STUDIA MATHEMATICA 238 (2) (2017)

Unitary subgroups and orbits of compact self-adjoint operators

by

Tamara Bottazzi (Buenos Aires) and
Alejandro Varela (Los Polvorines and Buenos Aires)

Abstract. Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ be the anti-Hermitian bounded diagonal operators in some fixed orthonormal basis and $\mathcal{K}(\mathcal{H})$ the compact operators. We study the group of unitary operators

$$\mathcal{U}_{k,d} = \{u \in \mathcal{U}(\mathcal{H}) : \exists D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}),\ u - e^D \in \mathcal{K}(\mathcal{H})\}$$

in order to obtain a concrete description of short curves in unitary Fredholm orbits $\mathcal{O}_b = \{e^K b e^{-K} : K \in \mathcal{K}(\mathcal{H})^{ah}\}$ of a compact self-adjoint operator b with spectral multiplicity one. We consider the rectifiable distance on $\mathcal{O}_b$ defined as the infimum of curve lengths measured with the Finsler metric defined by means of the quotient space $\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$. Then for every $c \in \mathcal{O}_b$ and $x \in T_c(\mathcal{O}_b)$ there exists a minimal lifting $Z_0 \in \mathcal{B}(\mathcal{H})^{ah}$ (in the quotient norm, not necessarily compact) such that $\gamma(t) = e^{tZ_0} c e^{-tZ_0}$ is a short curve on $\mathcal{O}_b$ in a certain interval.

1. Introduction. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on a separable Hilbert space $\mathcal{H}$, and $\mathcal{K}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ the compact and unitary operators respectively. If an orthonormal basis is fixed we can consider matricial representations of each $A \in \mathcal{B}(\mathcal{H})$ and the set of diagonal operators, which we denote by $\mathcal{D}(\mathcal{B}(\mathcal{H}))$. The subset of anti-Hermitian diagonal operators is denoted by $\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$.

Consider the following subset of the unitary group $\mathcal{U}(\mathcal{H})$:

$$\mathcal{U}_{k,d} = \{u \in \mathcal{U}(\mathcal{H}) : \exists D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}),\ u - e^D \in \mathcal{K}(\mathcal{H})\}. \tag{1.1}$$

We will prove that $\mathcal{U}_{k,d}$ is a subgroup of $\mathcal{U}(\mathcal{H})$. Moreover, $\mathcal{U}_{k,d}$ is closed and pathwise connected in the topology of $\mathcal{U}(\mathcal{H})$ given by the operator norm. Therefore $\mathcal{U}_{k,d}$ is a Lie subgroup in the sense of [9] and [10].

2010 *Mathematics Subject Classification*: Primary 22F30, 22E10, 51F25; Secondary 47B15.
Key words and phrases: unitary groups, Lie subgroups, unitary orbits, geodesic curves, minimal operators in quotient spaces.
Received 23 August 2016; revised 4 December 2016.
Published online 21 April 2017.

DOI: 10.4064/sm8690-12-2016

We have not found any reference to the subgroup $\mathcal{U}_{k,d}$ in the literature, and so we include a detailed study of it. In Theorem 3.18 we prove that $\mathcal{U}_{k,d}$ is a Lie subgroup of $\mathcal{U}(\mathcal{H})$. The Lie algebra of $\mathcal{U}_{k,d}$ turns out to be $\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$, which is not complemented in $\mathcal{B}(\mathcal{H})^{ah}$, and therefore a stronger notion of Lie subgroup cannot be used (see Proposition 3.16).

This subgroup admits a generalization to $\mathcal{U}_{\mathcal{J},\mathcal{A}}$ for certain ideals $\mathcal{J}$ and subalgebras $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})^h$ (see Remark 3.19).

Our particular interest in $\mathcal{U}_{k,d}$ relies on the geometric study of the orbits

$$\mathcal{O}_b^{\mathcal{V}} = \{ubu^* : u \in \mathcal{V}\}$$

where b a self-adjoint operator and $\mathcal{V} \subset \mathcal{B}(\mathcal{H})$. If the spectrum of b is finite, $\mathcal{O}_b$ is a complemented submanifold of $b + \mathcal{K}(\mathcal{H})$ (see [1]). If we consider a compact diagonal self-adjoint operator b with spectral multiplicity one then the orbit $\mathcal{O}_b$ can be given a smooth structure (see [6, Lemma 1]).

The subgroup $\mathcal{U}_{k,d}$ has the following properties.

- If $\mathcal{U}_k = \{u \in \mathcal{U}(\mathcal{H}) : u - 1 \in \mathcal{K}(\mathcal{H})\}$, then the following orbits coincide:
$$\mathcal{O}_b = \mathcal{O}_b^{\mathcal{U}_k} = \{ubu^* : u \in \mathcal{U}_k\} = \mathcal{O}_b^{\mathcal{U}_{k,d}} = \{ubu^* : u \in \mathcal{U}_{k,d}\}.$$
- The natural Finsler metrics defined in $T_1(\mathcal{O}_b^{\mathcal{U}_{k,d}})$ and $T_1(\mathcal{O}_b^{\mathcal{U}_k})$ by means of the quotient norm coincide if b is a compact self-adjoint diagonal operator and we consider the identifications of the tangent spaces with the quotients
$$\begin{aligned} T_1(\mathcal{O}_b^{\mathcal{U}_k}) &\cong T_c(\mathcal{O}_b) \cong T_1(\mathcal{U}_k)/T_1(\mathcal{I}_b) = \mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah}), \\ T_1(\mathcal{O}_b^{\mathcal{U}_{k,d}}) &\cong T_1(\mathcal{U}_{k,d})/T_1(\mathcal{I}_b) \\ &\cong (\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}))/\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}) \\ &\cong \mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}) \end{aligned}$$
with $c \in \mathcal{O}_b$ (see Remark 4.5 for details).

These properties allow the construction of minimum length curves of $\mathcal{O}_b$ considering the rectifiable distance defined in the Preliminaries (see (2.6)).

Next we describe minimal vectors of the tangent space and their relation to short curves in these homogeneous spaces. We say that a self-adjoint operator $Z \in \mathcal{B}(\mathcal{H})$ is *minimal for* a subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ if

$$\|Z\| = \inf_{D \in \mathcal{A}} \|Z + D\|, \tag{1.2}$$

for $\|\cdot\|$ the usual operator norm in $\mathcal{B}(\mathcal{H})$. Given a fixed Z we say that $D_0 \in \mathcal{A}$ is *minimal for* Z if $\|Z + D_0\| = \inf_{D \in \mathcal{A}} \|Z + D\|$, that is, $Z + D_0$ is minimal for $\mathcal{A}$. These minimal operators Z allow the concrete description of short curves $\gamma(t) = e^{itZ} A e^{-itZ}$ in the unitary orbit $\mathcal{O}_A$ of a fixed self-adjoint operator $A \in \mathcal{B}(\mathcal{H})^h$, when considered with a certain natural Finsler metric (see (2.4), [8], [1] and [6] for details and different examples).

If we fix an orthonormal basis in $\mathcal{H}$ we can consider matricial representations and diagonal operators in $\mathcal{B}(\mathcal{H})$. In [6] we studied the orbit $\mathcal{O}_A$ of a diagonal compact self-adjoint operator $b \in \mathcal{B}(\mathcal{H})$ under the action of the Fredholm unitary subgroup $\mathcal{U}_k = \{e^K : K \in \mathcal{K}(\mathcal{H})^{ah}\}$ where $\mathcal{K}(\mathcal{H})^{ah}$ denotes the compact anti-Hermitian operators. We used a particular element $Z_r \in \mathcal{K}(\mathcal{H})^{ah}$ with the property that there does not exist a compact diagonal D_0 such that the quotient norm $\|Z_r + D_0\| = \inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H}))} \|Z_r + D\|$ is attained. This example prompted an interesting geometric question, since the existence of such a minimal compact diagonal D_0 would allow the explicit description of a short path with initial velocity $[Z_r, b]$ (see [8, 5]).

Using the fact that $\lim (Z_r)_{jj}$ converges to a non-zero constant as $j \to \infty$, we showed in [6] that the curve parametrized by

$$\beta(t) = e^{tZ_r} b e^{-tZ_r}$$

with $|t| \leq \pi/(2\|Z_r\|)$ is still a geodesic, even though Z_r is not a minimal operator. Moreover, β can be approximated uniformly by minimal length curves of finite matrices β_n (with minimal initial velocity vectors) satisfying $\beta_n(0) = \beta(0) = b$ and $\beta_n'(0) = \beta'(0)$.

Nevertheless, in the same paper, we showed examples of compact operators Z_o whose unique minimal diagonals had several limits. In these cases the techniques used with Z_r were not enough to prove either that $\gamma(t) = e^{tZ_o} b e^{-tZ_o}$ was a short curve or that γ could be approximated by curves of matrices.

In the present work we describe short curves that include those cases. In order to do so we consider the unitary subgroup $\mathcal{U}_{k,d}$. The action of this group on a diagonal self-adjoint operator b produces the same orbit as $\mathcal{U}_k$ but permits a concrete description of geodesics using minimal operators of its Lie algebra $\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ (see Theorem 4.2 and Corollary 4.6).

2. Preliminaries. Let $(\mathcal{H}, \langle\,,\rangle)$ be a separable Hilbert space. As usual, $\mathcal{B}(\mathcal{H})$, $\mathcal{U}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ denote the sets of bounded, unitary and compact operators on $\mathcal{H}$. We denote by $\|\cdot\|$ the usual operator norm in $\mathcal{B}(\mathcal{H})$. It should be clear from the context whether $\|\cdot\|$ refers to the operator norm or the norm on the Hilbert space, $\|h\| = \langle h, h\rangle^{1/2}$ for $h \in \mathcal{H}$.

Given $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, we use the superscript ^{ah} (respectively ^{h}) to denote the subset of anti-Hermitian (respectively Hermitian) elements of $\mathcal{A}$.

Consider the Fredholm subgroup of $\mathcal{U}(\mathcal{H})$ defined as

$$\mathcal{U}_k = \{u \in \mathcal{U}(\mathcal{H}) : u - 1 \in \mathcal{K}(\mathcal{H})\} = \{u \in \mathcal{U}(\mathcal{H}) : \exists K \in \mathcal{K}(\mathcal{H})^{ah},\, u = e^K\}$$

(see [1] and Proposition 3.1).

