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THE MOMENTLESS THEORY OF ONE-SHEET HYPERBOLOIDAL SHELLS

This paper presents a method of statical calculations of a shell having the form of a one-sheet hyperboloid. Shells of that kind are used in industrial building, *e. g.* as cooling towers in electric power stations, foundries, etc. It is assumed that the shell works in the so called momentless state, *i. e.* that the internal forces lie in a plane that is tangent to the middle surface of the shell. The determination of the momentless state of stress-resultants in a hyperboloidal shell has been the subject of numerous works, *e. g.* [3], [5], [9]. In all of them either the solutions of the equilibrium equations are obtained by the expansion of the required stress-resultants in Fourier series, or numerical methods are used providing approximate solutions. The method of expansion in Fourier series requires the assumption that the functions in question have axial symmetry. In practical applications of that method only the first few terms of the expansions are used, and it is very seldom that an exact estimate of the error can be given.

In the method here presented it is not necessary to assume axial (or any other) symmetry of the functions under consideration. Exact solutions are obtained by means of quadratures even in those cases in which the method of expansion in Fourier series fails (an elliptic hyperboloid).

Besides, in the general case a certain hypothesis concerning the manner of co-action between the shell and its edge has been assumed. That hypothesis enables us to find the general and exact solution of a problem which has so far been open.

I wish to express my gratitude to Professor Dr S. Drobot for his suggestion that I should write this paper and for his valuable remarks.

§ 1. Geometry of the middle surface of hyperboloidal shell. The equation of the middle surface of a shell is usually expressed in geographical coordinates (meridians and parallels), which are convenient in applications but complicate unnecessarily the solution of the equilibrium equations. It is well known [10], that for ruled shells the solutions of the equilibrium equations can be obtained by quadratures if we take asymptotic lines as the curvilinear coordinates system on the middle surface. In this paper, for calculation reasons and also in view of the boundary conditions, we have adopted parallels of latitude and one family of asymptotic lines as the curvilinear coordinates system. The equilibrium equations are shown to separate also in this case and it is possible to obtain their general solutions by means of quadratures.

A one-sheet hyperboloid given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

is presented parametrically by the equations:

$$x = a(\cos u+v\sin u),\quad y = b(\sin u-v\cos u),\quad z = cv. \tag{1}$$

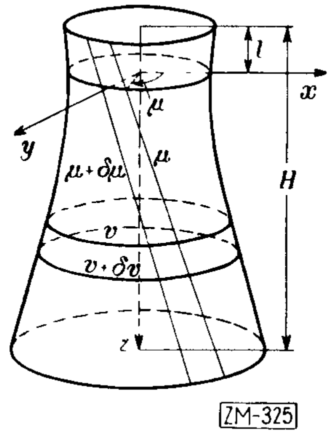

Fig. 1

The parametric lines $u = \text{const}$ are straight lines of one family of generators, and the lines $v = \text{const}$ are parallels of latitude. For the shell given in Fig. 1 the parameters change in the intervals

$$0 \leqslant u < 2\pi, \qquad -\frac{l}{c} \leqslant v \leqslant \frac{H-l}{c},$$

where H is the height of the shell and l the height of the part of the shell above the neck ellipse.

The vector equation of the middle surface is

$$\boldsymbol{r} = a(\cos u+v\sin u)\boldsymbol{i}+b(\sin u-v\cos u)\boldsymbol{j}+cv\boldsymbol{k}.$$

§ 2. Equilibrium equations of the momentless theory. We adopt the general equilibrium equations written in the following form [1], [8]:

$$\begin{gathered}
T^1_u+S_v+\left\{{11 \atop 1}\right\}T^1+2\left\{{12 \atop 1}\right\}S+\left\{{22 \atop 1}\right\}T^2+\frac{g_{22}\boldsymbol{r}_u-g_{12}\boldsymbol{r}_v}{\sqrt{g}}\boldsymbol{P}=0,\\
T^2_v+S_u+\left\{{11 \atop 2}\right\}T^2+2\left\{{12 \atop 2}\right\}S+\left\{{22 \atop 2}\right\}T^2+\frac{g_{11}\boldsymbol{r}_v-g_{12}\boldsymbol{r}_u}{\sqrt{g}}\boldsymbol{P}=0,\\
b_{11}T^1+2b_{12}S+b_{22}T^2+(\boldsymbol{r}_u\times\boldsymbol{r}_v)\boldsymbol{P}=0,
\end{gathered} \tag{2}$$

where

g_{11}, g_{12}, g_{22} — coefficients of the first quadratic form of the surface,

$g = g_{11}g_{22} - (g_{12})^2$ — discriminant of the first quadratic form,

$\left\{{11 \atop 1}\right\}, \left\{{12 \atop 1}\right\}, \left\{{22 \atop 1}\right\}, \ldots$ — Christoffel symbols,

$\boldsymbol{P}$ — external load of the shell per unit of middle surface area,

$S, T^1, T^2, S_u, S_v, T^1_u, T^2_v$ — required stress-resultants and their derivative with respect to parameters u and v.

