^{\infty}$ of $\{j\}_{j=1}^{\infty}$ is already chosen, we define $\alpha_n=\limsup_{j\to\infty}\nu_n(f_{J_{n-1}(j)},g_{J_{n-1}(j)})$, so that $\alpha_n\le\mu_n$. Now, we pick a subsequence $\{J_n(j)\}_{j=1}^{\infty}$ of $\{J_{n-1}(j)\}_{j=1}^{\infty}$ such that $\nu_n(f_{J_n(j)},g_{J_n(j)})\to\alpha_n$ as $j\to\infty$. Noting that $\{J_j(j)\}_{j=n}^{\infty}$ is a subsequence of $\{J_n(j)\}_{j=1}^{\infty}$ (for all $n\in\mathbb{N}$) and denoting the diagonal sequences $\{f_{J_j(j)}\}_{j=1}^{\infty}$ and $\{g_{J_j(j)}\}_{j=1}^{\infty}$ again by $\{f_j\}$ and $\{g_j\}$, respectively, we obtain (5.1).

In the following, the set of all nondecreasing bounded functions mapping T into $\mathbb{R}^+=[0,\infty)$ is denoted by $\mathrm{Mon}(T;\mathbb{R}^+)$.

STEP 2. In this step, we prove that there are subsequences of $\{f_j\}$ and $\{g_j\}$ from (5.1), again denoted by $\{f_j\}$ and $\{g_j\}$, respectively, and a sequence of functions $\{\beta_n\}_{n=1}^{\infty}\subset\mathrm{Mon}(T;\mathbb{R}^+)$ such that

$$\lim_{j\to\infty}\nu_n(f_j,g_j;T_t^-)=\beta_n(t)\quad\text{for all } n\in\mathbb{N}\text{ and } t\in T, \tag{5.2}$$

where $T_t^-=\{s\in T: s\le t\}$ for $t\in T$.

Note that, for each $n\in\mathbb{N}$, the function $t\mapsto\nu_n(f_j,g_j;T_t^-)$ is nondecreasing on T, and $\nu_n(f_j,g_j;T_t^-)\le\nu_n(f_j,g_j)$ for all $t\in T$ and $n\in\mathbb{N}$. By (5.1), there is a sequence $\{C_n\}_{n=1}^{\infty}\subset\mathbb{R}^+$ such that $\nu_n(f_j,g_j)\le C_n$ for all $n,j\in\mathbb{N}$. In what follows, we apply the diagonal procedure once again.

The sequence $\{t\mapsto\nu_1(f_j,g_j;T_t^-)\}_{j=1}^{\infty}\subset\mathrm{Mon}(T;\mathbb{R}^+)$ is uniformly bounded by C_1, and so, by Helly's Selection Principle, there are an increasing sequence $\{K_1(j)\}_{j=1}^{\infty}\subset\mathbb{N}$ (i.e., a subsequence of $\{j\}_{j=1}^{\infty}$) and a function $\beta_1\in\mathrm{Mon}(T;\mathbb{R}^+)$ such that $\nu_1(f_{K_1(j)},g_{K_1(j)};T_t^-)\to\beta_1(t)$ as $j\to\infty$ for all $t\in T$. The sequence $\{t\mapsto\nu_2(f_{K_1(j)},g_{K_1(j)};T_t^-)\}_{j=1}^{\infty}\subset\mathrm{Mon}(T;\mathbb{R}^+)$ is uniformly bounded on T by C_2, and so, again by Helly's Theorem, there are a subsequence $\{K_2(j)\}_{j=1}^{\infty}$ of $\{K_1(j)\}_{j=1}^{\infty}$ and a function $\beta_2\in\mathrm{Mon}(T;\mathbb{R}^+)$ such that $\nu_2(f_{K_2(j)},g_{K_2(j)};T_t^-)\to\beta_2(t)$ as $j\to\infty$ for all $t\in T$. Inductively, if $n\ge3$ and a subsequence $\{K_{n-1}(j)\}_{j=1}^{\infty}$ of $\{j\}_{j=1}^{\infty}$ and a function $\beta_{n-1}\in\mathrm{Mon}(T;\mathbb{R}^+)$ are already chosen, we apply the Helly Theorem to the sequence of functions $\{t\mapsto\nu_n(f_{K_{n-1}(j)},g_{K_{n-1}(j)};T_t^-)\}_{j=1}^{\infty}\subset\mathrm{Mon}(T;\mathbb{R}^+)$, which is uniformly bounded on T by C_n: there are a subsequence $\{K_n(j)\}_{j=1}^\infty$ of $\{K_{n-1}(j)\}_{j=1}^\infty$ and a function $\beta_n \in \mathrm{Mon}(T;\mathbb{R}^+)$ such that $\nu_n(f_{K_n(j)}, g_{K_n(j)}; T_t^-) \to \beta_n(t)$ as $j \to \infty$ for all $t \in T$. Since $\{K_j(j)\}_{j=n}^\infty$ is a subsequence of $\{K_n(j)\}_{j=1}^\infty$ (for all $n \in \mathbb{N}$), the diagonal sequences $\{f_{K_j(j)}\}_{j=1}^\infty$ and $\{g_{K_j(j)}\}_{j=1}^\infty$, again denoted by $\{f_j\}$ and $\{g_j\}$, respectively, satisfy condition (5.2).