$\mathcal{U}(\mathcal{H})$ is a Banach–Lie group and its Lie algebra $T_1(\mathcal{U}(\mathcal{H}))$ equals $\mathcal{B}(\mathcal{H})^{ah}$. We consider the usual analytical exponential map $\exp : \mathcal{B}(\mathcal{H})^{ah} \to \mathcal{U}(\mathcal{H})$, given for any $X \in \mathcal{B}(\mathcal{H})^{ah}$ by $\exp(X) = \sum_{n=0}^{\infty}(1/n!)X^n = e^X$. Then $\mathcal{B}(\mathcal{H})^{ah}$ can be made into a contractive Lie algebra (i.e., $\|[X,Y]\|_c \leq \|X\|_c\|Y\|_c$ for all $X, Y \in \mathcal{B}(\mathcal{H})^{ah}$) by defining $\|\cdot\|_c := 2\|\cdot\|$. Then, by [4, Proposition 1.29],

$$\|\log(e^X e^Y)\|_c \leq -\log(2 - e^{\|X\|_c + \|Y\|_c})$$

if $\|X\|_c + \|Y\|_c < \log 2$. Consequently, it can be proved that if

$$\|X\| + \|Y\| < (\log 2)/2 \tag{2.1}$$

then the Baker–Campbell–Hausdorff (B-C-H) series expansion converges absolutely for all $X, Y \in \mathcal{B}(\mathcal{H})^{ah}$. This B-C-H series can be defined as

$$\log(e^X e^Y) = \sum_{n=1}^{\infty} c_n(T),$$

where each c_n is a polynomial map of $\mathcal{B}(\mathcal{H})^{ah} \times \mathcal{B}(\mathcal{H})^{ah}$ into $\mathcal{B}(\mathcal{H})^{ah}$ of degree n. For instance, the first terms are:

$$\begin{cases} c_1(X,Y) = X + Y, \\ c_2(X,Y) = \frac{1}{2}[X,Y], \\ c_3(X,Y) = \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]]. \end{cases}$$

Also, each c_n with $n > 1$ is a sum of commutators. Therefore, the above series can be rewritten as follows

$$\log(e^X e^Y) = X + Y + \sum_{n=2}^{\infty} c_n(T). \tag{2.2}$$

For the complete general expression and other properties of the B-C-H series for Lie algebras see [4] or [12].

Definition 2.1. Given $X \in \mathcal{B}(\mathcal{H})^{ah}$ we will say that X is *sufficiently close to* 0 if $\|X\| < (\log 2)/4$.

We see that the B-C-H series (2.2) converges for every $X, Y \in \mathcal{B}(\mathcal{H})^{ah}$ sufficiently close to 0, since this condition implies (2.1).

We define the *unitary Fredholm orbit* of a fixed self-adjoint $A \in \mathcal{B}(\mathcal{H})$ as

$$\mathcal{O}_A = \{uAu^* : u \in \mathcal{U}_k\} \subset A + \mathcal{K}(\mathcal{H}). \tag{2.3}$$

With the action $\pi_b : \mathcal{U}_k \to \mathcal{O}_A$, $\pi_b(u) = L_u \cdot b = ubu^*$, the orbit $\mathcal{O}_A$ becomes a homogeneous space in some cases. If A has finite spectrum then $\mathcal{O}_A$ is a submanifold of $A + \mathcal{K}(\mathcal{H})$ (see [1, Theorem 4.4]), and if A is a compact operator with spectral multiplicity one then $\mathcal{O}_A$ has a smooth structure (see [6, Lemma 1]).

Let $[\cdot,\cdot]$ be the commutator operator in $\mathcal{B}(\mathcal{H})$, that is, $[T,S] = TS - ST$ for any $T, S \in \mathcal{B}(\mathcal{H})$.

For each $b \in \mathcal{O}_A$, the isotropy group $\mathcal{I}_b$ is

$$\begin{aligned}\mathcal{I}_b &= \{u \in \mathcal{U}_k : ubu^* = b\} = \{e^K \in \mathcal{U}_k : K \in \mathcal{K}(\mathcal{H})^{ah},\ [K,b]=0\} \\ &= \{b\}' \cap \mathcal{K}(\mathcal{H})^{ah},\end{aligned}$$

where $\{b\}'$ is the set of all operators T in $\mathcal{B}(\mathcal{H})$ that commute with b (i.e., $[T,b]=0$).

For each $b \in \mathcal{O}_A$, the tangent space $T_b(\mathcal{O}_A)$ equals $\{Yb - bY : Y \in \mathcal{K}(\mathcal{H})^{ah}\}$ and can be identified as follows:

$$T_b(\mathcal{O}_A) \cong T_1(\mathcal{U}_k)/T_1(\mathcal{I}_b) \cong \mathcal{K}(\mathcal{H})^{ah}/(\{b\}' \cap \mathcal{K}(\mathcal{H})^{ah}).$$

In this context we consider the following Finsler metric defined for $x \in T_b(\mathcal{O}_A)$:

$$\begin{aligned}\|x\|_b &= \inf\{\|Y\| : Y \in \mathcal{K}(\mathcal{H})^{ah},\ [Y,b]=x\} \\ &= \inf_{C \in (\{b\}' \cap \mathcal{K}(\mathcal{H})^{ah})} \|Y_0 + C\|.\end{aligned} \tag{2.4}$$

where $Y_0 + C$ is any element of the class $[Y_0] = \{Y \in \mathcal{K}(\mathcal{H})^{ah} : [Y,b]=x\}$. Note that this norm is invariant under the action of $\mathcal{U}_k$.

An element $Z \in \mathcal{B}(\mathcal{H})^{ah}$ such that $[Z,b]=x$ and $\|Z\| = \|x\|_b$ is called a *minimal lifting* for x. This operator Z may not be compact and/or unique (see [5]). Consider piecewise smooth curves $\beta : [r,s] \to \mathcal{O}_A$. We define

$$\mathrm{L}(\beta) = \int_r^s \|\beta'(t)\|_{\beta(t)}\, dt, \tag{2.5}$$

$$\mathrm{dist}(c_1,c_2) = \inf\{\mathrm{L}(\beta) : \beta \text{ is smooth, } \beta(r)=c_1,\ \beta(s)=c_2\}, \tag{2.6}$$

called the *rectifiable length* of β and the distance between $c_1, c_2 \in \mathcal{O}_A$, respectively. In this context, we define *short curves* to be those piecewise smooth curves $\gamma : [r,s] \to \mathcal{O}_A$ such that $\mathrm{L}(\gamma|_{[t_1,t_2]}) \le \mathrm{dist}(\gamma(t_1),\gamma(t_2))$ for every subinterval $[t_1,t_2] \subset [r,s]$.

If $\mathcal{A}$ is any C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ and $\{e_k\}_{k=1}^\infty$ is a fixed orthonormal basis of $\mathcal{H}$, we denote by $\mathcal{D}(\mathcal{A})$ the set of diagonal operators with respect to this basis, that is,

$$\mathcal{D}(\mathcal{A}) = \{T \in \mathcal{A} : \langle Te_i, e_j \rangle = 0 \text{ for all } i \ne j\}.$$

Given $Z \in \mathcal{A}$, if there exists $D_1 \in \mathcal{D}(\mathcal{A})$ such that

$$\|Z + D_1\| \le \|Z + D\|$$

for all $D \in \mathcal{D}(\mathcal{A})$, we say that D_1 is a *best approximant* of Z in $\mathcal{D}(\mathcal{A})$. The operator $Z + D_1$ satisfies

$$\|Z + D_1\| = \mathrm{dist}(Z, \mathcal{D}(\mathcal{A})),$$

and $Z + D_1$ is a *minimal operator* in the class $[Z]$ of the quotient space $\mathcal{A}/\mathcal{D}(\mathcal{A})$; we also say that D_1 is minimal for Z.

These minimal operators play an important role in the concrete description of minimal length curves on $\mathcal{O}_A$ (see [8] and [1]).

If Z is anti-Hermitian then

$$\operatorname{dist}(Z, \mathcal{D}(\mathcal{A})) = \operatorname{dist}(Z, \mathcal{D}(\mathcal{A}^{ah})),$$

since $\|\operatorname{Im}(X)\| \leq \|X\|$ for every $X \in \mathcal{A}$.

Let $T \in \mathcal{B}(\mathcal{H})$ and consider for the fixed basis of $\mathcal{H}$ the coefficients $T_{ij} = \langle Te_i, e_j\rangle$ for each $i, j \in \mathbb{N}$. This defines an infinite matrix $(T_{ij})_{i,j\in\mathbb{N}}$ whose jth column and ith row are the vectors in ℓ^2 given by $c_j(T) = (T_{1j}, T_{2j}, \ldots)$ and $f_j(T) = (T_{i1}, T_{i2}, \ldots)$, respectively.

We use $\sigma(T)$ and $R(T)$ to denote the spectrum and range of $T \in \mathcal{B}(\mathcal{H})^h$, respectively.

We define $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{D}(\mathcal{B}(\mathcal{H}))$, $\Phi(X) = \operatorname{Diag}(X)$, as the map that builds a diagonal operator with the same diagonal as X (i.e., $\Phi(X)_{ii} = \operatorname{Diag}(X)_{ii} = X_{ii}$ and 0 elsewhere). For a given bounded sequence $\{d_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ we denote by $\operatorname{Diag}(\{d_n\}_{n\in\mathbb{N}})$ the diagonal (infinite) matrix with $\{d_n\}_{n\in\mathbb{N}}$ on its diagonal and 0 elsewhere.

3. The unitary subgroup $\mathcal{U}_{k,d}$. Recall the unitary Fredholm group

$$\mathcal{U}_k = \{u \in \mathcal{U}(\mathcal{H}) : u - 1 \in \mathcal{K}(\mathcal{H})\}$$

(see [1] and [6]) and define the following subsets of $\mathcal{U}(\mathcal{H})$:

$$\begin{aligned} \mathcal{U}_{k,d} &= \{u \in \mathcal{U}(\mathcal{H}) : \exists D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}),\ u - e^D \in \mathcal{K}(\mathcal{H})\}, \\ \mathcal{U}_d &= \{u \in \mathcal{U}(\mathcal{H}) : \exists D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}),\ u = e^D\} = \mathcal{U}(\mathcal{H}) \cap \mathcal{D}(\mathcal{B}(\mathcal{H})), \\ \mathcal{U}_{k+d} &= \{u \in \mathcal{U}(\mathcal{H}) : \exists K \in \mathcal{K}(\mathcal{H})^{ah}\ \exists D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}),\ u = e^{K+D}\}. \end{aligned} \tag{3.1}$$

Also denote

$$\mathcal{O}_b^{\mathcal{F}} = \{ubu^* : u \in \mathcal{F}\},$$

where $\mathcal{F}$ is any of the sets of unitary operators defined in (3.1). The main purpose of this section is to study these unitary sets and their relations.

The following proposition has been proved in [6] using arguments of [1, Lemma 2.1].

Proposition 3.1. $\mathcal{U}_k = \{e^K : K \in \mathcal{K}(\mathcal{H})^{ah},\ \|K\| \leq \pi\}$.