In practice, instead of T^1, T^2, S we use physical stress-resultants T_1, T_2 and S and we have

$$T^1 = \sqrt{\frac{g_{22}}{g_{11}}}\,T_1, \qquad T^2 = \sqrt{\frac{g_{11}}{g_{12}}}\,T_2. \tag{3}$$

Equations (2) for a hyperboloidal shell give

$$T^1_u + S_v + \frac{1}{2g}\left(c^2\frac{\partial g_{11}}{\partial u} - 2a^2b^2\right)T^1 + \frac{1}{g}\left(c^2\frac{\partial g_{11}}{\partial u} + 2a^2b^2v\right)S + X(u, v) = 0, \tag{4}$$

$$T^2_v + S_u - \frac{a^2b^2}{g}v(1+v^2)T^1 + \frac{2a^2b^2}{g}vS + Y(u, v) = 0, \tag{5}$$

$$T^1 = \frac{1}{1+v^2}\left[2S + \frac{\sqrt{g}}{abc}Z(u, v)\right]. \tag{6}$$

X, Y and Z are functions dependent on the external load of the shell

$$X = \frac{g_{22}\boldsymbol{r}_u - g_{12}\boldsymbol{r}_v}{\sqrt{g}}\boldsymbol{P}, \qquad Y = \frac{g_{11}\boldsymbol{r}_v - g_{12}\boldsymbol{r}_u}{\sqrt{g}}\boldsymbol{P}, \qquad Z = (\boldsymbol{r}_u \times \boldsymbol{r}_v)\boldsymbol{P}. \tag{7}$$

On eliminating the unknown T^1 from equations (4) and (5) we obtain

$$\begin{gathered} 2S_u + (1+v^2)S_v + 2vS + \Phi(u, v) = 0, \\ S_u + T^2_v + Y - \frac{abv}{c\sqrt{g}}Z = 0, \end{gathered} \tag{8}$$

where

$$\Phi(u, v) = (1+v^2)X + \frac{1}{abc}\frac{\partial(\sqrt{g}Z)}{\partial u} + \frac{1}{2abc\sqrt{g}}\left(c^2\frac{\partial g_{11}}{\partial u} - 2a^2b^2v\right)Z.$$

The first equation of (8) is a partial differential equation of the first order with respect to S. Having the solution of the first equation we can solve the second equation by direct integration.

§ 3. Solutions of equilibrium equations. We shall first deal with the solution of the equation

$$2S_u+(1+v^2)S_v+2vS+\Phi(u,v)=0. \tag{9}$$

This equation is solved by the method of characteristics. The integral curves of the system of ordinary differential equations

$$\frac{du}{2}=\frac{dv}{1+v^2}=-\frac{dS}{2vS+\Phi(u,v)}$$

are the characteristics of equation (9). Integrating this system of equations we obtain the equations of the characteristics:

$$u=2\operatorname{arctg}v+\alpha,$$

$$S=-\frac{1}{1+v^2}\int\limits_{v_0}^{v}\Phi(u-2\operatorname{arctg}v+2\operatorname{arctg}t,\,t)\,dt+\frac{1+v_0^2}{1+v^2}\eta,$$

where $\alpha=u_0-2\operatorname{arctg}v_0$ and η are arbitrary constants. The characteristics are defined here as the intersection lines of two families of surfaces. These lines must be used to form an integration surface which for $v=v_0$ passes through the given curve $\eta=\eta(u_0)$. Hence

$$S=-\frac{1}{1+v^2}\int\limits_{v_0}^{v}\Phi(u-2\operatorname{arctg}v+2\operatorname{arctg}t,\,t)\,dt+ \\ +\frac{1+v_0^2}{1+v^2}\eta(u-2\operatorname{arctg}v+2\operatorname{arctg}v_0). \tag{10}$$

This is a solution of equation (9), satisfying the boundary condition

$$S(u,v_0)=\eta(u).$$

The first equation of (8) gives

$$S_u=-\left(\frac{1+v^2}{2}S\right)_v-\frac{1}{2}\Phi(u,v),$$

whence, on substituting S_u in the second equation of (8), we obtain

$$T^2=\tfrac{1}{2}\int\limits_{v_0}^{v}[\Psi(u,t)-\Phi(u-2\operatorname{arctg}v+2\operatorname{arctg}t,\,t)]\,dt+ \\ +\frac{1+v_0^2}{2}[\eta(u-2\operatorname{arctg}v+2\operatorname{arctg}v_0)-\eta(u)]+\psi(u) \tag{11}$$

where

$$\Psi(u, v) = \Phi(u, v) - 2Y + \frac{2abv}{c\sqrt{g}} Z$$

and $\psi(u)$ is an arbitrary function.

With given external loads solutions (10), (11) and (6) define the stresses in the shell, those solutions containing two arbitrary functions, η and ψ. We shall determine those functions from the boundary conditions.

§ 4. Change of the curvilinear system. Since in statical calculations it is more convenient to use an orthogonal curvilinear coordinates system consisting of meridians and parallels of latitude, let us transform our solutions into that system.

We use the following formulae for the change of system

$$\varphi = u - \operatorname{arctg} v, \qquad v = v.$$

Equations (1) will then assume the form

$$
\begin{aligned}
x &= a\sqrt{1+v^2}\cos\varphi, \\
y &= b\sqrt{1+v^2}\sin\varphi, \qquad (12) \\
z &= cv.
\end{aligned}
$$

This is a parametric representation of a hyperboloid where the parametric lines $\varphi = \text{const}$ and $v = \text{const}$ are respectively meridians and parallels of latitude.

$$T^{11} = \frac{T^1}{\sqrt{g}}, \qquad T^{12} = T^{21} = \frac{S}{\sqrt{g}}, \qquad T^{22} = \frac{T^2}{\sqrt{g}}$$

are contravariant components of the tensor (T^{ik}). Let $\overline{T}^{ik}$ denote the contravariant components of the tensor

$$(\overline{T}^{ik})$$

in the new system; the physical stresses in that system will then be expressed by

$$\overline{T}_1 = \sqrt{\frac{\bar{g}_{11}}{\bar{g}^{11}}}\,\overline{T}^{11}, \qquad \overline{S} = \sqrt{\bar{g}}\,\overline{T}^{12}, \qquad \overline{T}_2 = \sqrt{\frac{\bar{g}_{22}}{\bar{g}^{22}}}\,\overline{T}^{22},$$

where $\bar{g}_{11}$, $\bar{g}_{22}$ and $\bar{g}^{11}$, $\bar{g}^{22}$ are the components of the covariant and the contravariant metric tensor in the new system.