Step 3. Let Q be an at most countable dense subset of T. Note that Q contains all isolated (=nonlimit) points of T (i.e., points $t \in T$ such that the intervals $(t-\delta, t)$ and $(t, t+\delta)$ lie in $\mathbb{R} \setminus T$ for some $\delta > 0$). The set $Q_n \subset T$ of discontinuity points of the nondecreasing function β_n is at most countable. Setting $S = Q \cup \bigcup_{n=1}^\infty Q_n$, we find that S is an at most countable dense subset of T and

$$\beta_n \text{ is continuous at all points of } T \setminus S \text{ for all } n \in \mathbb{N}. \tag{5.3}$$

Since the set $\{f_j(t) : j \in \mathbb{N}\}$ is precompact in M for all $t \in T$, and $S \subset T$ is at most countable, we may assume (applying the diagonal procedure again and passing to a subsequence of $\{f_j\}$ if necessary) that, given $s \in S$, there is a point $f(s) \in M$ such that $d(f_j(s), f(s)) \to 0$ as $j \to \infty$. In this way, we obtain a function $f : S \to M$.

Step 4. Now, we show that, for every $t \in T \setminus S$, the sequence $\{f_j(t)\}_{j=1}^\infty$ converges in M. For this, we prove that this sequence is Cauchy in M, i.e., $d(f_j(t), f_k(t)) \to 0$ as $j, k \to \infty$. Fix $\varepsilon > 0$. By assumption (2.6), $\mu_n/n \to 0$ as $n \to \infty$, so we choose $n = n(\varepsilon) \in \mathbb{N}$ such that

$$\frac{\mu_{n+1}}{n+1} \le \varepsilon.$$

By property (5.1), there is $j_1 = j_1(\varepsilon, n) \in \mathbb{N}$ such that

$$\nu_{n+1}(f_j, g_j) \le \alpha_{n+1} + \varepsilon \le \mu_{n+1} + \varepsilon \quad \text{for all } j \ge j_1.$$

The definition of the set S and (5.3) imply that t is a limit point for T and, at the same time, a point of continuity of the function β_n. By the density of S in T, there is $s = s(\varepsilon, n, t) \in S$ such that

$$|\beta_n(t) - \beta_n(s)| \le \varepsilon.$$

It follows from (5.2) that there is $j_2 = j_2(\varepsilon, n, t, s) \in \mathbb{N}$ such that

$$|\nu_n(f_j, g_j; T_t^-) - \beta_n(t)| \le \varepsilon \quad \text{and} \quad |\nu_n(f_j, g_j; T_s^-) - \beta_n(s)| \le \varepsilon \quad \forall j \ge j_2.$$

Assuming that $s < t$ (for $t < s$ the argument is similar) and applying Lemma 2(a), (b), we get, for all $j \ge \max\{j_1, j_2\}$,

$$\begin{aligned}|(f_j,g_j)(\{s,t\})| &\le \nu_{n+1}(f_j,g_j;T_t^-)-\nu_n(f_j,g_j;T_s^-)\\ &\le \nu_{n+1}(f_j,g_j;T_t^-)-\nu_n(f_j,g_j;T_t^-)\\ &\quad+|\nu_n(f_j,g_j;T_t^-)-\beta_n(t)|+|\beta_n(t)-\beta_n(s)|\\ &\quad+|\beta_n(s)-\nu_n(f_j,g_j;T_s^-)|\\ &\le \frac{\nu_{n+1}(f_j,g_j;T_t^-)}{n+1}+\varepsilon+\varepsilon+\varepsilon\\ &\le \frac{\nu_{n+1}(f_j,g_j)}{n+1}+3\varepsilon\le\frac{\mu_{n+1}}{n+1}+\frac{\varepsilon}{n+1}+3\varepsilon\le 5\varepsilon.\end{aligned}$$