Remark 3.2. Let $S_0 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_0 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. Then the exponential series $\sum_{n=0}^{\infty} \frac{1}{n!}(S_0 + D_0)^n$ converges absolutely and

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{1}{n!}(S_0 + D_0)^n \\ &\quad = \sum_{n=0}^{\infty} \frac{1}{n!}\left(S_0^n + \binom{n}{1} S_0^{n-1} D_0 + \cdots + \binom{n}{n-1} S_0 D_0^{n-1} + D_0^n\right) \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} S_0 \left(S_0^{n-1} + \binom{n}{1} S_0^{n-2} D_0 + \cdots + \binom{n}{n-1} D_0^{n-1} \right) + \frac{1}{n!} D_0^n$$

$$= S_0 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n!} \left(S_0^{n-1} + \binom{n}{1} S_0^{n-2} D_0 + \cdots + \binom{n}{n-1} D_0^{n-1} \right)}_{\substack{\parallel \\ \Psi(S_0, D_0)}} + \sum_{n=0}^{\infty} \frac{1}{n!} D_0^n$$

$$= S_0 \Psi(S_0, D_0) + e^{D_0}$$

with $S_0 \Psi(S_0, D_0) \in \mathcal{K}(\mathcal{H})$.

Proposition 3.3. *$\mathcal{U}_{k,d}$ is a unitary subgroup of $\mathcal{U}(\mathcal{H})$ and*

$$\mathcal{U}_k \mathcal{U}_d = \{u \in \mathcal{U}(\mathcal{H}) : \exists K \in \mathcal{K}(\mathcal{H})^{ah},\ \|K\| \leq \pi,\ \exists D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}),\ u = e^K e^D\}.$$

Moreover

$$\mathcal{U}_{k,d} = \mathcal{U}_k \mathcal{U}_d = \mathcal{U}_d \mathcal{U}_k.$$

Proof. Let $u \in \mathcal{U}_{k,d}$. Then there exists $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that $u - e^D \in \mathcal{K}(\mathcal{H})$. Hence $ue^{-D} - 1 \in \mathcal{K}(\mathcal{H})$, so there is $K \in \mathcal{K}(\mathcal{H})^{ah}$ with $\|K\| \leq \pi$ such that $ue^{-D} = e^K$, and therefore $u \in \mathcal{U}_k \mathcal{U}_d$.

Conversely, if there exist $K' \in \mathcal{K}(\mathcal{H})^{ah}$ and $D' \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that $u = e^{K'} e^{D'} \in \mathcal{U}(\mathcal{H})$, then $ue^{-D'} = e^{K'} \in \mathcal{U}_k$, so $ue^{-D'} - 1 \in \mathcal{K}(\mathcal{H})$, and thus

$$(ue^{-D'} - 1)e^D = u - e^{D'} \in \mathcal{K}(\mathcal{H}).$$

These calculations prove that $\mathcal{U}_{k,d} = \mathcal{U}_k \mathcal{U}_d$.

Similar computations (with left multiplication by e^{-D}) lead to $\mathcal{U}_{k,d} = \mathcal{U}_d \mathcal{U}_k$.

Now we will prove that $\mathcal{U}_{k,d}$ is a group. Let $u, v \in \mathcal{U}_{k,d}$.

- There exists $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that $u - e^D \in \mathcal{K}(\mathcal{H})$. Then $u^* - e^{-D} \in \mathcal{K}(\mathcal{H})$, so $u^* \in \mathcal{U}_{k,d}$.
- Since $\mathcal{U}_{k,d} = \mathcal{U}_k \mathcal{U}_d$ we can write $u = e^{K_1} e^{D_1}$ and $v = e^{K_2} e^{D_2}$ with $K_1, K_2 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_1, D_2 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. Then by Remark 3.2,

$$\begin{aligned} uv = e^{D_1 + D_2} &+ K_1 \Psi(K_1, D_1) + e^{D_2} K_2 \Phi(K_2, D_2) \\ &+ K_1 \Psi(K_1, D_1) K_2 \Phi(K_2, D_2). \end{aligned}$$

  Therefore $uv - e^{D_1 + D_2} \in \mathcal{K}(\mathcal{H})$, which implies that $uv \in \mathcal{U}_{k,d}$.

Thus, $\mathcal{U}_{k,d}$ is a unitary subgroup of $\mathcal{U}(\mathcal{H})$. ■

Proposition 3.4. *Let $\mathcal{U}_{k,d}$, $\mathcal{U}_d$ and $\mathcal{U}_{k+d}$ be as defined in* (3.1). *Then:*

(1) $\mathcal{U}_d \subsetneq \mathcal{U}_{k,d}$.
(2) $\mathcal{U}_k \subsetneq \mathcal{U}_{k,d}$.
(3) $\mathcal{U}_{k+d} \subseteq \mathcal{U}_{k,d}$.
(4) *If $u \in \mathcal{U}_{k,d}$ then $u = K' + D'$ with some $K' \in \mathcal{K}(\mathcal{H})$ and $D \in \mathcal{D}(\mathcal{U}(\mathcal{H}))$.*

(5) *For every* $K \in \mathcal{K}(\mathcal{H})^{ah}$ *and* $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ *there exists* $K' \in \mathcal{K}(\mathcal{H})^{ah}$ *such that* $e^K e^D = e^D e^{K'}$.
(6) $\mathcal{U}_k \subsetneq \mathcal{U}_{k+d}$.
(7) $\mathcal{U}_k = \{u \in \mathcal{U}(\mathcal{H}) : \exists D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah}),\ u - e^D \in \mathcal{K}(\mathcal{H})\}$.

Proof. (1) This is apparent.

(2) $u \in \mathcal{U}_k \Leftrightarrow u - 1 \in \mathcal{K}(\mathcal{H}) \Leftrightarrow u - e^0 \in \mathcal{K}(\mathcal{H})$.

(3) Let $e^{K+D} \in \mathcal{U}_{k+d}$ with $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. Then

$$e^{K+D} = 1 + (K + D) + \frac{1}{2!}(K + D)^2 + \cdots = e^D + K\Psi(K, D),$$

so $e^{K+D} - e^D \in \mathcal{K}(\mathcal{H})$, and hence $e^{K+D} \in \mathcal{U}_{k,d}$.

(4) $u \in \mathcal{U}_{k,d} \Rightarrow u - e^D \in \mathcal{K}(\mathcal{H})$ with $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}) \Rightarrow \exists K' \in \mathcal{K}(\mathcal{H})$, $u = K' + e^D$.

(5) If $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ then Proposition 3.1 implies that $e^K - 1 \in \mathcal{K}(\mathcal{H})$, which gives $(e^K - 1)e^D = e^K e^D - e^D \in \mathcal{K}(\mathcal{H})$ and $e^{-D}e^K e^D - 1 \in \mathcal{K}(\mathcal{H})$. Then $e^{-D}e^K e^D \in \mathcal{U}_k$ and there exists $K' \in \mathcal{K}(\mathcal{H})^{ah}$ such that $e^{-D}e^K e^D = e^{K'}$. The result follows easily.

(6) This is apparent.

(7) If $u \in \mathcal{U}_k$ then $u - 1 = u - e^0 \in \mathcal{K}(\mathcal{H})$ with $0 \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$, and so $u \in \{u \in \mathcal{U}(\mathcal{H}) : \exists D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah}),\ u - e^D \in \mathcal{K}(\mathcal{H})\}$. Conversely, let $u \in \mathcal{U}(\mathcal{H})$ and $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ be such that $u - e^D \in \mathcal{K}(\mathcal{H})$. Then

$$u - 1 = u - e^D + e^D - 1 \in \mathcal{K}(\mathcal{H}),$$

since $e^D \in \mathcal{U}_k$, which completes the proof. ∎

Proposition 3.5. *Let* $K_1, K_2 \in \mathcal{K}(\mathcal{H})^{ah}$ *and* $D_1, D_2 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. *Then the following statements are equivalent:*

(a) $e^{K_1}e^{D_1} = e^{K_2}e^{D_2}$.
(b) *There exists* $d \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ *such that*

$$e^{K_2} = e^{K_1}e^{-d} \quad \text{and} \quad e^{D_2} = e^d e^{D_1} = e^{d+D_1}.$$

Proof. (b)⇒(a) is apparent after computing $e^{K_2}e^{D_2}$.

Let us consider (a)⇒(b).

If $e^{K_1}e^{D_1} = e^{K_2}e^{D_2}$ then $e^{D_1-D_2} = e^{-K_1}e^{K_2}$. Since e^{-K_1}, e^{K_2} are in $\mathcal{U}_k$ which is a group, there exists $K_{1,2} \in \mathcal{K}(\mathcal{H})^{ah}$ such that $\|K_{1,2}\| \le \pi$ and $e^{-K_1}e^{K_2} = e^{K_{1,2}}$ (see Proposition 3.1). Moreover, there exists $D_{1,2} \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ with $\|D_{1,2}\| \le \pi$ such that $e^{D_1-D_2} = e^{D_{1,2}}$. Therefore

$$e^{K_{1,2}} = e^{-K_1}e^{K_2} = e^{D_1-D_2} = e^{D_{1,2}}$$

with $K_{1,2} \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_{1,2} \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. Using [7, Theorem 3.1] we conclude that $|K_{1,2}| = |D_{1,2}|$, which implies that $K_{1,2}$ and $D_{1,2}$ are both diagonal and compact operators. If we choose $-d = D_{1,2} \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$, then

$$e^{K_2} = e^{K_1}e^{D_1-D_2} = e^{K_1}e^{D_{1,2}} = e^{K_1}e^{-d},$$
$$e^{D_2} = e^{D_2-D_1}e^{D_1} = e^{-D_{1,2}}e^{D_1} = e^{d}e^{D_1},$$

which proves the proposition. ■

PROPOSITION 3.6. *Let $u \in \mathcal{U}(\mathcal{H})$. Then the following statements are equivalent:*

(a) $u \in \mathcal{U}_{k,d}$.
(b) $u - \operatorname{Diag}(u) \in \mathcal{K}(\mathcal{H})$ *and* $|u_{jj}| \to 1$ *as* $j \to \infty$.

Proof. (a)⇒(b). If $u \in \mathcal{U}_{k,d}$ then there exists $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that $u - e^D \in \mathcal{K}(\mathcal{H})$. Then $\operatorname{Diag}(u - e^D)_{jj} = u_{jj} - e^D_{jj} \to 0$ as $j \to \infty$, and therefore $\operatorname{Diag}(u) - e^D \in \mathcal{K}(\mathcal{H})$ and $|u_{jj}| \to 1$ as $j \to \infty$. Then since

$$u - e^D = u - \operatorname{Diag}(u) + \underbrace{\operatorname{Diag}(u) - e^D}_{\in\mathcal{K}(\mathcal{H})} \in \mathcal{K}(\mathcal{H}),$$

we deduce that $u - \operatorname{Diag}(u) \in \mathcal{K}(\mathcal{H})$.