On applying well-known transformation formulae and using the equality $\bar{g}(\varphi, v) = g(u, v)$, we obtain the following relations between the physical stresses in the shell with reference to a curvilinear coordinates system consisting of meridians and parallels and the solutions of the system of equilibrium equations (10), (11) and (6):

$$\bar{S} = S - \frac{1}{1+v^2} T^2,$$

$$\bar{T}_1 = \sqrt{\frac{\bar{g}_{11}}{\bar{g}_{22}}} \left[T^1 - \frac{2}{1+v^2} S + \frac{1}{(1+v^2)^2} T^2 \right], \tag{13}$$

$$\bar{T}_2 = \sqrt{\frac{\bar{g}_{22}}{\bar{g}_{11}}} T^2,$$

where

$$\bar{g}_{11} = (1+v^2)(a^2 \sin^2\varphi + b^2 \cos^2\varphi), \qquad \bar{g}_{22} = \frac{v^2}{1+v^2}(a^2\cos^2\varphi + b^2 \sin^2\varphi) + c^2.$$

Equations (13), on substituting in them solutions (6), (10) and (11), give

$$\bar{S}(\varphi, v) = -\frac{1}{2(1+v^2)} \Big\{ \int_{v_0}^{v} [\Phi(\varphi - \operatorname{arctg} v + 2 \operatorname{arctg} t, t) + \tag{14}$$
$$+ \Psi(\varphi + \operatorname{arctg} v, t)] dt - (1+v_0^2)[\eta(\varphi - \operatorname{arctg} v + 2\operatorname{arctg} v_0) +$$
$$+ \eta(\varphi + \operatorname{arctg} v)] + 2\psi(\varphi + \operatorname{arctg} v) \Big\}$$

$$\bar{T}_2(\varphi, v) = \frac{1}{2}\sqrt{\frac{\bar{g}_{22}}{\bar{g}_{11}}} \Big\{ \int_{v_0}^{v} [\Psi(\varphi + \operatorname{arctg} v, t) - \Phi(\varphi - \operatorname{arctg} v + \tag{15}$$
$$+ 2 \operatorname{arctg} t, t)] dt + (1+v_0^2)[\eta(\varphi - \operatorname{arctg} v + 2 \operatorname{arctg} v_0) -$$
$$- \eta(\varphi + \operatorname{arctg} v)] + 2\psi(\varphi + \operatorname{arctg} v) \Big\},$$

$$\bar{T}_1(\varphi, v) = \frac{1}{1+v^2} \sqrt{\frac{\bar{g}_{11}}{\bar{g}_{22}}} \left[\frac{1}{1+v^2} \sqrt{\frac{\bar{g}_{11}}{\bar{g}_{22}}} \bar{T}_2 + \frac{\sqrt{g}}{abc} Z(\varphi, v) \right]. \tag{16}$$

They are the general solutions of the equilibrium equations of a hyperboloidal shell in geographical coordinates.

§ 5. Solutions of the equilibrium equations with a free upper edge of the shell. The upper edge of the shell is free if the stresses occurring on that edge are equal to zero, *i. e.* for $v = v_0$

$$\bar{S}(\varphi, v_0) \equiv 0 \quad \text{and} \quad \bar{T}_2(\varphi, v_0) \equiv 0 \quad (v_0 = -\frac{l}{c}).$$

From these conditions we shall determine arbitrary functions η and ψ occurring in the solutions of the equilibrium equations. The stresses on the upper edge of the shell are obtained by substituting $v = v_0$ in solutions (14) and (15), and we have

$$\bar{S}(\varphi, v_0) = \eta(\varphi + \operatorname{arctg} v_0) - \psi(\varphi + \operatorname{arctg} v_0) \equiv 0,$$

$$\bar{T}_2(\varphi, v_0) = \psi(\varphi + \operatorname{arctg} v_0) \equiv 0.$$

Hence functions η and ψ must be identically equal to zero. Consequently formulae (14), (15) and (16) give

$$\bar{S} = -\frac{1}{2(1+v^2)} \int_{v_0}^{v} [\Phi(\varphi - \operatorname{arctg} v + 2\operatorname{arctg} t, t) + \Psi(\varphi + \operatorname{arctg} v, t)]\,dt, \tag{17}$$

$$\bar{T}_2 = \frac{1}{2}\sqrt{\frac{\bar{g}_{22}}{\bar{g}_{11}}} \int_{v_0}^{v} [\Psi(\varphi + \operatorname{arctg} v, t) - \Phi(\varphi - \operatorname{arctg} v + 2\operatorname{arctg} t, t)]\,dt, \tag{18}$$

$$\bar{T}_1 = \frac{1}{1+v^2}\sqrt{\frac{\bar{g}_{11}}{\bar{g}_{22}}}\left[\frac{1}{1+v^2}\sqrt{\frac{\bar{g}_{11}}{\bar{g}_{22}}}\,\bar{T}_2 + \frac{\sqrt{g}}{abc} Z\right]. \tag{19}$$

These formulae define the stress-resultants in a shell in the form of a one-sheet elliptical hyperboloid (the methods used so far have permitted the determination of stress-resultants in a hyperboloidal shell of revolution only).