Being convergent (see (b)), the sequences $\{f_j(s)\}_{j=1}^\infty$, $\{g_j(s)\}_{j=1}^\infty$ and $\{g_j(t)\}_{j=1}^\infty$ are Cauchy in M, and so there is $j_3=j_3(\varepsilon,s,t)\in\mathbb{N}$ such that, for all $j,k\ge j_3$, we have

$$d(f_j(s),f_k(s))\le\varepsilon,\quad d(g_j(s),g_k(s))\le\varepsilon,\quad\text{and}\quad d(g_j(t),g_k(t))\le\varepsilon.$$

By (2.3), we get

$$|(g_j,g_k)(\{s,t\})|\le d(g_j(s),g_k(s))+d(g_j(t),g_k(t))\le 2\varepsilon\qquad\forall j,k\ge j_3.$$

Setting $j_4=\max\{j_1,j_2,j_3\}$ and applying Lemma 1(e), (c), (b), we find

$$\begin{aligned}d(f_j(t),f_k(t)) &\le d(f_j(s),f_k(s))+|(f_j,f_k)(\{s,t\})|\\ &\le d(f_j(s),f_k(s))+|(f_j,g_j)(\{s,t\})|+|(g_j,g_k)(\{s,t\})|\\ &\quad+|(g_k,f_k)(\{s,t\})|\\ &\le\varepsilon+5\varepsilon+2\varepsilon+5\varepsilon=13\varepsilon\qquad\text{for all } j,k\ge j_4.\end{aligned}$$

Since j_4 depends only on ε (and t), this proves that $\{f_j(t)\}_{j=1}^\infty$ is a Cauchy sequence in M, which together with assumption (a) establishes its convergence in M to an element denoted by $f(t)\in M$.

Here and at the end of Step 3, we have shown that the function $f:T=S\cup(T\setminus S)\to M$ is a pointwise limit on T of a subsequence $\{f_{j_k}\}_{k=1}^\infty$ of the original sequence $\{f_j\}_{j=1}^\infty$. Since $g_{j_k}\to g$ pointwise on T as $k\to\infty$ as well, we conclude from Lemma 2(c) that

$$\nu_n(f,g)\le\liminf_{k\to\infty}\nu_n(f_{j_k},g_{j_k})\le\limsup_{j\to\infty}\nu_n(f_j,g_j)=\mu_n\qquad\forall n\in\mathbb{N},$$

and so $\nu_n(f,g)=o(n)$, or $f\in\mathrm{R}(g)$. This completes the proof of Theorem 1. ∎

REMARK 3. (a) Condition (b) in Theorem 1 may be replaced by the following one: $\{g_j(t)\}_{j=1}^\infty$ is a *Cauchy sequence* in M for every $t\in T$. However, if (M,d) is not complete, we may no longer infer the property $\nu_n(f,g)\le\mu_n$, $n\in\mathbb{N}$, of the pointwise limit f (as there may be no g).

(b) Condition (2.6) is *necessary* for the uniformly convergent sequences $\{f_j\}$ and $\{g_j\}$: in fact, if $f_j\rightrightarrows f$ and $g_j\rightrightarrows g$ on T, and $\nu_n(f,g)=o(n)$, then it follows from Lemma 2(d) that $\lim_{j\to\infty}\nu_n(f_j,g_j)=\nu_n(f,g)=o(n)$.

