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Paracompactness and product spaces

by

K. Morita (Tokyo)

As is well known, the topological product of normal spaces is not generally normal. This fact suggests the possibilities of characterizing a certain type of topological spaces by requiring the normality of the product of them with a prescribed normal space. Indeed, C. H. Dowker [4] proved that a topological space X is countably paracompact and normal if and only if the product space $X\times I$ is normal. Here I means the closed line interval [0, 1].

The purpose of this paper is to deal with the problem: What topological space can be characterized if we require the normality of its product with $I^{\mathfrak{m}}$? Here $\mathfrak{m}$ is an infinite cardinal number and $I^{\mathfrak{m}}$ means the product space of $\mathfrak{m}$ copies of I.

To state our result it is necessary to introduce the notion of $\mathfrak{m}$-paracompactness. For any infinite cardinal number $\mathfrak{m}$, a topological space X is said to be $\mathfrak{m}$-*paracompact* if any open covering of X with power $\leqslant \mathfrak{m}$ (i.e. consisting of at most $\mathfrak{m}$ sets) admits a locally finite open covering as its refinement [1]. In case $\mathfrak{m}=\aleph_0$, "$\aleph_0$-paracompact" is nothing else "countably paracompact" and for a topological space X with an open base of power $\leqslant\mathfrak{m}$ the statement "X is $\mathfrak{m}$-paracompact" is equivalent to "X is paracompact". Here X is said to be *paracompact* if X is $\mathfrak{m}$-paracompact for any cardinal number $\mathfrak{m}$. For a Hausdorff space X paracompactness of X implies normality of X. While it is an open question whether every normal Hausdorff space is countably paracompact or not, it is easy to see that the notion of $\aleph_\alpha$-paracompactness is different from that of $\aleph_{\alpha+1}$-paracompactness for normal Hausdorff spaces. In fact, for any ordinal α the linearly ordered space $W(\omega_{\alpha+1})$, that consists of all ordinals less than the initial ordinal $\omega_{\alpha+1}$ of the power $\aleph_{\alpha+1}$ and has the interval topology, is a normal Hausdorff space which is $\aleph_\alpha$-paracompact (or more precisely $\aleph_\alpha$-compact) but not $\aleph_{\alpha+1}$-paracompact.

As a generalization of paracompact spaces, $\mathfrak{m}$-paracompact spaces have many analogous properties. Some of these are included in § **1**.

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[1] A family of subsets of X is called *locally finite* if every point of X has a neighborhood that meets only a finite number of sets of the family.

Now our main theorem, which will be proved in § 2, reads as follows: A topological space X is $\mathfrak{m}$-paracompact and normal if and only if the product space $X\times I^{\mathfrak{m}}$ is normal.

As a special case we obtain a theorem that a topological space X is paracompact and normal if and only if $X\times I^{\mathfrak{m}}$ is normal for a cardinal number $\mathfrak{m}$ not less than the power of an open base of X. In the case of completely regular Hausdorff spaces, another formulation is possible: A completely regular Hausdorff space X is paracompact if and only if the product space $X\times T$ is normal for a compact Hausdorff space T containing X as a subspace. As an immediate consequence of our theorem we have a theorem of H. Tamano [16] asserting that a completely regular Hausdorff space X is paracompact if and only if $X\times\beta(X)$ is normal where $\beta(X)$ is the Stone-Čech compactification of X.

As an application of our main theorem we shall prove in § 3 that if a topological space X has the weak topology with respect to a closed covering $\{A_\lambda\}$ such that each A_λ is $\mathfrak{m}$-paracompact and normal, then X is $\mathfrak{m}$-paracompact and normal.

Finally, in § 4 we shall prove two theorems asserting $\mathfrak{m}$-paracompactness of some topological spaces treated in recent literatures.

§ 1. Some basic properties of $\mathfrak{m}$-paracompact spaces. We begin by stating the following characterizations of $\mathfrak{m}$-paracompact spaces.

THEOREM 1.1. *Let X be a topological space. Then the following statements are equivalent.*

(a) *X is $\mathfrak{m}$-paracompact and normal.*

(b) *Every open covering of X with power $\leqslant\mathfrak{m}$ is a normal covering* (in the sense of Tukey [17]).

(c) *Every open covering of X with power $\leqslant\mathfrak{m}$ admits a locally finite closed covering as its refinement.*

(d) *Every open covering of X with power $\leqslant\mathfrak{m}$ admits a closure-preserving closed covering as its refinement.*

(e) *Every open covering of X with power $\leqslant\mathfrak{m}$ admits a σ-locally finite open covering as its refinement and X is countably paracompact normal.*

The implications (a)$\to$(c), (c)$\to$(d), (e)$\to$(a) are obvious. Now assume (d). Then the first part of (e) is proved as in Michael [10]. The second part is obvious if we apply the condition (d) to the case $\mathfrak{m}=\aleph_0$. Thus (d)$\to$(e) is proved. The second part of (e) seems necessary to assure the equivalence (a)$\rightleftarrows$(e); but it is not necessary for a cardinal number $\mathfrak{m}$ not less than the power of an open base of X (= a base for the open sets of X).