(b)⇐(a) If $u_{jj} \to 1$ as $j \to \infty$ then there exists $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that $u_{jj} - e^D_{jj} \to 0$ as $j \to \infty$ (for example take $e^D_{jj} = (1/|u_{jj}|)u_{jj}$ when j is sufficiently large). Then $\operatorname{Diag}(u) - e^D \in \mathcal{K}(\mathcal{H})$. Hence

$$\underbrace{u - \operatorname{Diag}(u)}_{\in\mathcal{K}(\mathcal{H})} + \underbrace{\operatorname{Diag}(u) - e^D}_{\in\mathcal{K}(\mathcal{H})} = u - e^D \in \mathcal{K}(\mathcal{H}),$$

and therefore $u \in \mathcal{U}_{k,d}$. ■

REMARK 3.7. The previous proposition allows us to prove easily that $\mathcal{U}_{k,d} \subsetneq \mathcal{U}(\mathcal{H})$ since the block diagonal symmetry

$$u = \begin{pmatrix} s & 0 & 0 & \dots \\ 0 & s & 0 & \dots \\ 0 & 0 & s & \\ \vdots & \vdots & & \ddots \end{pmatrix} \quad \text{with} \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

clearly does not satisfy condition (b) of Proposition 3.6, but $u \in \mathcal{U}(\mathcal{H})$.

The following proposition is a consequence of the results of [7].

PROPOSITION 3.8. *Let $K, K' \in \mathcal{K}(\mathcal{H})^{ah}$ satisfy $\|K\|, \|K'\| \le \pi$ and let $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ be such that $e^De^K = e^{K'}e^D$. Then $\|K\| = \|K'\|$ and*

(a) *if* $\|K\| = \|K'\| = \pi$, *then*
  (1) $|K| = e^{-D}|K'|e^D$,
  (2) $v \in \mathcal{H}$ *is an eigenvector of* K *with eigenvalue* $\lambda \in i\mathbb{R}$, $|\lambda| < \pi$ ⇔ e^Dv *is an eigenvector of* K' *with eigenvalue* $\lambda \in i\mathbb{R}$, $|\lambda| < \pi$,

(3) *if E_X is the spectral measure of the operator X, then*

$$K - e^{-D}K'e^D = 2\pi i\big(E_K(\mathbb{R}+i\pi) - E_{e^{-D}K'e^D}(\mathbb{R}+i\pi)\big),$$

(b) *if $\|K\| = \|K'\| < \pi$ then $K = e^{-D}K'e^D$.*

Proof. Observe first that since $e^De^K = e^{K'}e^D$, we have

$$e^K = e^{-D}e^{K'}e^D = e^{e^{-D}K'e^D}, \tag{3.2}$$

and therefore $|K| = |e^{-D}K'e^D| = e^{-D}|K'|e^D$ (see [7, Theorem 3.1(i)]), which implies $\|K\| = \|K'\|$.

(a)(1) This is a direct consequence of (3.2), the fact that $\sigma(K)$ and $\sigma(e^{-D}K'e^D)$ are contained in $\mathcal{S} = \{z \in \mathbb{C} : -\pi \le \mathrm{Im}(z) \le \pi\}$, and [7, Theorem 3.1(i)].

(a)(2) Consider $\lambda \in \sigma(K) \subset i\mathbb{R}$, $|\lambda| < \pi$ and $v \in \mathcal{H}$ such that $Kv = \lambda v$. Then $e^Kv = e^\lambda v$ and (3.2) imply that e^λ is an eigenvalue of $e^{e^{-D}K'e^D}$ with eigenvector v. Then $\lambda \in \sigma(e^{-D}K'e^D)$ (because $|\lambda| < 1$) with eigenvector v. Therefore $\lambda \in \sigma(K')$ with eigenvector e^Dv. The other implication follows similarly.

(a)(3) This statement follows from $\sigma(K), \sigma(e^{-D}K'e^D) \subset \mathcal{S}$ and [7, Remark 2.4 and Theorem 4.1].

(b) If $\|K\| = \|K'\| < \pi$ then (3.2) and [7, Corollary 4.2(iii)] imply directly that $K = e^{-D}K'e^D$. ∎

Corollary 3.9. *Let $K, K' \in \mathcal{K}(\mathcal{H})^{ah}$ with $\|K\|, \|K'\| \le \pi$, and let $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. Then*

(a) *if $\|K\| = \|K'\| = \pi$, then*

$$e^De^K = e^{K'}e^D \;\Leftrightarrow\; K - e^{-D}K'e^D = 2\pi i\big(E_K(\mathbb{R}+i\pi) - E_{e^{-D}K'e^D}(\mathbb{R}+i\pi)\big),$$

(b) *if $\|K\|, \|K'\| < \pi$, then*

$$e^De^K = e^{K'}e^D \;\Leftrightarrow\; K = e^{-D}K'e^D.$$

Proof. (a) If $e^De^K = e^{K'}e^D$ then

$$K - e^{-D}K'e^D = 2\pi i\big(E_K(\mathbb{R}+i\pi) - E_{e^{-D}K'e^D}(\mathbb{R}+i\pi)\big)$$

follows from Proposition 3.8(a)(3).

The converse is proved using the fact that

$$K - 2\pi iE_K(\mathbb{R}+i\pi) = e^{-D}K'e^D - E_{e^{-D}K'e^D}(\mathbb{R}+i\pi)$$

implies that

$$e^{K-2\pi iE_K(\mathbb{R}+i\pi)} = e^{e^{-D}K'e^D-2\pi iE_{e^{-D}K'e^D}(\mathbb{R}+i\pi)},$$

and since K commutes with E_K, and $e^{-D}K'e^D$ with $E_{e^{-D}K'e^D}$, we obtain

$$e^Ke^{-2\pi iE_K(\mathbb{R}+i\pi)} = e^{e^{-D}K'e^D}e^{-2\pi iE_{e^{-D}K'e^D}(\mathbb{R}+i\pi)}.$$

Since $e^{-2\pi i E_K(\mathbb{R}+i\pi)} = e^{-2\pi i E_{e^{-D}K'e^D}(\mathbb{R}+i\pi)} = 1$, we have

$$e^K = e^{e^{-D}K'e^D} = e^{-D}e^{K'}e^D,$$

which ends the proof.

(b) This is apparent by using Proposition 3.8(b) and the fact that $e^{e^{-D}K'e^D} = e^{-D}e^{K'}e^D$. ∎

Thompson [11] proved that for any $X, Y \in M_n(\mathbb{C})^{ah}$ there exist unitaries U, V such that

$$e^X e^Y = e^{U^*XU+V^*YV}. \tag{3.3}$$

Subsequently, Antezana et al. [3] proved a generalization of (3.3) for compact operators: given $K_1, K_2 \in \mathcal{K}(\mathcal{H})^{ah}$, there exist unitaries U_n, V_n, for $n \in \mathbb{N}$, such that

$$e^{K_1}e^{K_2} = \lim_{n\to\infty} e^{U_n^*K_1U_n+V_n^*K_2V_n}$$

with convergence in the usual operator norm. The following proposition adds a new simple case where this equality holds.

Proposition 3.10. *Let $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ and suppose that there exists $\lambda \in i\mathbb{R}$ such that $\lim_{n\to\infty} D_{nn} = \lambda$. Then there exist unitaries U_n, V_n, for $n \in \mathbb{N}$, such that*

$$e^K e^D = \lim_{n\to\infty} e^{U_n^*KU_n+V_n^*DV_n}.$$

Proof. Observe that $D - \lambda I \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$. Then by [3, Theorem 3.1 and Remark 3.3] there exist unitaries U_n, V_n, for $n \in \mathbb{N}$, such that

$$\begin{aligned} e^K e^D e^{-\lambda I} = e^K e^{D-\lambda I} &= \lim_{n\to\infty} e^{U_n^*KU_n+V_n^*(D-\lambda I)V_n} \\ &= \lim_{n\to\infty} e^{U_n^*KU_n+V_n^*DV_n}e^{-\lambda I}. \end{aligned}$$

Therefore, $e^K e^D = \lim_{n\to\infty} e^{U_n^*KU_n+V_n^*DV_n}$. ∎

Proposition 3.11. *Let $K_1, K_2 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_1, D_2 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ be such that $K_1 + D_1$ and $K_2 + D_2$ are sufficiently close to* 0 (*see Definition* 2.1). *Then there exists $K \in \mathcal{K}(\mathcal{H})^{ah}$ such that*

$$e^{K_1+D_1}e^{K_2+D_2} = e^{K+D_1+D_2}.$$

Proof. Using the B-C-H formula (2.2), we have

$$X = \log(e^{K_1+D_1}e^{K_2+D_2}) = K_1 + D_1 + K_2 + D_2 + \sum_{n\ge 2} c_n(K_1 + D_1, K_2 + D_2).$$

Also, observe that $c_n(K_1 + D_1, K_2 + D_2) \in \mathcal{K}(\mathcal{H})^{ah}$ for every n, since $\mathcal{K}(\mathcal{H})$ is a two-sided closed ideal and $[D_1, D_2] = 0$. Therefore, $X = K + D$ with $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ and

$$e^{K_1+D_1}e^{K_2+D_2} = e^{K+D} \in \mathcal{U}_{k+d}. \tag{3.4}$$

In particular $D = D_1 + D_2$, since each c_n is a sum of commutators and $\mathrm{Diag}([A, B]) = 0$ for every $A, B \in \mathcal{B}(\mathcal{H})^{ah}$. ∎

Corollary 3.12. *Let $K_1, K_2 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_1, D_2 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$.*

(1) *If $K_1 + D_1$ and $K_2 + D_2$ are sufficiently close to* 0 (*see Definition* 2.1), *then*

$$e^{K_1+D_1} e^{K_2+D_2} = e^{\tilde{K}+D_1+D_2}$$

with

$$\tilde{K} = K_1 + K_2 + \sum_{n\geq 2} c_n(K_1 + D_1, K_2 + D_2) \in \mathcal{K}(\mathcal{H})^{ah}$$

and $\mathrm{Diag}(\tilde{K}) = \mathrm{Diag}(K_1 + K_2)$.

(2) *If K_1 and D_1 are sufficiently close to* 0, *there exist $K', K'' \in \mathcal{K}(\mathcal{H})^{ah}$ such that*

$$e^{K_1} e^{D_1} = e^{D_1} e^{K'} = e^{K''+D_1}. \tag{3.5}$$

Proof. These equalities are due to Proposition 3.4(3), Proposition 3.11 and equalities used in its proof. ∎

Theorem 3.13. *$\mathcal{U}_{k,d}$ is pathwise connected and closed in $\mathcal{U}(\mathcal{H})$.*

Proof. Every $u = e^K e^D \in \mathcal{U}_{k,d}$ (with $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$) is connected to 1 by the curve $\gamma(t) = e^{tK} e^{tD}$ for $t \in [0, 1]$.

To prove the closedness of $\mathcal{U}_{k,d}$, let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}_{k,d}$, $u_n = e^{K_n} e^{D_n}$ for $n \in \mathbb{N}$, with $K_n \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_n \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ be a sequence such that $\lim_{n\to\infty} u_n = u_0$ in the operator norm. We will prove that $u_0 \in \mathcal{U}_{k,d}$.

Since $u_0 - u_n = u_0 - e^{D_n} + e^{D_n} - u_n \to 0$ as $n \to \infty$, and $e^{D_n} - u_n \in \mathcal{K}(\mathcal{H})$ for all $n \in \mathbb{N}$, we see that $\mathrm{dist}(\{u_0 - e^{D_n}\}_{n\in\mathbb{N}}, \mathcal{K}(\mathcal{H})) = 0$.