(c) Condition (2.6) is 'almost necessary' in the following sense. Suppose $T \subset \mathbb{R}$ is a measurable set with Lebesgue measure $\mathcal{L}(T) < \infty$, $\{f_j\}$, $\{g_j\} \subset M^T$ are two sequences of measurable functions which converge pointwise (or almost everywhere) on T to functions $f, g \in M^T$, respectively, such that $\nu_n(f,g) = o(n)$. By Egorov's Theorem, given $\varepsilon > 0$, there exists a measurable set $E_\varepsilon \subset T$ such that $\mathcal{L}(T \setminus E_\varepsilon) \le \varepsilon$, $f_j \rightrightarrows f$ and $g_j \rightrightarrows g$ on E_ε. So, as in the previous remark (b), we have

$$\lim_{j\to\infty} \nu_n(f_j, g_j; E_\varepsilon) = \nu_n(f, g; E_\varepsilon) \le \nu_n(f,g) = o(n).$$

Example 2. (a) Condition (2.6) is *not necessary* for pointwise convergence even if $g_j = c$ for all $j \in \mathbb{N}$. To see this, let $T = [0,1]$ and $x, y \in M$ with $x \ne y$. Given $j \in \mathbb{N}$, define $f_j : T \to M$ by: $f_j(t) = x$ if $j!t$ is an integer, and $f_j(t) = y$ otherwise, $t \in [0,1]$. The pointwise precompact sequence $\{f_j\} \subset M^T$ consists of bounded regulated functions (in fact, $\nu_n(f_j, c) = o(n)$, and so $f_j \in \mathrm{Reg}([0,1]; M) = \mathrm{R}(c)$ for all $j \in \mathbb{N}$). It converges pointwise on T to the Dirichlet-type function $\mathcal{D}_{x,y}$ from (4.3). Note that $\nu_n(\mathcal{D}_{x,y}, c) = nd(x,y)$, and so $\mathcal{D}_{x,y} \notin \mathrm{R}(c)$. Since the usual Jordan variation $V(f_j)$ of f_j on $T = [0,1]$ is equal to $2 \cdot j! d(x,y)$, we find

$$\nu_n(f_j, c) = d(x,y) \begin{cases} n & \text{if } n < 2 \cdot j!, \\ 2 \cdot j! & \text{if } n \ge 2 \cdot j!, \end{cases} \qquad n, j \in \mathbb{N}.$$

Thus, $\lim_{j\to\infty} \nu_n(f_j, c) = d(x,y) \cdot n$, i.e., condition (2.6) does not hold.

(b) Under the assumptions of Theorem 1, condition (2.6) does not in general imply $\limsup_{j\to\infty} \nu_n(f_j, g) = o(n)$. To see this, let $g_j = f_j$ be as in example (a) above, so that $g = \mathcal{D}_{x,y}$. Given $n, j \in \mathbb{N}$, choose a collection $\{I_i\}_{i=1}^n \prec (0, 1/j!)$ with $I_i = \{s_i, t_i\}$ such that s_i is rational and t_i is irrational for all $i = 1, \dots, n$. Noting that, by (2.1),

$$|(f_j, g)(\{s_i, t_i\})| = \sup_{z\in M} |d(y,z) - d(x,z)| = d(y,x),$$

we get

$$\nu_n(f_j, g) \ge \sum_{i=1}^n |(f_j, g)(I_i)| = nd(y,x) \quad \text{for all } n, j \in \mathbb{N}.$$

(c) The choice of an appropriate sequence $\{g_j\}$ is essential in Theorem 1. Let $\{x_j\}$, $\{y_j\} \subset M$ be two sequences which converge in M to $x, y \in M$, respectively, with $x \ne y$. Define $f_j : T = [0,1] \to M$ by $f_j = \mathcal{D}_{x_j, y_j}$, $j \in \mathbb{N}$ (cf. (4.3)). Clearly, $\{f_j\}$ converges uniformly on T to $\mathcal{D}_{x,y}$ (so $\{f_j\}$ is pointwise precompact on T), and $\nu_n(f_j, c) = nd(x_j, y_j)$ for all $n, j \in \mathbb{N}$. Since

$$|d(x_j, y_j) - d(x,y)| \le d(x_j, x) + d(y_j, y) \to 0 \quad \text{as } j \to \infty,$$

we find $\lim_{j\to\infty} \nu_n(f_j, c) = nd(x,y)$, condition (2.6) is not satisfied, and Theorem 1 is inapplicable with $g_j = c$, $j \in \mathbb{N}$.