§ 6. Cooling tower in the form of a one-sheet hyperboloid of revolution loaded with its own weight. We shall now deal with the application of the results obtained to statical calculations of cooling towers in the form a one-sheet hyperboloid of revolution. The stress occurring in the shell will be obtained from a superposition of the stresses resulting from the loading of the shell with its own weight and with the wind under the assumption that the upper edge of the shell is free and the stresses resulting from the loading of the upper edge of the shell with a circular ring.

A parametric representation of a one-sheet hyperboloid of revolution will be obtained by assuming $b = a$ in equations (1). The coefficients and the discriminant of the metric form of that surface are

$$g_{11} = a^2(1+v^2), \quad g_{12} = -a^2, \quad g_{22} = a^2+c^2, \quad g = a^2[c^2+(a^2+c^2)v^2].$$

The loading of a hyperboloidal shell with its own weight corresponds to the vertical load

$$\mathbf{P} = P\mathbf{k}, \tag{20}$$

where P is the specific weight of the shell per unit of the middle surface area. We assume that the weight P of the shell varies only with height, *i. e.* that it is only a function of parameter v. With load (20) the components of external load (7) are

$$X = \frac{a^2c}{\sqrt{g}}P, \quad Y = \frac{a^2c(1+v^2)}{\sqrt{g}}P, \quad Z = -a^2vP.$$

The functions $\Phi(u,v)$ and $\Psi(u,v)$ are found from the formulae

$$\Phi(u,v) = (1+v^2)X + \frac{1}{a^2c}\cdot\frac{\partial(\sqrt{g}Z)}{\partial u} - \frac{a^2v}{c\sqrt{g}}Z, \tag{21}$$

$$\Psi(u,v) = (1+v^2)X - 2Y + \frac{1}{a^2c}\cdot\frac{\partial(\sqrt{g}Z)}{\partial u} + \frac{a^2v}{c\sqrt{g}}Z. \tag{22}$$

Hence, introducing additional symbols,

$$\frac{c}{a} = \lambda, \quad 1+\lambda^2 = \varepsilon^2$$

we obtain

$$\Phi(u,v) = \frac{aP}{\lambda}\sqrt{\lambda^2+\varepsilon^2v^2}, \quad \Psi(u,v) = -\frac{aP}{\lambda}\sqrt{\lambda^2+\varepsilon^2v^2}.$$

Substituting these functions and the component of the load Z in formulae (17), (18) and (19), we obtain

$$\begin{aligned} \bar{S} &= 0, \\ \bar{T}_2 &= -\frac{a\sqrt{\lambda^2+\varepsilon^2v^2}}{\lambda(1+v^2)}\int_{v_0}^{v} P\sqrt{\lambda^2+\varepsilon^2t^2}\,dt, \\ \bar{T}_1 &= \frac{1}{\lambda^2+\varepsilon^2v^2}\bar{T}_2 - \frac{av}{\lambda}P. \end{aligned} \tag{23}$$

These formulae give the stress-resultants in the shell under consideration provided the vertical load $P(v)$ is given. That load can of course be assumed without any restrictions according to the construction conditions. For illustration we shall give three detailed examples of the application of formulae (23).

In the first example we shall assume that the weight of the shell per unit of the middle surface area is constant: then $P = \text{const}$ and formulae (23) give

$$\bar{S} = 0,$$

$$\bar{T}_2 = -\frac{aP\sqrt{\lambda^2+\varepsilon^2 v^2}}{2\lambda(1+v^2)}\left[t\sqrt{\lambda^2+\varepsilon^2 t^2}+\frac{\lambda^2}{\varepsilon}\ln\left(\varepsilon t+\sqrt{\lambda^2+\varepsilon^2 t^2}\right)\right]_{v_0}^{v},$$

$$\bar{T}_1 = \frac{1}{\lambda^2+\varepsilon^2 v^2}\bar{T}_2 - \frac{aP}{\lambda}v.$$

In the second example, which is applicable in cooling towers, we assume that the wall thickness varies with the height and has its maximum value at the base of the tower. In particular it is assumed that the thickness of the wall of the tower is a linear function of the height; then

$$h = h_0(mv+n),$$

where $m = \mu c/H$, $n = \mu l/H+1$, h_0 is the wall thickness at the top of the tower and the dimensionless coefficient $\mu = \Delta h/h_0$ is the ratio of the increasing Δh of the wall thickness at the base of the tower to the wall thickness at the top of the tower.