The equivalence (a)$\rightleftarrows$(b) is an immediate consequence of the following theorem.

THEOREM 1.2. *Let $\mathfrak{G}$ be an open covering of a topological space X. Then the following statements are equivalent.*

(a) *$\mathfrak{G}$ is a normal covering.*

(b) *There exists a continuous mapping f from X into a metric space Y such that $\mathfrak{G}$ is refined by the inverse image of some open covering of Y.*

(c) *$\mathfrak{G}$ admits a locally finite open normal covering as its refinement.*

(d) *$\mathfrak{G}$ admits as its refinement a locally finite open covering $\{H_\lambda\}$ each set of which is expressed as $H_\lambda=\{x\mid f_\lambda(x)>0\}$ with a continuous function f_λ: $X\to I$.*

(e) *$\mathfrak{G}$ has a locally finite partition of unity subordinated to it.*

(f) *$\mathfrak{G}$ has a partition of unity subordinated to it.*

(g) *There exists a normal open covering $\{U_\gamma\}$ of X such that on each of the subspaces U_γ the covering $\{G\cap U_\gamma\mid G\in\mathfrak{G}\}$ is normal.*

Proof. The implications (a)$\to$(b), (b)$\to$(c), (c)$\to$(d), (d)$\to$(e), (e)$\to$(f), and (a)$\to$(g) are known or easy to prove (cf. A. H. Stone [15], E. Michael [8]). We shall prove only two implications (f)$\to$(a) and (g)$\to$(a); (f)$\to$(a) is stated in Michael [8] without proof.

(f)$\to$(a). Let $\{\varphi_\lambda\mid\lambda\in\Lambda\}$ be a partition of unity on X which is subordinated to $\mathfrak{G}$; i.e. each φ_λ is a continuous map from X into $I=[0,1]$ such that $\sum\varphi_\lambda(x)=1$ for every point x of X and $\{x\mid\varphi_\lambda(x)>0\}$ is contained in some member of $\mathfrak{G}$. Let us consider a metric space M which consists of sets $\{x_\lambda\mid\lambda\in\Lambda\}$ with $x_\lambda\in I$ such that $\sum x_\lambda=1$ and has a metric $d(x,y)=\sum|x_\lambda-y_\lambda|$. For every point x of X we assign $\varphi(x)=\{\varphi_\lambda(x)\mid\lambda\in\Lambda\}$. Then φ is a continuous mapping of X into M. To prove this, for a given point x_0 of X and a positive number ε, we find a finite subset Γ of Λ and a neighborhood U_0 of x_0 such that $\sum_{\lambda\notin\Gamma}\varphi_\lambda(x_0)<\varepsilon$ and $\sum_{\lambda\in\Gamma}|\varphi_\lambda(x)-\varphi_\lambda(x_0)|<\varepsilon$ for $x\in U_0$. Then we have $\sum_{\lambda\notin\Gamma}\varphi_\lambda(x)=\sum_{\lambda\in\Gamma}\big(\varphi_\lambda(x_0)-\varphi_\lambda(x)\big)+\sum_{\lambda\notin\Gamma}\varphi_\lambda(x_0)\leqslant\sum_{\lambda\in\Gamma}|\varphi_\lambda(x)-\varphi_\lambda(x_0)|+\sum_{\lambda\notin\Gamma}\varphi_\lambda(x_0)<2\varepsilon$ and hence $d\big(\varphi(x),\varphi(x_0)\big)<\sum_{\lambda\notin\Gamma}|\varphi_\lambda(x)-\varphi_\lambda(x_0)|+\varepsilon\leqslant\sum_{\lambda\notin\Gamma}\big(\varphi_\lambda(x)+\varphi_\lambda(x_0)\big)+\varepsilon<2\varepsilon+\varepsilon+\varepsilon=4\varepsilon$ for every point x of U_0. If we put $V_\lambda=\big\{\{x_\mu\}\mid x_\lambda>0\big\}$, V_λ are open sets of M and $\varphi(X)\subset\bigcup V_\lambda$, $\varphi^{-1}(V_\lambda)=\{x\mid\varphi_\lambda(x)>0\}$. Hence by (b) $\mathfrak{G}$ is a normal covering.