On the other hand, set $g_j = \mathcal{D}_{x,y}$ for all $j \in \mathbb{N}$. Given $\{s,t\} \subset T$, we have, by (2.3),

$$|(f_j, g_j)(\{s,t\})| \le d(f_j(s), g_j(s)) + d(f_j(t), g_j(t)) \le 2\varepsilon_j,$$

where $\varepsilon_j = \max\{d(x_j, x), d(y_j, y)\} \to 0$ as $j \to \infty$. This implies $\nu_n(f_j, g_j) \le 2n\varepsilon_j$, and so (2.6) is fulfilled.

(d) The following question is natural: Is condition (2.6) invariant under equivalent metrics on M? (Recall that two metrics d and d' on M are *equivalent* if, given $\{x_j\} \subset M$ and $x \in M$, the assertions $d(x_j, x) \to 0$ and $d'(x_j, x) \to 0$ are equivalent.) We answer this question in the negative by constructing an appropriate example. Let d be an unbounded metric on a set M, that is, $\sup_{x,y\in M} d(x,y) = \infty$ (e.g., $M = \mathbb{R}$ and $d(x,y) = |x-y|$). Since this is equivalent to $\sup_{y\in M} d(x,y) = \infty$ for all $x \in M$, fix $x_0 \in M$ and pick a sequence $\{y_j\} \subset M$ such that $d(x_0, y_j) \to \infty$ as $j \to \infty$ (e.g., if $M = \mathbb{R}$, we may set $x_0 = 0$ and $y_j = j$). Given $j \in \mathbb{N}$, define $f_j : T = [0,1] \to M$ as follows: $f_j(t) = y_j$ if $t = 1/(j+1)$, and $f_j(t) = x_0$ otherwise, $t \in [0,1]$. The sequence $\{f_j\} \subset M^T$ converges pointwise on T to the constant function $f(t) = x_0$ for all $t \in [0,1]$. Clearly (cf. (2.7)), $\nu_1(f_j) = d(x_0, y_j)$ and $\nu_n(f_j) = 2d(x_0, y_j)$ for all $n \ge 2$. Thus, condition (2.6) does not hold even if $g_j = c$ for all $j \in \mathbb{N}$.

On the other hand, the function d' given by $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$, $x, y \in M$, is a metric on M, which is equivalent to d. Calculating the quantity (2.7) with respect to d', we find

$$\nu_1(f_j) = \frac{d(x_0, y_j)}{1 + d(x_0, y_j)} \quad \text{and} \quad \nu_n(f_j) = \frac{2d(x_0, y_j)}{1 + d(x_0, y_j)} \quad \text{for all } n \ge 2.$$

It follows that $\lim_{j\to\infty} \nu_n(f_j, c)$ is equal to 1 if $n = 1$ and equal to 2 for $n \ge 2$, and so condition (2.6) is satisfied.

(e) Furthermore, condition (2.6) may not hold for *any* equivalent metric on M under which the sequence $\{f_j\} \subset M^T$ is pointwise convergent on T (with all $g_j = c$). In order to see this, we let $T = [0, 2\pi]$ and $M = \mathbb{R}$, and define $\{f_j\} \subset \mathbb{R}^{[0,2\pi]}$ by (see [11, Example 4]) $f_j(t) = \sin(j^2 t)$ if $0 \le t < 2\pi/j$, and $f_j(t) = 0$ if $2\pi/j \le t \le 2\pi$, $j \in \mathbb{N}$. Clearly, $\{f_j\}$ is pointwise convergent on $[0, 2\pi]$ to the constant function $f \equiv 0$ with respect to the usual metric $(x,y) \mapsto |x-y|$ on $\mathbb{R}$ as well as *any* metric d on $\mathbb{R}$ equivalent to it. Given $j, k \in \mathbb{N}$, we set

$$s_{j,k} = \frac{1}{j^2}\left(-\frac{3\pi}{2} + 2\pi k\right) \quad \text{and} \quad t_{j,k} = \frac{1}{j^2}\left(-\frac{\pi}{2} + 2\pi k\right),$$

so that $f_j(s_{j,k}) = 1$ and $f_j(t_{j,k}) = -1$. Moreover, we have $0 < s_{j,1} < t_{j,1} < s_{j,2} < t_{j,2} < \cdots < s_{j,j} < t_{j,j} < 2\pi/j$ for all $j \in \mathbb{N}$. Taking into account (2.7), we find, for all $j \ge n$,

$$\nu_n(f_j) = \nu_n(f_j, c) \ge \sum_{k=1}^{n} d\big(f_j(s_{j,k}), f_j(t_{j,k})\big) = nd(1, -1).$$

Hence $\limsup_{j\to\infty} \nu_n(f_j) \ge nd(1, -1)$, and so (2.6) is not fulfilled.