Observe that

$$\begin{aligned} &\mathrm{dist}\big(\{\mathrm{Diag}(u_0) - e^{D_n}\}_{n\in\mathbb{N}}, \mathcal{K}(\mathcal{H})\big) \\ &\quad= \mathrm{dist}\big(\{\mathrm{Diag}(u_0) - \mathrm{Diag}(u_n) + \mathrm{Diag}(u_n) - e^{D_n}\}_{n\in\mathbb{N}}, \mathcal{K}(\mathcal{H})\big) \\ &\quad\leq \inf_{K\in\mathcal{K}(\mathcal{H})} \big(\|\mathrm{Diag}(u_0) - \mathrm{Diag}(u_n)\| + \|\mathrm{Diag}(u_n) - e^{D_n} - K\|\big) \end{aligned} \tag{3.6}$$

for any $n \in \mathbb{N}$.

Note that $u_n - e^{D_n} \in \mathcal{K}(\mathcal{H})$, which implies that $\mathrm{Diag}(u_n - e^{D_n}) = \mathrm{Diag}(u_n) - e^{D_n} \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$. Since $u_n \to u_0$, we have $\mathrm{Diag}(u_n) \to \mathrm{Diag}(u_0)$. Then the first summand in (3.6) can be arbitrarily small for large n and the infimum of the second term is zero because $\mathrm{Diag}(u_n) - e^{D_n} \in \mathcal{K}(\mathcal{H})$. Thus

$$\begin{aligned} &\mathrm{dist}\big(\{\mathrm{Diag}(u_0) - e^{D_n}\}_{n\in\mathbb{N}}, \mathcal{K}(\mathcal{H})\big) = 0 \\ &\qquad\qquad = \mathrm{dist}\big(\{\mathrm{Diag}(u_0) - e^{D_n}\}_{n\in\mathbb{N}}, \mathcal{D}(\mathcal{K}(\mathcal{H}))\big). \end{aligned} \tag{3.7}$$

Moreover, since $u_n - e^{D_n} \in \mathcal{K}(\mathcal{H})$, for every $K \in \mathcal{K}(\mathcal{H})$ we have

$$\begin{aligned}(3.8)\quad \operatorname{dist}(u_0 - \operatorname{Diag}(u_0), \mathcal{K}(\mathcal{H})) &= \inf_{K\in\mathcal{K}(\mathcal{H})} \|u_0 - \operatorname{Diag}(u_0) - K\| \\ &\le \|u_0 - \operatorname{Diag}(u_0) - u_n + e^{D_n} - K\| \\ &\le \|u_0 - u_n\| + \|e^{D_n} - \operatorname{Diag}(u_0) - K\|.\end{aligned}$$

Here both summands can be arbitrarily small, since $u_n \to u_0$ and the distance from $\{\operatorname{Diag}(u_0) - e^{D_n}\}_{n\in\mathbb{N}}$ to $\mathcal{K}(\mathcal{H})$ is null as seen above in (3.7). Thus $\operatorname{dist}(u_0 - \operatorname{Diag}(u_0), \mathcal{K}(\mathcal{H})) = 0$, and therefore

$$(3.9)\qquad u_0 - \operatorname{Diag}(u_0) \in \mathcal{K}(\mathcal{H}).$$

If there exists $\delta > 0$ such that for a subsequence $\{e^{D_{n_k}}\}_{k\in\mathbb{N}}$ we have $|(\operatorname{Diag}(u_0) - e^{D_{n_k}})_{jj}| \ge \delta$ for all $k \in \mathbb{N}$ and infinitely many $j \in \mathbb{N}$, then we obtain a contradiction to (3.7). Therefore, given $\delta > 0$, only finitely many $n \in \mathbb{N}$ satisfy $|(\operatorname{Diag}(u_0) - e^{D_n})_{jj}| \ge \delta$ for infinitely many $j \in \mathbb{N}$. Thus, if $k \in \mathbb{N}$ and we choose $\delta = 1/k$, there exists $n_k \in \mathbb{N}$ such that if $n \ge n_k$ then $|(\operatorname{Diag}(u_0) - e^{D_n})_{jj}| \ge 1/k$ only for finitely many $j \in \mathbb{N}$. Observe that the subsequence n_k could be chosen to be strictly increasing. For each $k \in \mathbb{N}$, let $j_k \in \mathbb{N}$ be such that $|(\operatorname{Diag}(u_0) - e^{D_{n_k}})_{jj}| < 1/k$ for all $j \ge j_k$. We can choose j_k to be strictly increasing in k and $j_1 > 1$. Therefore, for each $k \in \mathbb{N}$, there exist $n_k, j_k \in \mathbb{N}$ such that

$$(3.10)\qquad |(\operatorname{Diag}(u_0) - e^{D_{n_k}})_{jj}| < 1/k \quad \text{for all } j \ge j_k.$$

Then define the following unitary diagonal matrix e^D:

$$(3.11)\qquad (e^D)_{jj} = \begin{cases} 1 & \text{if } 1 \le j < j_1, \\ (e^{D_{n_1}})_{jj} & \text{if } j_1 \le j < j_2, \\ (e^{D_{n_2}})_{jj} & \text{if } j_2 \le j < j_3, \\ \dots \end{cases}$$

D can be chosen as the anti-Hermitian diagonal matrix formed with the corresponding parts of $0, D_{n_1}, D_{n_2}, \dots$.

If we define $j_0 = 1$ and take any $j \in \mathbb{N}$, then (3.10) and (3.11) imply that

$$|(\operatorname{Diag}(u_0) - e^D)_{jj}| = |(\operatorname{Diag}(u_0) - e^{D_{n_k}})_{jj}| < 1/k \quad \text{if } j_k \le j < j_{k+1}.$$

Then $(\operatorname{Diag}(u_0) - e^D)_{jj} \to 0$ as $j \to \infty$, and therefore $\operatorname{Diag}(u_0) - e^D \in \mathcal{K}(\mathcal{H})$ since it is a diagonal matrix.

Using $u_0 - \operatorname{Diag}(u_0) \in \mathcal{K}(\mathcal{H})$ (see (3.9)) we conclude that $(u_0 - \operatorname{Diag}(u_0)) + (\operatorname{Diag}(u_0) - e^D) = u_0 - e^D \in \mathcal{K}(\mathcal{H})$, which implies that $u_0 \in \mathcal{U}_{k,d}$, and therefore $\mathcal{U}_{k,d}$ is closed. ∎

LEMMA 3.14. *There exists $\varepsilon_0 > 0$ such that if $u \in \mathcal{U}_{k,d}$ and $\|u - 1\| < \varepsilon_0$ then $u \in \mathcal{U}_{k+d}$. Moreover, there exist $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that*

(a) *$u = e^{K+D}$ with $K, D \in \exp^{-1}(B(1, 3\varepsilon_0))$,*
(b) *K, D are sufficiently close to 0,*
(c) *$K + D \in \exp^{-1}(B(1, \varepsilon_0)) \cap \mathcal{B}(\mathcal{H})^{ah}$.*

Proof. Fix $\delta_0 > 0$ that fulfills two conditions: first, if $V \in B(0, \delta_0) \cap \mathcal{B}(\mathcal{H})^{ah}$ then V is sufficiently close to 0, and secondly, $\exp : B(0, \delta_0) \cap \mathcal{B}(\mathcal{H})^{ah} \to \exp(B(0, \delta_0)) \cap \mathcal{U}(\mathcal{H})$ is a diffeomorphism under the usual operator norm. The last requirement can be fulfilled after applying the inverse map theorem for Banach spaces.

Choose $\varepsilon_0 = \varepsilon > 0$ such that

$$B(1, \varepsilon) \subset B(1, 3\varepsilon) \subset \exp(B(0, \delta_0)). \tag{3.12}$$

If $u \in \mathcal{U}_{k,d} \cap B(1, \varepsilon)$, then there exist $K_1 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_1 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that $u = e^{K_1} e^{D_1}$. Observe that the j, j entry of the diagonal of $u = e^{K_1} e^{D_1}$ is $e^{K_1}_{jj} e^{D_1}_{jj}$.

Now $\|u - 1\| < \varepsilon$ implies that $|e^{K_1}_{jj} e^{D_1}_{jj} - 1| < \varepsilon$ for all $j \in \mathbb{N}$. Suppose that $|e^{D_1}_{jj} - 1| \geq 2\varepsilon$ for infinitely many $j \in \mathbb{N}$. Then, using $|e^{D_1}_{jj}| = 1$ we obtain $|e^{-D_1}_{jj} - 1| = |e^{D_1}_{jj}(e^{-D_1}_{jj} - 1)| = |1 - e^{D_1}_{jj}| \geq 2\varepsilon$, and $|e^{K_1}_{jj} - e^{-D_1}_{jj}| = |(e^{K_1}_{jj} - e^{-D_1}_{jj}) e^{D_1}_{jj}| = |e^{K_1}_{jj} e^{D_1}_{jj} - 1| < \varepsilon$ for infinitely many $j \in \mathbb{N}$. Therefore, there must exist infinitely many $j \in \mathbb{N}$ such that

$$|e^{K_1}_{jj} - e^{-D_1}_{jj}| < \varepsilon \quad \text{and} \quad |e^{-D_1}_{jj} - 1| \geq \varepsilon \tag{3.13}$$

(see Figure 1).

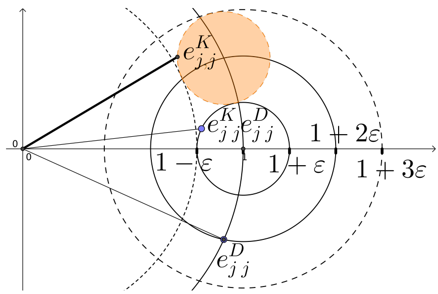

Fig. 1

Hence

$$\begin{aligned} |e^{K_1}_{jj} - 1| &= |e^{K_1}_{jj} - e^{-D_1}_{jj} + e^{-D_1}_{jj} - 1| \geq \big| |e^{K_1}_{jj} - e^{-D_1}_{jj}| - |e^{-D_1}_{jj} - 1| \big| \\ &= |e^{-D_1}_{jj} - 1| - |e^{K_1}_{jj} - e^{-D_1}_{jj}| \geq 2\varepsilon - \varepsilon = \varepsilon. \end{aligned}$$

for infinitely many $j \in \mathbb{N}$, where we have used (3.13) in the last equality and inequality. This is a contradiction because $e^{K_1} \in \mathcal{U}_k$ and so $e^{K_1} - 1 \in \mathcal{K}(\mathcal{H})$, which implies that the diagonal of e^{K_1} tends to 1. Thus $|e^{D_1}_{jj} - 1| \geq 2\varepsilon$ only for finitely many $j \in \mathbb{N}$. Choosing appropriately a compact anti-Hermitian diagonal d we can construct $D_2 = D_1 + d \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ and $K_2 \in \mathcal{K}(\mathcal{H})^{ah}$ such that $u = e^{K_1}e^{D_1} = e^{K_2}e^{D_2}$ and $|e^{D_2}_{jj} - 1| \leq 2\varepsilon$ for all $j \in \mathbb{N}$ (see Proposition 3.5). Then $\|e^{D_2} - 1\| < 2\varepsilon$ and $\|e^{D_2} - 1\| = \|e^{-D_2}(e^{D_2} - 1)\| = \|1 - e^{-D_2}1\| = \|e^{-D_2} - 1\| < 2\varepsilon$. Moreover, since e^{D_2} is unitary,

$$\begin{aligned} \|e^{K_2} - 1\| &\leq \|e^{K_2} - e^{-D_2}\| + \|e^{-D_2} - 1\| \\ &= \|(e^{K_2} - e^{-D_2})e^{D_2}\| + \|e^{-D_2} - 1\| \\ &= \|e^{K_2}e^{D_2} - 1\| + \|e^{-D_2} - 1\| \\ &= \|u - 1\| + \|e^{-D_2} - 1\| < \varepsilon + 2\varepsilon = 3\varepsilon. \end{aligned} \tag{3.14}$$