To the above change in the thickness of the wall of the tower corresponds the following variable load of the shell:

$$P(v) = P_0(mv+n), \tag{24a}$$

where $P = \gamma h_0$ is the weight of the shell per unit of the middle surface area on the upper edge of the shell (γ — constant specific weight of the material of the tower). In this case formulae (23) on substituting load (24a) give

$$\bar{S} = 0,$$

$$\bar{T}_2 = -\frac{aP_0\sqrt{\lambda^2+\varepsilon^2 v^2}}{\lambda(1+v^2)}\left\{\left[mt^2+(\tfrac{1}{2}n-m)t-\frac{\lambda^2}{\varepsilon^2}m\right]\sqrt{\lambda^2+\varepsilon^2 t^2}+\right.$$

$$\left.+(m+\tfrac{1}{2}n)\frac{\lambda^2}{\varepsilon}\ln\left(\varepsilon t+\sqrt{\lambda^2+\varepsilon^2 t^2}\right)\right\}\Bigg|_{v_0}^{v},$$

$$\bar{T}_1 = \frac{1}{\lambda^2+\varepsilon^2 v^2}\bar{T}_2 - \frac{aP_0}{\lambda}(mv+n)v.$$

In the third example we assume that the weight of the shell varies in the following way:

$$P(v) = \frac{P_0}{\lambda}\sqrt{\lambda^2+\varepsilon^2 v^2}. \tag{24b}$$

Under this load we obtain from formulae (23) particularly simple expressions for the stress-resultants. The above load can be realised if we assume that the wall thickness of the tower varies according to the formula

$$h = \frac{h_0}{\lambda}\sqrt{\lambda^2+\varepsilon^2 v^2}.$$

This corresponds to the increment of the wall thickness towards the base of the tower, while the difference between the wall thickness at the top of the tower and the thickness of the neck circle is usually so small that it can be neglected in practical applications.

EXAMPLE. For a tower with dimensions (Fig. 1) $H = 52.20$ m, $l = 8.10$ m, neck circle radius $a = 11.90$ m and base radius $r_d = 20.95$ m, the wall thickness of the tower assumes the values:

at the top of the tower $h = 1.03\,h_0$,
on the neck circle $h = h_0$,
at the base of the tower $h = 1.85\,h_0$.

From the load assumed above and from formulae (23) it follows that

$$\bar{S} = 0,$$

$$\bar{T}_2 = -\frac{aP_0\sqrt{\lambda^2+\varepsilon^2 v^2}}{\lambda^2(1+v^2)}\left[\lambda^2 t + \tfrac{1}{3}\varepsilon^2 t^3\right]_{v_0}^{v},$$

$$\bar{T}_1 = \frac{1}{\lambda^2+\varepsilon^2 v^2}\bar{T}_2 - \frac{aP_0}{\lambda^2}v\sqrt{\lambda^2+\varepsilon^2 v^2}.$$

§ 7. Cooling tower loaded with the wind. In statical calculations of cooling towers we usually apply the following experimental formula for the distribution of the wind load [4], [5], [9]:

$$W = w(\omega_1\cos\varphi + \omega_2\cos 2\varphi + \omega_3). \tag{25}$$

The angle φ appearing in this formula is reckoned along a parallel of latitude from the meridian plane determined by the direction of the wind. The coefficient w is the basic load per 1 m² of a plane perpendicular to the direction of the wind and the coefficients ω_1, ω_2, ω_3 depend on the roughness of the shell.

We assume that the wind pressure is perpendicular to the middle surface of the shell ([5], [9]) and we take the plane xOz as the meridian plane determined by the direction of the wind. If $\boldsymbol{n}$ is a unit vector normal to the central surface, then the wind load per unit of middle surface area is

$$\boldsymbol{W} = -W\boldsymbol{n}.$$

With the accepted parametrisation (1) we find the following dependence of angle φ on the parameters u and v:

$$\varphi = u - \operatorname{arctg} v.$$

Since in the case of tall buildings the variability of wind pressure according to height must also be taken into consideration, we assume that the coefficient w is a linear function of the height (as has been accepted in the standards):

$$w = w_0(mv+n), \tag{26}$$

where $m = -\nu c/H$, $n = \nu(H-l)/H+1$, w_0 is the load per 1 m^2 of a plane perpendicular to the direction of the wind at the base of the tower and the dimensionless coefficient $\nu = \Delta w/w_0$ is the ratio of the increasing Δw of basic load w at the top of the tower to basic load w_0 at the base of the tower. The formula for the wind load distribution then assumes the form

$$W = w_0(mv+n)[\omega_1\cos(u-\operatorname{arctg} v)+\omega_2\cos 2(u-\operatorname{arctg} v)+\omega_3], \tag{27}$$

and the external load components are expressed as

$$X = 0, \quad Y = 0, \quad Z = -\sqrt{g}\,W.$$

Functions $\Phi(u, v)$ and $\Psi(u, v)$ are determined from formulae (21) and (22), whence

$$\Phi(u, v) = \frac{a}{\lambda}\left[vW-(\lambda^2+\varepsilon^2 v^2)\frac{\partial W}{\partial u}\right],$$

$$\Psi(u, v) = -\frac{a}{\lambda}\left[vW+(\lambda^2+\varepsilon^2 v^2)\frac{\partial W}{\partial u}\right].$$

Formulae (17) and (18) after integration and formula (19) after substituting the load component Z give

$$\bar{S} = -\frac{aw_0}{\lambda(1+v^2)}\left[\omega_1 \frac{Av+B}{\sqrt{1+v^2}}\sin\varphi + \omega_2 \frac{2Cv+D(1-v^2)}{1+v^2}\sin 2\varphi\right],$$

$$(28) \qquad \bar{T}_2 = -\frac{aw_0\sqrt{\lambda^2+\varepsilon^2 v^2}}{\lambda(1+v^2)}\left[\omega_1 \frac{A-Bv}{\sqrt{1+v^2}}\cos\varphi + \right.$$

$$\left. + \omega_2 \frac{C(1-v^2)-2Dv}{1+v^2}\cos 2\varphi + \omega_3 E\right],$$

$$\bar{T}_1 = \frac{1}{\lambda^2+\varepsilon^2 v^2}\bar{T}_2 - \frac{aw_0}{\lambda}\sqrt{\lambda^2+\varepsilon^2 v^2}(mv+n)(\omega_1\cos\varphi + \omega_2\cos 2\varphi + \omega_3),$$