(g)$\to$(a). Let $\{V_\lambda\}$ be a normal open covering of X which is a star-refinement of $\{U_\gamma\}$. Let $\{\varphi_\lambda\mid\lambda\in\Lambda\}$ be a partition of unity on X which is subordinated to $\{V_\lambda\}$; we assume that $\{x\mid\varphi_\lambda(x)>0\}\subset V_\lambda$ for each $\lambda\in\Lambda$. Since $\overline{V}_\lambda$ is contained in some element U_γ of $\{U_\gamma\}$, $\{G\cap\overline{V}_\lambda\mid G\in\mathfrak{G}\}$ is a normal covering of the subspace $\overline{V}_\lambda$.

Let $\{f_{\lambda\mu} \mid \mu \in \Gamma_\lambda\}$ be a partition of unity on $\overline{V}_\lambda$ which is subordinated to this covering. We put

$$\varphi_{\lambda\mu}(x) = \begin{cases} \varphi_\lambda(x) f_{\lambda\mu}(x) & \text{for} \quad x \in \overline{V}_\lambda, \\ 0 & \text{for} \quad x \in X - \overline{V}_\lambda. \end{cases}$$

Since we have $\varphi_\lambda(x) = 0$ for $x \in \overline{V}_\lambda - V_\lambda$, $\varphi_{\lambda\mu}(x)$ is continuous over X. Since $\varphi_\lambda(x) = \sum_{\mu \in \Gamma_\lambda} \varphi_{\lambda\mu}(x)$ for $x \in X$, we have $\sum_{\lambda,\mu} \varphi_{\lambda\mu}(x) = 1$ for $x \in X$. Moreover, we have $\{x \mid \varphi_{\lambda\mu}(x) > 0\} \subset \{x \mid \varphi_\lambda(x) > 0\} \cap \{x \mid f_{\lambda\mu}(x) > 0\}$, and $\{x \mid f_{\lambda\mu}(x) > 0\}$ is contained in some element of $\mathfrak{G}$. Hence $\{\varphi_{\lambda\mu} \mid \lambda \in \Lambda, \mu \in \Gamma_\lambda\}$ is a partition of unity on X which is subordinated to $\mathfrak{G}$. Therefore $\mathfrak{G}$ is normal by (f).

COROLLARY 1.3. *Let $\{G_\alpha \mid \alpha \in \Omega\}$ be an open covering of a normal space X. If there exists a normal open covering $\{U_\gamma\}$ of X such that each set U_γ is contained in a sum of a finite number of sets of $\{G_\alpha\}$, then $\{G_\alpha\}$ is a normal covering of X.*

Proof. Considering a normal star-refinement of $\{U_\gamma\}$ we see that this corollary is an immediate consequence of Theorem 1.2. However, a direct proof is rather simple. $\{U_\gamma\}$ has a locally finite open refinement $\{V_\lambda \mid \lambda \in \Lambda\}$. Each V_λ is contained in $\bigcup \{G_\alpha \mid \alpha \in \Gamma_\lambda\}$ with a finite subset Γ_λ of Ω. Then $\{V_\lambda \cap G_\alpha \mid \lambda \in \Lambda, \alpha \in \Gamma_\lambda\}$ is a locally finite open refinement of $\{G_\alpha \mid \alpha \in \Omega\}$.

In view of Theorem 1.1, we obtain at once the following theorem as in Michael [10].

THEOREM 1.4. *Let f be a closed continuous mapping of a topological space X onto another topological space Y. If X is $\mathfrak{m}$-paracompact and normal, so is also Y.*

THEOREM 1.5. *Let A be a subset of an $\mathfrak{m}$-paracompact normal space X. If for any open set G containing A there exists a family $\{H_\lambda\}$ of open F_σ-sets of X such that $A \subset \bigcup H_\lambda \subset G$ and $\{H_\lambda\}$ is locally finite in A, then the subspace A is $\mathfrak{m}$-paracompact and normal.*

Proof. By assumption, for each λ there exist a countable number of closed sets $F_{\lambda i}$ such that $H_\lambda = \bigcup \{F_{\lambda i} \mid i = 1, 2, \ldots\}$. Take closed sets $C_{\lambda i}$ so that $F_{\lambda i} \subset \operatorname{Int} C_{\lambda i}$, $C_{\lambda i} \subset H_\lambda$. Since each $C_{\lambda i}$ is $\mathfrak{m}$-paracompact normal, H_λ is $\mathfrak{m}$-paracompact normal by Theorem 3.5 below. Now let $\{G_\gamma \cap A\}$ be any open covering of X with power $\leqslant \mathfrak{m}$; G_γ are open sets of X. Put $G = \bigcup G_\gamma$. For this G we find $\{H_\lambda\}$ with the property stated in the theorem. Then $\{H_\lambda \cap A\}$ is a normal covering of A since it satisfies condition (d) of Theorem 1.2, and $\{G_\gamma \cap H_\lambda \cap A \mid \gamma\}$ is a normal covering of the subspace $H_\lambda \cap A$ since H_λ is $\mathfrak{m}$-paracompact and normal as is shown above. Hence $\{G_\gamma \cap A\}$ is a normal covering of X by Theorem 1.2 (²).