We have shown that $\|e^{D_2} - 1\| < 2\varepsilon$ and $\|e^{K_2} - 1\| < 3\varepsilon$, which implies that $e^{D_2}, e^{K_2} \in \exp(B(0, \delta_0))$ (see the definition of ε in (3.12). Therefore, since $\exp : B(0, \delta_0) \cap \mathcal{B}(\mathcal{H})^{ah} \to \exp(B(0, \delta_0)) \cap \mathcal{U}(\mathcal{H})$ is a diffeomorphism, there exist unique $D, K \in \exp^{-1}(B(0, 3\varepsilon)) \cap \mathcal{B}(\mathcal{H})^{ah} \subset B(0, \delta_0) \cap \mathcal{B}(\mathcal{H})^{ah}$ such that $e^D = e^{D_2}$ and $e^K = e^{K_2}$. Standard calculations show that D must be diagonal and K compact. Hence $D \in \exp^{-1}(B(0, 3\varepsilon)) \cap \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ and $K \in \exp^{-1}(B(0, 3\varepsilon)) \cap \mathcal{K}(\mathcal{H})^{ah}$. Moreover, since $D, K \in B(0, \delta_0)$, they are sufficiently close to 0. Then using (3.5) we get

$$u = e^K e^D = e^{K+D} \in \mathcal{U}_{k+d}$$

with $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$, $K, D \in \exp^{-1}(B(0, 3\varepsilon))$ and K, D sufficiently close to 0 as required in (a) and (b).

Since $e^{K+D} = u \in \mathcal{U}_{k,d}$, and K and D are sufficiently close to 0, we have $\|K+D\| < \pi$. Hence, as $\exp : \exp^{-1}(B(1, \varepsilon)) \to B(1, \varepsilon)$ is a diffeomorphism, there exists $V \in \exp^{-1}(B(1, \varepsilon))$ such that $e^V = u = e^{K+D}$. Then [7, Corollary 4.2] implies that $V = K + D$, and therefore $K + D \in \exp^{-1}(B(1, \varepsilon))$ as required in (c). ∎

Proposition 3.15. *There exists an open neighborhood $\mathcal{V} \subset \mathcal{B}(\mathcal{H})^{ah}$ of* 0 *such that*

$$\exp\big(\mathcal{V} \cap (\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}))\big) = \exp(\mathcal{V}) \cap \mathcal{U}_{k,d}.$$

Proof. Take $\mathcal{V} = \exp^{-1}(B(1, \varepsilon_0)) \cap \mathcal{B}(\mathcal{H})^{ah}$, where ε_0 is as in Lemma 3.14. Then, as seen in that lemma, for every $u \in \mathcal{U}_{k,d}$ such that $u = e^{K_1}e^{D_1} \in B(1, \varepsilon_0)$ with $K_1 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_1 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$, there exist $K \in \mathcal{K}(\mathcal{H})^{ah} \cap \exp^{-1}(B(1, 3\varepsilon_0))$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}) \cap \exp^{-1}(B(1, 3\varepsilon_0))$ such that

$$u = e^{K+D} \quad \text{with} \quad K + D \in \exp^{-1}(B(1, \varepsilon_0)) \cap \mathcal{B}(\mathcal{H})^{ah}. \tag{3.15}$$

Suppose first that $V \in \mathcal{V}$ and $e^V \in \exp(\mathcal{V}) \cap \mathcal{U}_{k,d}$. Then, as in (3.15), there exist $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that $e^V = e^{K+D}$. Since the exponential is a diffeomorphism when restricted to $\mathcal{V}$, it follows that $V = K + D$. Therefore

$$\exp(\mathcal{V}) \cap \mathcal{U}_{k,d} \subset \exp\big(\mathcal{V} \cap (\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}))\big).$$

Now suppose that $V \in \mathcal{V}$ and

$$e^V = e^{K+D} \in \exp\big(\mathcal{V} \cap (\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}))\big)$$

with $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. Then clearly $e^V \in \exp(\mathcal{V})$, and using Proposition 3.4(3) we find that also $e^V = e^{K+D} \in \mathcal{U}_{k,d}$. This proves that $\exp(\mathcal{V} \cap (\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}))) \subset \exp(\mathcal{V}) \cap \mathcal{U}_{k,d}$, which concludes the proof. ∎

Proposition 3.16. $\{X \in \mathcal{B}(\mathcal{H})^{ah} : e^{tX} \in \mathcal{U}_{k,d}, \forall t \in \mathbb{R}\} = \mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$.

Proof. Proposition 3.4(3) directly implies that $e^{t(K+D)} = e^{tK+tD} \in \mathcal{U}_{k,d}$ for all $t \in \mathbb{R}$, and therefore $\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}) \subset L(\mathcal{U}_{k,d})$.

Suppose now that $X \neq 0$ (0 is a trivial case) and let $X \in L(\mathcal{U}_{k,d})$. Then $e^{tX} \in \mathcal{U}_{k,d}$ for all $t \in \mathbb{R}$. In particular $e^{tX} \in \mathcal{U}_{k,d}$ for small $|t|$, for example for $t_0 = \delta_0/(2\|X\|) < \delta_0/\|X\|$ where $\delta_0 > 0$ is as in the proof of Lemma 3.14. Then $\|t_0 X\| = |t_0|\,\|X\| < \delta_0$ and $u = e^{t_0 X} \in \mathcal{U}_{k,d}$. Therefore by Lemma 3.14, there exist $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that

$$e^{t_0 X} = e^{K+D}.$$

The constant δ_0 of the proof of Lemma 3.14 is chosen such that $\exp : B(0,\delta_0) \cap \mathcal{B}(\mathcal{H})^{ah} \to \exp(B(0,\delta_0)) \cap \mathcal{U}(\mathcal{H})$ is a diffeomorphism. Then $t_0 X = K + D$ and therefore $X = (1/t_0)(K+D) = (1/t_0)K + (1/t_0)D \in \mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ as required. ∎

Remark 3.17. Following [9, V.2.3] and [10, p. 428] we call H a *Lie subgroup* of a Banach–Lie group G if H is a closed subgroup of G which is itself a Lie group relative to the induced topology.

The previous results allow us to state the following.

Theorem 3.18. *$\mathcal{U}_{k,d}$ is a Lie subgroup of $\mathcal{U}(\mathcal{H})$ and its Lie algebra is $L(\mathcal{U}_{k,d}) = \mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$.*

Proof. According to the definition of Lie subgroup in Remark 3.17, Theorem 3.13 and Proposition 3.15 imply that $\mathcal{U}_{k,d}$ is a Lie subgroup of $\mathcal{U}(\mathcal{H})$.

The equality $L(\mathcal{U}_{k,d}) = \mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ follows from [9, Corollary V.2.2] and Proposition 3.16. ∎

Although there exist stronger notions of Lie subgroup, they cannot be used for $\mathcal{U}_{k,d}$ since its Lie algebra $\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ is not complemented in $\mathcal{B}(\mathcal{H})^{ah}$ (the Lie algebra of $\mathcal{U}(\mathcal{H})$).

Remark 3.19 (Generalization of the $\mathcal{U}_{k,d}$ group). The proofs of some of the basic properties we use in the study of $\mathcal{U}_{k,d}$ require that the exponential $\exp : \mathcal{K}(\mathcal{H})^{ah} \to \mathcal{U}_k$ be surjective. This is why the following generalization involves ideals $\mathcal{J}$ with this property.

If $\mathcal{J} \subset \mathcal{B}(\mathcal{H})$ is either of the two-sided closed ideals of p-Schatten operators (for $p \in [0,\infty)$) or $\mathcal{K}(\mathcal{H})$, and $\mathcal{A}$ is any C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, then the following subsets of $\mathcal{U}(\mathcal{H})$ can be defined, by analogy with (3.1):

$$\begin{aligned}
\mathcal{U}_{\mathcal{J}} &= \{u \in \mathcal{U}(\mathcal{H}) : u - 1 \in \mathcal{J}\}, \\
\mathcal{U}_{\mathcal{J},\mathcal{A}} &= \{u \in \mathcal{U}(\mathcal{H}) : \exists A \in \mathcal{A}^{ah},\ u - e^A \in \mathcal{J}\}, \\
\mathcal{U}_{\mathcal{A}} &= \{u \in \mathcal{U}(\mathcal{H}) : \exists A \in \mathcal{A}^{ah},\ u = e^A\}, \\
\mathcal{U}_{\mathcal{J}+\mathcal{A}} &= \{u \in \mathcal{U}(\mathcal{H}) : \exists J \in \mathcal{J}^{ah}\ \exists A \in \mathcal{A}^{ah},\ u = e^{J+A}\}.
\end{aligned}$$

The groups $\mathcal{U}_{\mathcal{J}}$, where $\mathcal{J}$ is any p-Schatten ideal of $\mathcal{B}(\mathcal{H})$, were studied in [2].