where

$$A = \varepsilon^2\left\{\left[\tfrac{1}{4}m(t^3+\tfrac{1}{2}t)+\tfrac{1}{3}n(t^2+1)\right]\sqrt{1+t^2} - \tfrac{1}{8}m\ln\left(t+\sqrt{1+t^2}\right)\right\}\Big|_{v_0}^{v},$$

$$B = \lambda^2\left\{\left[\tfrac{1}{3}m(t^2+1)+\tfrac{1}{2}nt\right]\sqrt{1+t^2} + \tfrac{1}{2}n\ln\left(t+\sqrt{1+t^2}\right)\right\}\Big|_{v_0}^{v},$$

$$C = \left[(4\varepsilon^2-1)(\tfrac{1}{3}mt^3+\tfrac{1}{2}nt^2) - n\ln(1+t^2) - 2m(t-\operatorname{arctg} t)\right]_{v_0}^{v},$$

$$D = \left[2\varepsilon^2(nt+\tfrac{1}{2}mt^2-\tfrac{1}{3}nt^3-\tfrac{1}{4}mt^4) - m\ln(1+t^2) - 2n\operatorname{arctg} t\right]_{v_0}^{v},$$

$$E = \left[\tfrac{1}{3}mt^3+\tfrac{1}{2}nt^2\right]_{v_0}^{v}.$$

If we assume that basic pressure w is constant along the whole height of the shell $w = w_0$, then $m = 0$, $n = 1$, and the coefficients occurring in formulae (28) are the following:

$$A = \tfrac{1}{3}\varepsilon^2(t^2+1)\sqrt{t^2+1}\Big|_{v_0}^{v}, \qquad B = \tfrac{1}{2}\lambda^2\left[t\sqrt{1+t^2}+\ln\left(t+\sqrt{1+t^2}\right)\right]_{v_0}^{v},$$

$$C = \tfrac{1}{2}\left[(4\varepsilon^2-1)t^2-\ln(1+t^2)\right]_{v_0}^{v}, \qquad D = \tfrac{2}{3}\left[\varepsilon^2(3t-t^3)-3\operatorname{arctg} t\right]_{v_0}^{v},$$

$$E = \tfrac{1}{2}t^2\Big|_{v_0}^{v}.$$

§ 8. Effect of a circular ring connected monolithically with the upper edge of a hyperboloidal shell upon the stress-resultants in the shell. The solutions of the equilibrium equations in the preceding sections have been obtained under the assumption that the upper edge of the shell is free. Usually the upper edge of the shell ends in a circular ring, owing to which the boundary conditions for the system of equilibrium equations undergo a change. The object of our further considerations will be the determination of the boundary conditions for the system of equilibrium equations in the case of the ring being connected monolithically with the shell, which in turn will allow us to calculate the effect of the ring upon the stresses in the shell.

We shall conduct those calculations with reference to a curvilinear coordinates system consisting of meridians and parallels of latitude. We assume that the axis of the circular ring is given by the vector equation

$$\boldsymbol{R}(\varphi) = a\sqrt{1+v_0^2}(\cos\varphi\boldsymbol{i}+\sin\varphi\boldsymbol{j})+z_0\boldsymbol{k} \quad \left(v_0 = -\frac{l}{c}\right).$$

The unit vectors tangent to the parametric lines of the central surface (12) on the upper edge of the shell, *i. e.* for $v = v_0$, are

$$\frac{\boldsymbol{r}_\varphi}{\sqrt{E}} = -\sin\varphi\boldsymbol{i}+\cos\varphi\boldsymbol{j},$$

$$\frac{\boldsymbol{r}_v}{\sqrt{G}} = \frac{1}{\sqrt{G}}\left(\frac{av_0}{\sqrt{1+v_0^2}}\cos\varphi\,\boldsymbol{i}+\frac{av_0}{\sqrt{1+v_0^2}}\sin\varphi\boldsymbol{j}+c\boldsymbol{k}\right),$$

where

$$E = a^2(1+v_0^2), \qquad G = \frac{a^2v_0^2}{1+v_0^2}+c^2.$$

In our further considerations we shall use a movable orthogonal system whose versors are

$$\boldsymbol{e}_1 = \frac{\boldsymbol{r}_\varphi}{\sqrt{E}}, \quad \boldsymbol{e}_2 = \boldsymbol{e}_1\times\boldsymbol{k}, \quad \boldsymbol{e}_3 = \boldsymbol{k}.$$

The vector equilibrium equation of the circular ring has the form (see [2], [7])

$$\frac{d\boldsymbol{N}}{d\varphi}-\sqrt{E}\,\boldsymbol{q} = 0, \tag{29}$$

where $\boldsymbol{N}$ is the internal force in the ring and $\boldsymbol{q}$ is the external load per unit of the ring exis arc. The force $\boldsymbol{N}$ and the external load $\boldsymbol{q}$ are decomposed into components in the movable system:

$$\boldsymbol{N} = N_1\boldsymbol{e}_1+N_2\boldsymbol{e}_2+N_3\boldsymbol{e}_3, \qquad \boldsymbol{q} = q_1\boldsymbol{e}_1+q_2\boldsymbol{e}_2+q_3\boldsymbol{e}_3.$$