6. Extensions of known selection theorems. In this section, we consider extensions of two selection theorems from [25] and [22, 33]. The other selection theorems from the references in the Introduction were shown to be particular cases of [11–13] (see Remark 5).

Since $\nu_n(f_j, g_j)/n \le V(f_j, g_j)/n$, instead of condition (2.6) in Theorem 1 we may assume that $\limsup_{j\to\infty} V(f_j, g_j) < \infty$ or $\sup_{j\in\mathbb{N}} V(f_j, g_j) < \infty$; in both cases the resulting pointwise limit f of a subsequence of $\{f_j\}$ satisfies the regularity condition of the form $V(f, g) < \infty$.

Making use of the notion of ε-variation (Section 4), we get the following

Theorem 3. *Given $\emptyset \ne T \subset \mathbb{R}$ and a metric space (M, d), let $\{f_j\} \subset M^T$ be a pointwise precompact sequence of functions such that*

$$\limsup_{j\to\infty} V_\varepsilon(f_j) < \infty \quad \text{for all } \varepsilon > 0. \tag{6.1}$$

Then there is a subsequence $\{f_{j_k}\}$ of $\{f_j\}$ which converges pointwise on T to a regulated function $f \in \mathrm{R}(c)$.

Proof. Taking into account Theorem 1, it suffices to verify that (6.1) implies $\limsup_{j\to\infty} \nu_n(f_j, c) = o(n)$, which is (2.6) with $g_j = c$ for all $j \in \mathbb{N}$. In fact, by (6.1), for every $\varepsilon > 0$ there are $j_0 = j_0(\varepsilon) \in \mathbb{N}$ and $C(\varepsilon) > 0$ such that $V_\varepsilon(f_j) \le C(\varepsilon)$ for all $j \ge j_0$. Definition (4.1) yields the existence of $g_j \in \mathrm{BV}(T; M)$ such that $d_\infty(f_j, g_j) \le \varepsilon$ and $V(g_j) \le V_\varepsilon(f_j) + 1/j \le C(\varepsilon) + 1$ for all $j \ge j_0$. By (3.7) (where we replace g_j and g by c, and f by g_j),

$$\frac{\nu_n(f_j, c)}{n} \le 2d_\infty(f_j, g_j) + \frac{\nu_n(g_j, c)}{n} \le 2\varepsilon + \frac{V(g_j)}{n} \le 2\varepsilon + \frac{C(\varepsilon) + 1}{n}$$

for all $j \ge j_0$ and $n \in \mathbb{N}$. Consequently,

$$\frac{1}{n} \limsup_{j\to\infty} \nu_n(f_j, c) \le \frac{1}{n} \sup_{j\ge j_0} \nu_n(f_j, c) \le 2\varepsilon + \frac{C(\varepsilon) + 1}{n} \qquad \forall \varepsilon > 0,\ n \in \mathbb{N}.$$

This implies that the left-hand side tends to zero as $n \to \infty$: given $\eta > 0$, we set $\varepsilon = \eta/4$ and choose $n_0 = n_0(\eta) \in \mathbb{N}$ such that $(C(\varepsilon) + 1)/n_0 \le \eta/2$, which yields $2\varepsilon + (C(\varepsilon) + 1)/n \le \eta$ for all $n \ge n_0$. ∎

Remark 4. (a) If $M = \mathbb{R}^N$ in Theorem 3, we may infer that $V_\varepsilon(f)$ does not exceed the lim sup from (6.1): in fact, it follows from [25, Proposition 3.6] that $V_\varepsilon(f) \le \liminf_{k\to\infty} V_\varepsilon(f_{j_k})$ for all $\varepsilon > 0$.

(b) It is worth mentioning that when the sequence $\{f_j\}$ from Theorem 3 belongs to the Skorokhod space $\mathbb{D} = \mathbb{D}([a, b]; \mathbb{R}^N)$ of càdlàg functions (i.e., functions continuous from the right and having limits from the left at each point of $(a, b]$), the subsequence extracted via Theorem 3 (and [25, Theorem 3.8]) is convergent in some nonmetrizable topology (the so-called topology S) on $\mathbb{D}$. For more details on the topology S we refer to [32].