It can be proved that the above sets have the following properties:

(1) $\mathcal{U}_{\mathcal{J}} = \{e^J \in \mathcal{U}(\mathcal{H}) : J \in \mathcal{J}^{ah},\ \|J\| \le \pi\}$.
(2) $\mathcal{U}_{\mathcal{J},\mathcal{A}}$ is a group, equals

$$\mathcal{U}_{\mathcal{J},\mathcal{A}} = \{u \in \mathcal{U}(\mathcal{H}) : \exists J \in \mathcal{J}^{ah},\ \|J\| \le \pi,\ \exists A \in \mathcal{A}^{ah},\ u = e^J e^A\},$$

and $\mathcal{U}_{\mathcal{J},\mathcal{A}} = \mathcal{U}_{\mathcal{J}}\mathcal{U}_{\mathcal{A}} = \mathcal{U}_{\mathcal{A}}\mathcal{U}_{\mathcal{J}}$.
(3) $\mathcal{U}_{\mathcal{A}} \subsetneq \mathcal{U}_{\mathcal{J},\mathcal{A}}$.
(4) $\mathcal{U}_{\mathcal{J}} \subsetneq \mathcal{U}_{\mathcal{J},\mathcal{A}}$.
(5) $\mathcal{U}_{\mathcal{J}+\mathcal{A}} \subseteq \mathcal{U}_{\mathcal{J},\mathcal{A}}$.
(6) If $u \in \mathcal{U}_{\mathcal{J},\mathcal{A}}$ then $u = J' + A'$ with some $J' \in \mathcal{J}$ and $A' \in \mathcal{U}_{\mathcal{A}}$.
(7) For every $J \in \mathcal{J}^{ah}$ and $A \in \mathcal{A}^{ah}$ there exists $J' \in \mathcal{J}^{ah}$ such that $e^J e^A = e^A e^{J'}$.
(8) $\mathcal{U}_{\mathcal{J}} \subsetneq \mathcal{U}_{\mathcal{J}+\mathcal{A}}$.
(9) If $\mathcal{A}^{ah} \cap \mathcal{J} \neq \emptyset$, then

$$\mathcal{U}_{\mathcal{J}} = \{u \in \mathcal{U}(\mathcal{H}) : \exists A \in \mathcal{A}^{ah} \cap \mathcal{J},\ u - e^A \in \mathcal{J}\}.$$

(10) For every $J \in \mathcal{J}^{ah}$ and $A \in \mathcal{A}^{ah}$ sufficiently close to 0, there exist $J'' \in \mathcal{J}^{ah}$ and $A' \in \mathcal{A}^{ah}$ such that

$$e^J e^A = e^{J''+A'}.$$

(11) For every $J_1, J_2 \in \mathcal{J}^{ah}$ and $A_1, A_2 \in \mathcal{A}^{ah}$ the following statements are equivalent:
- $e^{J_1} e^{A_1} = e^{J_2} e^{A_2}$.
- There exists $a \in \mathcal{A}^{ah} \cap \mathcal{J}$ such that $e^{J_2} = e^{J_1} e^{-a}$ and $e^{A_2} = e^a e^{A_1} = e^{a+A_1}$.

Property (1) has been proved for p-Schatten ideals in [2, Remark 3.1], and for $\mathcal{K}(\mathcal{H})$ see Proposition 3.1. Properties (2)–(9) and (11) may be proved in much the same way as Propositions 3.3–3.5. Property (10) involves the B-C-H series expansion $\log(e^J e^A)$ for the Lie algebra $\mathcal{B}(\mathcal{H})^{ah}$ (see the Preliminaries).

4. Minimal length curves in the orbit of a compact self-adjoint operator. Consider the unitary Fredholm orbit of a compact operator

$$b = \mathrm{Diag}(\{\lambda_i\}_{i\in\mathbb{N}}) \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$$

with $\lambda_i \neq \lambda_j$ for each $i \neq j$, and the orbit

$$\mathcal{O}_b = \{ubu^* : u \in \mathcal{U}_k\}.$$

The isotropy subgroup of $c = e^{K_0}be^{-K_0} \in \mathcal{O}_b$ with $K_0 \in \mathcal{K}(\mathcal{H})^{ah}$ for the action $L_u \cdot c = ucu^*$, with $u \in \mathcal{U}_k$, is $\mathcal{I}_c = \{e^{K_0}e^d e^{-K_0} : d \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})\} = \{e^{e^{K_0}de^{-K_0}} : d \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})\}$. $T_c(\mathcal{O}_b)$ can be identified with the quotient space $\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ for every $c \in \mathcal{O}_b$. The projection to the quotient $\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ defines a Finsler metric as

$$\|x\|_{e^{K_0}e^d e^{-K_0}} = \|[Y]\| = \inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Y + e^{K_0}De^{-K_0}\|$$

for each class $[Y] = \{Y + e^{K_0}De^{-K_0} : D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})\}$ and $x = Yc - cY \in T_c(\mathcal{O}_b)$. This metric is invariant under the action of L_{e^K} for $e^K \in \mathcal{U}_k$ (see [6]). Assuming that $0 \in [r, s]$, this invariance implies that the curve $\gamma : [r, s] \to \mathcal{O}_b$ such that $\gamma(0) = b$ and $\dot\gamma(0) = x$ has the same length as $\beta = L_{e^K} \cdot \gamma : [r, s] \to \mathcal{O}_b$ and satisfies $\beta(0) = L_{e^K} \cdot b$, $\dot\beta(0) = L_{e^K} \cdot \dot\gamma(0) = L_{e^K} \cdot x$, for $e^K \in \mathcal{U}_k$.

Proposition 4.1. *Let $b = \mathrm{Diag}(\{\lambda_i\}_{i\in\mathbb{N}}) \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$ with $\lambda_i \neq \lambda_j$ for each $i \neq j$ and $Z_0 = S_0 + D_0$ with $S_0 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_0 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. Then $e^{Z_0}be^{-Z_0} \in \mathcal{O}_b$.*

Proof. Observe that D_0 is not necessarily compact. Using Remark 3.2 we can rewrite the exponential $e^{Z_0} = e^{S_0+D_0}$ as

$$e^{Z_0} = e^{D_0} + S_0\Psi(S_0, D_0).$$

Then $(e^{D_0} + S_0\Psi(S_0, D_0))e^{-D_0}$ is unitary and $S_0\Psi(S_0, D_0)e^{-D_0} = e^{D_0-D_0} - 1 + S_0\Psi(S_0, D_0)e^{-D_0} \in \mathcal{K}(\mathcal{H})$ since $S_0 \in \mathcal{K}(\mathcal{H})$. Moreover

$$(e^{D_0} + S_0\Psi(S_0, D_0))e^{-D_0} - 1 \in \mathcal{K}(\mathcal{H}),$$

which implies that $e^{S_0+D_0}e^{-D_0} - 1 \in \mathcal{K}(\mathcal{H})$. Therefore, by [6, Proposition 3] there exists $K \in \mathcal{K}(\mathcal{H})^{ah}$ such that

$$e^{S_0+D_0}e^{-D_0} = e^K, \quad \text{and therefore} \quad e^{S_0+D_0} = e^K e^{D_0}.$$

Then

$$e^{Z_0}be^{-Z_0} = e^{S_0+D_0}be^{-S_0-D_0} = e^K e^{D_0}be^{-D_0}e^{-K} = e^K be^{-K} \in \mathcal{O}_b. \ \blacksquare$$

Theorem 4.2. *Let $Z_0 = S_0 + D_0$ with $S_0 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_0 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ and let $b = \mathrm{Diag}(\{\lambda_i\}_{i\in\mathbb{N}}) \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$ with $\lambda_i \neq \lambda_j$ for each $i \neq j$, and $\gamma(t) = e^{tZ_0}be^{-tZ_0}$ for all $t \in \mathbb{R}$. Then*

(a) *$\gamma(t) \in \mathcal{O}_b$ for all $t \in \mathbb{R}$, and*
(b) *if Z_0 is minimal for $\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ (see (1.2) and the Preliminaries) then $\gamma : [-\pi/(2\|Z_0\|), \pi/(2\|Z_0\|)] \to \mathcal{O}_b$ is a minimal length curve on $\mathcal{O}_b$ relative to the distance (2.6).*

Proof. (a) follows directly from Proposition 4.1. Note that (a) holds even though Z_0 may not be compact.

In order to prove (b) consider $\mathcal{P}_b = \{ubu^* : u \in \mathcal{U}(\mathcal{H})\}$; then by [8, Theorem II], since Z_0 is minimal, the curve γ has minimal length over all the smooth curves in $\mathcal{P}_b$ that join $\gamma(0) = b$ and $\gamma(t)$ with $|t| \le \pi/(2\|Z_0\|)$. Since clearly $\mathcal{O}_b \subseteq \mathcal{P}_b$, for each $t_0 \in [-\pi/(2\|Z_0\|), \pi/(2\|Z_0\|)]$ the curve γ is minimal in $\mathcal{O}_b$, that is,

$$\mathrm{L}(\gamma) = \mathrm{dist}(b, \gamma(t_0)),$$

where dist is the rectifiable distance defined in (2.6). ∎

REMARK 4.3. Recall that for every S_0 there always exists a minimal $Z_0 \in \mathcal{B}(\mathcal{H})^{ah}$ as in Theorem 4.2 although it may not be compact (see [8], [5]).

PROPOSITION 4.4. *If* $b = \mathrm{Diag}(\{\lambda_i\}_{i\in\mathbb{N}}) \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$ *with* $\lambda_i \ne \lambda_j$ *for each* $i \ne j$*, then* $\mathcal{O}_b = \mathcal{O}_b^{\mathcal{U}_{k,d}}$ *and* $\mathcal{O}_b^{\mathcal{U}_{k+d}} \subseteq \mathcal{O}_b$.

Proof. Since b is diagonal and $e^K e^D b e^{-D} e^{-K} = e^K b e^{-K}$, it follows that $\mathcal{O}_b = \mathcal{O}_b^{\mathcal{U}_{k,d}}$. The inclusion $\mathcal{O}_b^{\mathcal{U}_{k+d}} \subseteq \mathcal{O}_b$ is trivial because b is diagonal and $\mathcal{U}_{k,d} \supset \mathcal{U}_{k+d}$ (see Proposition 3.4(3)). ∎

REMARK 4.5. Under the assumptions of Proposition 4.4, if $c \in \mathcal{O}_b$ the following identifications can be made:

$$T_c(\mathcal{O}_b) \cong T_1(\mathcal{U}_k)/T_1(\mathcal{I}_b) = \mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$$

and

$$\begin{aligned} T_c(\mathcal{O}_b^{\mathcal{U}_{k,d}}) &\cong T_1(\mathcal{U}_{k,d})/T_1(\mathcal{I}_b) \\ &\cong \big(\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})\big)/\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}). \end{aligned}$$

Moreover the norm on each quotient coincides on every class since for $K \in \mathcal{K}(\mathcal{H})^{ah}$ we have $\|[K]\| = \inf_{d\in\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|K + d\| = \inf_{D\in\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})} \|K + D\|$ (see for example [5, Proposition 5]). Therefore the Finsler metrics defined by the subgroups $\mathcal{U}_k$ and $\mathcal{U}_{k,d}$ coincide on $\mathcal{O}_b$.