Writing in full form the equation (29) and using the relations

$$\frac{d\boldsymbol{e}_1}{d\varphi} = -\boldsymbol{e}_2, \quad \frac{d\boldsymbol{e}_2}{d\varphi} = \boldsymbol{e}_1, \quad \frac{d\boldsymbol{e}_3}{d\varphi} = 0$$

we obtain the following system of differential equations:

$$\frac{dN_1}{d\varphi} + N_2 = \sqrt{E}\, q_1,$$

$$\frac{dN_2}{d\varphi} - N_1 = \sqrt{E}\, q_2, \tag{30}$$

$$\frac{dN_3}{d\varphi} = \sqrt{E}\, q_3.$$

As the external load of the ring we take the reactions of the stresses-resultants occurring on the upper edge on the shell

$$\bar{S}\boldsymbol{e}_1, \quad \bar{T}_2 \frac{\boldsymbol{r}_v}{\sqrt{G}}$$

and also the load of its own weight and the horizontal wind load

$$Q\boldsymbol{e}_3, \quad -W\boldsymbol{e}_2.$$

Here Q is the specific weight of the ring per unit of the ring axis arc and

$$W = pw(\omega_1 \cos\varphi + \omega_2 \cos 2\varphi + \omega_3)$$

is the wind load per unit of the ring axis arc (p — the height of the ring).

Equations (30), on substituting the external load components, are the following:

$$\frac{dN_1}{d\varphi} + N_2 = \sqrt{E}\,\bar{S}, \tag{31}$$

$$\frac{dN_2}{d\varphi} - N_1 = \frac{a^2 v_0}{\sqrt{G}} \bar{T}_2 - \sqrt{E}\, W, \tag{32}$$

$$\frac{dN_3}{d\varphi} = \sqrt{E}\left(\frac{c}{\sqrt{G}} \bar{T}_2 + Q\right). \tag{33}$$

Since the ring is carried by the shell, we have

$$N_3 = 0. \tag{34}$$

For a monolythic connection of the ring with the shell we assume [6] that the mean stress component in the direction

$$-\frac{\boldsymbol{r}_v}{\sqrt{G}}$$

is equal to the tangent stress occuring in the same direction on the edge of the shell:

$$\frac{N}{\Omega}\cdot\frac{r_v}{\sqrt{G}}=\frac{\bar{S}}{h},$$

where Ω is the area of intersection of the ring by a plane perpendicular to the ring axis. This equation when written in full will assume the form

$$\frac{a^2 v_0}{\sqrt{EG}}N_2+\frac{c}{\sqrt{G}}N_3=\chi\bar{S},\qquad \chi=\frac{\Omega}{h}. \tag{35}$$

In equations (31), (32), (33), (34) and (35) we have as unknowns three components of the internal force of the ring N_1, N_2, N_3 and the unknown stress-resultants $\bar{S}$ and $\bar{T}_2$ on the upper edge of the shell. Equations (33) and (34) imply

$$\bar{T}_2=-\frac{\sqrt{G}}{c}Q. \tag{36}$$

On substituting relations (36) in equation (32) and eliminating from equations (31), (32) and (33) the unknowns N_1 and N_2 we obtain for $\bar{S}$ the following differential equation of the second order with constant coefficients:

$$\frac{d^2\bar{S}}{d\varphi^2}+\left(1-\frac{a^2v_0}{\chi\sqrt{G}}\right)\bar{S}=pw\frac{a^2v_0}{\chi\sqrt{G}}(\omega_1\sin\varphi+2\omega_2\sin2\varphi).$$

The general solution of this equation has the form

$$\bar{S}=C_1\cos k\varphi+C_2\sin k\varphi-pw\left(\omega_1\sin\varphi+\frac{2a^2v_0\omega_2}{a^2v_0+3\chi\sqrt{G}}\sin2\varphi\right),$$

where

$$k=\sqrt{1-\frac{a^2v_0}{\chi\sqrt{G}}},$$

and C_1 and C_2 are arbitrary constants. On account of the nature of the load of the ring, $\bar{S}$ must be an odd periodic function with period 2π, which under the assumption that k is not an integer implies

$$C_1=C_2=0.$$

In the case of a monolithical connection of the upper edge of a hyperboloidal shell with a circular ring loaded with its own weight and the wind, the following conditions must be satisfied on the upper edge of the shell:

$$
\begin{aligned}
&\bar{S}(\varphi, v_0) = -pw\left(\omega_1 \sin\varphi + \frac{2a^2 v_0 \omega_2}{a^2 v_0 + 3\chi\sqrt{G}} \sin 2\varphi\right),\\
(37)\qquad &\\
&\bar{T}_2(\varphi, v_0) = -\frac{\sqrt{G}}{c} Q.
\end{aligned}
$$

Let $\bar{S}_r$, $\bar{T}_{1r}$, $\bar{T}_{2r}$ denote the stresses in the shell resulting from non-homogeneous (*i. e.* from non-zero) boundary conditions; then it follows from solutions (14), (15) and (16) on assuming zero loads of the shell that

$$
\begin{aligned}
\bar{S}_r &= \frac{1+v_0^2}{2(1+v^2)}[\eta(\varphi - \operatorname{arctg} v + 2\operatorname{arctg} v_0) + \eta(\varphi + \operatorname{arctg} v)] - \\
&\qquad - \frac{1}{1+v^2}\psi(\varphi + \operatorname{arctg} v),\\
\bar{T}_{2r} &= \frac{\sqrt{\lambda^2 + \varepsilon^2 v^2}}{1+v^2}\left\{\frac{1+v_0^2}{2}[\eta(\varphi - \operatorname{arctg} v + 2\operatorname{arctg} v_0) - \eta(\varphi + \operatorname{arctg} v)] + \right.\\
&\qquad \left. + \psi(\varphi + \operatorname{arctg} v)\right\},\\
\bar{T}_{1r} &= \frac{1}{\lambda^2 + \varepsilon^2 v^2}\bar{T}_{2r}.
\end{aligned}
$$