(c) Theorem 3 extends Theorem 3.8 from [25], which has been established for $T = [a, b]$ and $M = \mathbb{R}^N$ under the assumption that $\sup_{j\in\mathbb{N}} V_\varepsilon(f_j) < \infty$ for every $\varepsilon > 0$. The last assumption on the uniform boundedness of ε-variations is more restrictive than condition (6.1), as the following example shows.

Example 3. Let $\{x_j\}$ and $\{y_j\}$ be two sequences from M such that $x_j \neq y_j$ for all $j \in M$ and, for some $x \in M$, $x_j \to x$ and $y_j \to x$ in M as $j \to \infty$. We set $f_j = \mathcal{D}_{x_j,y_j}$, $j \in \mathbb{N}$, and $f(t) = x$ for all $t \in T = [0, 1]$. The sequence $\{f_j\} \subset M^T$ converges uniformly on T to the constant function f:

$$d_\infty(f_j, f) = \max\{d(x_j, x), d(y_j, x)\} \to 0 \quad \text{ as } j \to \infty.$$

Given $\varepsilon > 0$, there is $j_0 = j_0(\varepsilon) \in \mathbb{N}$ such that $d(x_j, y_j) \le \varepsilon$ for all $j \ge j_0$, and so, by (4.4), $V_\varepsilon(f_j) = 0$ for all $j \ge j_0$, which implies condition (6.1):

$$\limsup_{j\to\infty} V_\varepsilon(f_j) \le \sup_{j\ge j_0} V_\varepsilon(f_j) = 0.$$

On the other hand, if $k \in \mathbb{N}$ is fixed and $0 < \varepsilon < d(x_k, y_k)/2$, then (4.4) gives $V_\varepsilon(f_k) = \infty$, and so, $\sup_{j\in\mathbb{N}} V_\varepsilon(f_j) = \infty$.

Now, we are going to present an extension of a Helly-type selection theorem from [33, Section 4, Theorem 1] and [22, Theorem 2].

Let $\kappa : [0, 1] \to [0, 1]$ be a continuous, increasing and concave function such that $\kappa(0) = 0$, $\kappa(1) = 1$, and $\kappa(\tau)/\tau \to \infty$ as $\tau \to +0$ (e.g., $\kappa(\tau) = \tau(1 - \log \tau)$, $\kappa(\tau) = \tau^\alpha$ with $0 < \alpha < 1$, or $\kappa(\tau) = 1/\bigl(1 - \frac{1}{2}\log \tau\bigr)$, see [34]).

Let $T = [a, b]$ be a closed interval in $\mathbb{R}$, $a < b$. We set $|T| = b - a$, and if $\{t_i\}_{i=0}^n \prec [a, b]$ is a partition of T (i.e., $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$), we also set $I_i = \{t_{i-1}, t_i\}$ and $|I_i| = t_i - t_{i-1}$, $i = 1, \ldots, n$.

The *joint κ-variation* of functions $f, g \in M^T = M^{[a,b]}$ is defined by

$$V_\kappa(f, g) = \sup\Bigl\{\sum_{i=1}^n |(f, g)(I_i)| \Big/ \sum_{i=1}^n \kappa(|I_i|/|T|) : n \in \mathbb{N} \text{ and } \{t_i\}_{i=0}^n \prec [a, b]\Bigr\},$$

where $|(f, g)(I_i)| = |(f, g)(\{t_{i-1}, t_i\})|$ is given by (2.1).

Since $|(f, c)(I_i)| = d(f(t_{i-1}), f(t_i))$ is independent of a constant function $c : [a, b] \to M$, the quantity $V_\kappa(f) \equiv V_\kappa(f, c)$ is the *Korenblum κ-variation* of $f \in M^{[a,b]}$, introduced in [33, p. 191] and [34, Section 5] for $M = \mathbb{R}$.

The following theorem is a generalization of [22, Theorem 2], established for real functions of bounded κ-variation under the assumption that $\sup_{j\in\mathbb{N}} V_\kappa(f_j) < \infty$ and used in the proof of the decomposition of any $f \in \mathbb{R}^{[a,b]}$ with $V_\kappa(f) < \infty$ into the difference of two real κ-decreasing functions.