Let $c = L_{e^{K_0}} \cdot b = e^{K_0} b e^{-K_0} \in \mathcal{O}_b$ (for $K_0 \in \mathcal{K}(\mathcal{H})^{ah}$) and $x \in T_c(\mathcal{O}_b)$. Then there always exists a vector $z_c = L_{e^{K_0}} \cdot Z_0$ with $z_c c - c z_c = x$ minimal for $\{F \in \mathcal{B}(\mathcal{H})^{ah} : Fc - cF = 0\} = \{F \in \mathcal{B}(\mathcal{H})^{ah} : F = e^{K_0} D e^{-K_0}$ for some $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})\}$ such that Z_0 is minimal for $\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ as in Theorem 4.2(b) and Remark 4.3. That is,

$$\begin{aligned} \|x\|_c = \|[z_c]\|_c &= \inf_{F\in\mathcal{B}(\mathcal{H})^{ah}\cap\{c\}'} \|z_c + F\| \\ &= \inf_{D\in\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})} \|e^{K_0} Z_0 e^{-K_0} + e^{K_0} D e^{-K_0}\| \\ &= \inf_{D\in\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})} \|Z_0 + D\| = \|[Z_0]\|. \end{aligned}$$

This equality and the left invariance of the action L_{e^K} imply that the curve

$$\beta(t) = e^{tz_c} c e^{-tz_c}$$

for $t \in [-\pi/(2\|z_c\|), \pi/(2\|z_c\|)]$ satisfies $\beta(0) = c$, $\dot\beta(0) = x = z_c c - c z_c$ and $\mathrm{L}(\beta|_{[r,s]}) = \mathrm{L}(\gamma|_{[r,s]})$ for the curve γ mentioned in Theorem 4.2 and every $[r,s] \subset [-\pi/(2\|z_c)\|, \pi/(2\|z_c\|)]$. The previous comments and results allow us to prove the following.

Corollary 4.6. *Let $b = \mathrm{Diag}(\{\lambda_i\}_{i\in\mathbb{N}}) \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ for each $i \neq j$, $c = e^{K_0} b e^{-K_0} \in \mathcal{O}_b$, with $K_0 \in \mathcal{K}(\mathcal{H})^{ah}$, and $x \in T_c(\mathcal{O}_b)$. Then there exists $Z_0 \in \mathcal{B}(\mathcal{H})^{ah}$ minimal for $\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that $\beta(t) = e^{tz_c} c e^{-tz_c} \in \mathcal{O}_b$ for all $t \in \mathbb{R}$, $z_c = e^{K_0} Z_0 e^{-K_0}$ and $x = L_{e^{K_0}} \cdot (Z_0 b - b Z_0)$. Moreover, $\beta : [-\pi/(2\|z_c\|), \pi/(2\|z_c\|)] \to \mathcal{O}_b$ is a minimal length curve in $\mathcal{O}_b$ such that $\beta(0) = c$, $\dot\beta(0) = x$ relative to the distance* (2.6).

Proof. Given $x \in T_c(\mathcal{O}_b)$ we can choose $Z_0 \in \mathcal{K}(\mathcal{H})^{ah}$ such that $Z_0 b - b Z_0 = L_{e^{-K_0}} \cdot x \in T_b(\mathcal{O}_b)$ and $\|Z_0\| = \inf_{D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})} \|Z_0 + D\|$ as in Theorem 4.2 and Remark 4.3. Z_0 is minimal for $\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ and therefore Theorem 4.2(b) applies and $\gamma(t) = e^{tZ_0} b e^{-tZ_0}$ is a short curve for $t \in [-\pi/(2\|Z_0\|), \pi/(2\|Z_0\|)]$. Direct calculations show that

$$\begin{aligned} x &= L_{e^{K_0}} \cdot (Z_0 b - b Z_0) \\ &= (L_{e^{K_0}} \cdot Z_0)(L_{e^{K_0}} \cdot b) - (L_{e^{K_0}} \cdot b)(L_{e^{K_0}} \cdot Z_0). \end{aligned}$$

If $z_c = L_{e^{K_0}} \cdot Z_0 = e^{K_0} Z_0 e^{-K_0}$ it is apparent that if $\beta(t) = e^{tz_c} c e^{-tz_c}$ for $t \in \mathbb{R}$ and $c = L_{e^K_0} \cdot b$, then $\beta(0) = c$ and $\dot\beta(0) = z_c c - c z_c = L_{e^{K_0}} \cdot (Z_0 b - b Z_0) = x$.

Similar considerations to those in Proposition 4.1 using the fact that

$$\begin{aligned} \beta(t) &= e^{K_0} e^{tZ_0} e^{-K_0} e^{K_0} b e^{K_0} e^{-K_0} e^{-tZ_0} e^{-K_0} = L_{e^{K_0}}(e^{tZ_0} b e^{-tZ_0}) \\ &= L_{e^{K_0}}(\gamma(t)) \end{aligned}$$

imply that $\beta(t) \in \mathcal{O}_b$ for all $t \in \mathbb{R}$.

Standard arguments for homogeneous spaces (invariance of the Finsler metric) imply that β is a curve of minimal length when defined on the interval $[-\pi/(2\|z_c\|), \pi/(2\|z_c\|)] = [-\pi/(2\|Z_0\|), \pi/(2\|Z_0\|)]$ as γ is. ∎

Remark 4.7. Theorem 4.2, Remark 4.5 and Corollary 4.6 allow us to describe short curves β in $\mathcal{O}_b$ with initial condition $\beta(0) = c$ even for velocity vectors $x \in T_c(\mathcal{O}_b)$ that do not have a minimal compact lifting Z_0. Thus $\mathcal{U}_k$ is an example of a group whose action on $\mathcal{O}_b$ has short curves that need not be described with minimal vectors F in $\{F \in \mathcal{B}(\mathcal{H})^{ah} : Fc - cF = 0\}$. Nevertheless there exists another group $\mathcal{U}_{k,d}$ acting on $\mathcal{O}_b$ such that its b-orbit coincides with that of $\mathcal{U}_k$, defines the same Finsler metric on it and where every short curve can be described by means of a minimal lifting.

The previous geometric properties yield the following results relating the quotient norm $\|[K]\| = \inf_{D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})} \|K + D\|$ of two anti-Hermitian compact operators.

Proposition 4.8. *Let $K_1, K_2 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_1, D_2 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ be such that $e^{tK_1}e^{D_1} = e^{tK_2}e^{D_2}$ for all $t \in [0,1]$. Then*

$$\|[K_1]\| = \|[K_2]\|.$$

Proof. Let $b = \mathrm{Diag}(\{\lambda_i\}_{i\in\mathbb{N}}) \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ with $\lambda_i \neq \lambda_j$ for each $i \neq j$. The equality $e^{tK_1}e^{D_1} = e^{tK_2}e^{D_2}$ implies that

$$e^{tK_1}be^{-tK_1} = e^{tK_2}be^{-tK_2}$$

for all $t \in [0,1]$. If we consider $\alpha, \beta : [0,1] \to \mathcal{O}_b$ defined by

$$\beta(t) = e^{tK_1}be^{-tK_1} \quad \text{and} \quad \alpha(t) = e^{tK_2}be^{-tK_2},$$

then $\mathrm{L}(\beta) = \mathrm{L}(\alpha)$, so $\int_0^1 \|\beta'(t)\|_{\beta(t)}\,dt = \int_0^1 \|\alpha'(t)\|_{\alpha(t)}\,dt$, and therefore $\|[K_1]\| = \|[K_2]\|$. ∎

Proposition 4.9. *Let $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ be such that $K + D$ is minimal and $\|K + D\| < \pi/2$. If $K' \in \mathcal{K}(\mathcal{H})^{ah}$ is such that $e^{K+D} = e^{K'}e^D$ then*

$$\|[K]\| \leq \|[K']\|.$$

Proof. Let $b = \mathrm{Diag}(\{\lambda_i\}_{i\in\mathbb{N}}) \in \mathcal{D}(\mathcal{B}(\mathcal{H})^h)$ with $\lambda_i \neq \lambda_j$ for each $i \neq j$, and consider $\alpha, \beta : [0,1] \to \mathcal{O}_b$ defined by

$$\beta(t) = e^{t(K+D)}be^{-t(K+D)} \quad \text{and} \quad \alpha(t) = e^{tK'}be^{-tK'}.$$

Observe that since $e^{K+D} = e^{K'}e^D$, we have $\beta(0) = \alpha(0)$ and $\beta(1) = \alpha(1)$.

But β has minimal length between all rectifiable unitary curves that join b to $\beta(1) = e^{K+D}be^{-(K+D)} = e^{K'}be^{-K'} = \alpha(1)$ (see Corollary 4.6 and [8]). Therefore $\mathrm{L}(\beta) \leq \mathrm{L}(\alpha)$, so

$$\int_0^1 \|\beta'(t)\|_{\beta(t)}\,dt = \|Kb - bK\|_b = \|[K]\| \leq \int_0^1 \|\alpha'(t)\|_{\alpha(t)}\,dt$$
$$= \|K'b - bK'\|_b = \|[K']\|. \ ∎$$

Acknowledgments. We thank Professors Esteban Andruchow, Gabriel Larotonda and Eduardo Chiumiento for fruitful conversations and suggestions along the preparation of this manuscript.

References

[1] E. Andruchow and G. Larotonda, *The rectifiable distance in the unitary Fredholm group*, Studia Math. 196 (2010), 151–178.

[2] E. Andruchow, G. Larotonda and L. Recht, *Finsler geometry and actions of the p-Schatten unitary groups*, Trans. Amer. Math. Soc. 362 (2010), 319–344.

[3] J. Antezana, G. Larotonda and A. Varela, *Thompson-type formulae*, J. Funct. Anal. 262 (2012), 1515–1528.

[4] D. Beltiţă, *Smooth Homogeneous Structures in Operator Theory*, Chapman & Hall/ CRC Monogr. Surveys Pure Appl. Math. 137, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[5] T. Bottazzi and A. Varela, *Best approximation by diagonal compact operators*, Linear Algebra Appl. 439 (2013), 3044–3056.

[6] T. Bottazzi and A. Varela, *Minimal length curves in unitary orbits of a Hermitian compact operator*, Differential Geom. Appl. 45 (2016), 1–22.

[7] E. Chiumiento, *On normal operator logarithms*, Linear Algebra Appl. 439 (2013), 455–462.

[8] C. Durán, L. Mata-Lorenzo and L. Recht, *Metric geometry in homogeneous spaces of the unitary group of a C^*-algebra. I. Minimal curves*, Adv. Math. 184 (2004), 342–366.

[9] K.-H. Neeb, *Monastir Summer School: Infinite-Dimensional Lie Groups*, lecture notes, 2005.

[10] T. S. Ratiu, *Coadjoint orbits and the beginnings of a geometric representation theory*, in: Developments and Trends in Infinite-Dimensional Lie Theory, Progr. Math. 288, Birkhäuser Boston, Boston, MA, 2011, 417–457.

[11] R. Thompson, *Proof of a conjectured exponential formula*, Linear Multilinear Algebra 19 (1986), 187–197.

[12] V. S. Varadarajan, *Lie Groups, Lie Algebras and Their Representations*, Grad. Texts in Math. 102, Springer, 1984.

Tamara Bottazzi
Instituto Argentino de Matemática
"Alberto P. Calderón"
Saavedra 15 3er piso
(C1083ACA) Buenos Aires, Argentina
E-mail: tpbottaz@ungs.edu.ar

Alejandro Varela
Instituto de Ciencias
Universidad Nacional de General Sarmiento
J. M. Gutierrez 1150
(B1613GSX) Los Polvorines
Pcia. de Buenos Aires, Argentina
and
Instituto Argentino de Matemática
"Alberto P. Calderón"
Saavedra 15 3er piso
(C1083ACA) Buenos Aires, Argentina
E-mail: avarela@ungs.edu.ar