Hence, imposing boundary conditions (37), we obtain the following formulae for the additional stress-resultants in the shell caused by the ring:

$$
\begin{aligned}
&\bar{S}_r = -pw\frac{1+v_0^2}{1+v^2}(\omega_1 \sin\varphi\cos\beta + A\omega_2 \sin 2\varphi \cos 2\beta),\\
(38)\qquad &\bar{T}_{2r} = \frac{1+v_0^2}{1+v^2}\sqrt{\lambda^2+\varepsilon^2 v^2}\left[pw(\omega_1\cos\varphi\sin\beta + A\omega_2 \cos 2\varphi\sin 2\beta) - \right.\\
&\qquad \left. - \frac{1}{\lambda\sqrt{1+v_0^2}} Q\right],\\
&\bar{T}_{1r} = \frac{1}{\lambda^2+\varepsilon^2 v^2}\bar{T}_{2r},
\end{aligned}
$$

where

$$
\beta = \operatorname{arctg} v - \operatorname{arctg} v_0, \qquad A = \frac{2av_0\sqrt{1+v_0^2}}{3\chi\sqrt{\lambda^2+\varepsilon^2 v_0^2} + v_0\sqrt{1+v_0^2}}.
$$

If we disregard the wind load of the ring, formulae (38) imply

$$\overline{S}_r = 0,$$

$$\overline{T}_{2r} = -\frac{\sqrt{(1+v_0^2)(\lambda^2+\varepsilon^2 v^2)}}{\lambda(1+v^2)} Q,$$

$$\overline{T}_{1r} = -\frac{1}{\lambda(1+v^2)} \sqrt{\frac{1+v_0^2}{\lambda^2+\varepsilon^2 v^2}} Q.$$

The components N_1, N_2 of the internal force in the ring are determined from equations (32), (35) and (36). Equations (30) may also be used to calculate the internal force in the thrust ring at the base of the tower.

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BEZMOMENTOWA TEORIA POWŁOK O KSZTAŁCIE HIPERBOLOIDY JEDNOPOWŁOKOWEJ

STRESZCZENIE

Wyznaczenie bezmomentowego stanu naprężeń w powłoce hiperboloidalnej było przedmiotem wielu prac. W pracach tych uzyskuje się rozwiązania równań równowagi przez rozwinięcia szukanych napięć w szeregi Fouriera; spotyka się również rozwiązania przybliżone otrzymane metodami numerycznymi. Zwykle równania powierzchni środkowej powłoki hiperboloidalnej przyjmuje się we współrzędnych geograficznych (południki i równoleżniki), które są wygodne w zastosowaniach, natomiast niepotrzebnie komplikują rozwiązanie równań równowagi. W pracy tej ze względów rachunkowych, a także z uwagi na warunki brzegowe, za siatkę współrzędnych przyjęto jedną rodzinę linii asymptotycznych i równoleżniki. Okazuje się że i w tym przypadku równania równowagi separują się i można otrzymać przez kwadratury ogólne rozwiązania tych równań. W pracy niniejszej przedstawiono dokładne rozwiązania równań równowagi powłoki hiperboloidalnej za pomocą kwadratur, przy czym uzyskano rozwiązania w takich przypadkach, w których metoda rozwinięć w szereg Fouriera zawodzi (hiperboloida eliptyczna). Z uwagi na zastosowania, rozwiązania te przetransformowano na współrzędne geograficzne i podano rozwiązanie zadania brzegowego przy założeniu, że górny brzeg powłoki jest swobodny. Otrzymane wyniki zastosowano do obliczeń statycznych wież chłodniczych w kształcie jednopowłokowej hiperboloidy obrotowej, obciążonych ciężarem własnym i wiatrem. Między innymi podano wzory na naprężenia panujące w wieżach chłodniczych przy liniowej zmianie grubości ścianki wieży i liniowym wzroście parcia podstawowego wiatru wraz z wysokością (jak przyjęto w normach).

Oprócz tego w pracy tej przyjęto pewną hipotezę o sposobie współpracy powłoki z jej brzegiem. Hipoteza ta pozwoliła na wyznaczenie warunków brzegowych dla układu równań równowagi, przy monolitycznym połączeniu górnego brzegu powłoki z pierścieniem kołowym. Warunki te, po nałożeniu ich na rozwiązania ogólne, pozwoliły na obliczenie wpływu na naprężenia w powłoce pierścienia kołowego obciążonego ciężarem własnym i wiatrem.

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БЕЗМОМЕНТНАЯ ТЕОРИЯ ОБОЛОЧЕК О ФОРМЕ ОДНОПОЛОГО ГИПЕРБОЛОИДА

РЕЗЮМЕ

В статье представлено точное решение уравнений равновесия гиперболоидальной оболочки при помощи квадратур, причём получено решение в тех случаях, в которых метод развития в ряд Фурье отказывает (эллиптический гиперболоид). Полученные результаты применено к статическим исчислениям холодильных башен нагруженных собственным весом и ветром. Кроме того в статье принято некоторую гипотезу о взаимодействии оболочки с её краем. Гипотеза эта разрешила исчислить влияние на усилия в оболочке кругового кольца соединенного монолитно с верхним краем оболочки.