Theorem 4. *Under the assumptions of Theorem* 1, *suppose that condition* (2.6) *is replaced by* $\limsup_{j\to\infty} V_\kappa(f_j, g_j) < \infty$. *Then there is a subsequence of* $\{f_j\}$ *which converges pointwise on* $T = [a, b]$ *to a function* $f \in \mathrm{R}(g)$ *such that* $V_\kappa(f, g) < \infty$.

Proof. In order to show that (2.6) is satisfied, let $n \in \mathbb{N}$, $\{I_i\}_{i=1}^n \prec [a, b]$ with $I_i = \{s_i, t_i\}$, and set $I_i' = \{t_i, s_{i+1}\}$ and $|I_i'| = s_{i+1} - t_i$, $i = 1, \ldots, n-1$. By the definition of $V_\kappa(f_j, g_j)$ and the concavity of κ, we have

$$
\begin{aligned}
\sum_{i=1}^n |(f_j, g_j)(I_i)| &\le |(f_j, g_j)(\{a, s_1\})| + \sum_{i=1}^n |(f_j, g_j)(I_i)| \\
&\qquad + \sum_{i=1}^{n-1} |(f_j, g_j)(I_i')| + |(f_j, g_j)(\{t_n, b\})| \\
&\le \left[\kappa\Big(\frac{s_1 - a}{|T|}\Big) + \sum_{i=1}^n \kappa\Big(\frac{|I_i|}{|T|}\Big) + \sum_{i=1}^{n-1} \kappa\Big(\frac{|I_i'|}{|T|}\Big) + \kappa\Big(\frac{b - t_n}{|T|}\Big)\right] V_\kappa(f_j, g_j) \\
&\le (2n+1)\kappa\bigg(\frac{1}{(2n+1)(b-a)}\bigg[(s_1 - a) + \sum_{i=1}^n (t_i - s_i) \\
&\qquad + \sum_{i=1}^{n-1} (s_{i+1} - t_i) + (b - t_n)\bigg]\bigg) V_\kappa(f_j, g_j) \\
&\le (2n+1)\kappa\Big(\frac{1}{2n+1}\Big) V_\kappa(f_j, g_j).
\end{aligned}
$$

Thus,

$$
\frac{\nu_n(f_j, g_j)}{n} \le \Big(2 + \frac{1}{n}\Big)\kappa\Big(\frac{1}{2n+1}\Big) V_\kappa(f_j, g_j) \quad \text{for all } j, n \in \mathbb{N},
$$

and so (2.6) is satisfied.

Let $f \in \mathrm{R}(g)$ be the pointwise limit of a subsequence $\{f_{j_m}\}$ of $\{f_j\}$. Arguing as in the proof of Lemma 2(c), we get

$$
V_\kappa(f, g) \le \liminf_{m\to\infty} V_\kappa(f_{j_m}, g_{j_m}) \le \limsup_{j\to\infty} V_\kappa(f_j, g_j) < \infty. \ \blacksquare
$$

Remark 5. Since Theorem 1 is an extension of results from [11, 12], it also contains as particular cases all pointwise selection principles based on various notions of generalized variation. These principles may be further generalized in the spirit of Theorem 4, by replacing the increment $|f(I_i)| = d(f(s_i), f(t_i))$ applied in [11–13] by the joint increment $|(f, g)(I_i)|$ from (2.1).

Finally, it is worth mentioning that the joint modulus of variation (2.4), defined by means of (2.1), plays an important role in the extension of a result from [17] to metric space valued functions:

THEOREM 5. *Given $\emptyset \neq T \subset \mathbb{R}$ and a metric space (M, d), let $\{f_j\} \subset M^T$ be a pointwise precompact sequence of functions such that*

$$\lim_{N\to\infty} \sup_{j,k\geq N} \nu_n(f_j, f_k) = o(n).$$

Then there is a subsequence of $\{f_j\}$ which converges pointwise on T.

Taking into account Lemmas 1 and 2, the proof of this theorem follows the same lines as in [17, Theorem 1] (where $(M, d, +)$ is a metric semigroup and $\nu_n(f_j, f_k)$ is defined via (3.1)), and so it is omitted. Note that the limit of a pointwise convergent subsequence of $\{f_j\}$ in Theorem 5 may be a nonregulated function. For more details, examples and relations with previously known 'regular' and 'irregular' versions of pointwise selection principles from [23, 41, 43] we refer to [14, Section 5.2], [17, 18, 35].

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