COROLLARY 1.6. *Let X be an $\mathfrak{m}$-paracompact, normal space. If A is a generalized F_σ-set of X (i.e. for every open set G with $A \subset G$ there exists an F_σ-set C with $A \subset C \subset G$) then A is $\mathfrak{m}$-paracompact and normal.*

Proof. Suppose that $A \subset G$ for an open set G. Then there exist closed sets F_i, $i = 1, 2, \ldots$, such that $A \subset \bigcup F_i \subset G$. Since X is normal, for each i there exists a continuous map f_i: $X \to I$ such that $f_i(x) = 1$ or 0 according as $x \in F_i$ or $x \in X - G$. If we put $f(x) = \sum_{i=1}^{\infty} 2^{-i} f_i(x)$, then $H = \{x \mid f(x) > 0\}$ is an open F_σ-set and $A \subset H \subset G$. Thus the hypothesis of Theorem 1.5 holds for A. Hence Corollary 1.6 holds.

This corollary is due to Y. Smirnov [14] for the case of normality and to E. Michael [8] for the case of paracompactness.

Recently H. Corson [2] has shown that paracompact spaces are characterized by the property of possessing a uniformity with a certain type of completeness. His idea can be applied to our case.

Let $\{\mathfrak{U}_\lambda\}$ be a uniformity of X in the sense of Tukey [17] (³). A family $\mathfrak{F}$ of subsets of X having the finite intersection property is called a *weakly Cauchy family with respect to* $\{\mathfrak{U}_\lambda\}$ if for each λ there exists a set U of the covering $\mathfrak{U}_\lambda$ such that $\{\mathfrak{F}, U\}$ has the finite intersection property. Then the following theorem holds.

THEOREM 1.7. *A normal Hausdorff space X is $\mathfrak{m}$-paracompact if and only if there exists a uniformity $\{\mathfrak{U}_\lambda\}$ of X for which every weakly Cauchy family of power $\leqslant \mathfrak{m}$ (i.e. consisting of at most $\mathfrak{m}$ sets) has a cluster point.*

Proof. Suppose that X is $\mathfrak{m}$-paracompact. Let $\{\mathfrak{U}_\lambda\}$ be the family of all the open coverings of X with power $\leqslant \mathfrak{m}$. Then $\{\mathfrak{U}_\lambda\}$ is a uniformity of X, as is easily seen from Morita [12], Theorem 1.2. Suppose that a weakly Cauchy family $\mathfrak{F}$ of power $\leqslant \mathfrak{m}$ (with respect to this uniformity) has no cluster point. Then $\mathfrak{U} = \{X - \overline{F} \mid F \in \mathfrak{F}\}$ is an open covering of X with power $\leqslant \mathfrak{m}$. Hence $\mathfrak{U}$ is equal to some $\mathfrak{U}_\lambda$. Since $\mathfrak{F}$ is weakly Cauchy, there exists a set U_0 of $\mathfrak{U}_\lambda$ such that $U_0 \cap F \neq \emptyset$ for every $F \in \mathfrak{F}$. However, this is impossible since $U_0 = X - \overline{F}_0$ for some $F_0 \in \mathfrak{F}$. Thus the "only if" part is proved.

Conversely, assume that there exists a uniformity $\{\mathfrak{U}_\lambda\}$ of X for which every weakly Cauchy family of power $\leqslant \mathfrak{m}$ has a cluster point. Let $\mathfrak{G}$ be any open covering of X with power $\leqslant \mathfrak{m}$. Suppose that $\mathfrak{G}$ admits no locally finite open refinement. We put $\mathfrak{F} = \{X - G \mid G \in \mathfrak{G}\}$. We shall

(²) We can prove similarly the following: A subspace A of a paracompact Hausdorff space is paracompact if and only if for any open set G containing A there exists a normal (or locally finite) open covering $\{V_\lambda\}$ of A such that $\overline{V}_\lambda \subset G$ for each λ.

(³) Corson uses uniformities in the sense of Weil.

prove that $\mathfrak{F}$ is a weakly Cauchy family of power $\leqslant \mathfrak{m}$. For each $\mathfrak{U}_\lambda$ there exists a set U of $\mathfrak{U}_\lambda$ such that U is not contained in a sum of any finite number of sets of $\mathfrak{G}$; otherwise $\mathfrak{G}$ would admit a locally finite open refinement by Corollary 1.3. Hence $\{\mathfrak{F}, U\}$ has the finite intersection property. Therefore $\mathfrak{F}$ is weakly Cauchy. But $\mathfrak{F}$ has no cluster point. This is a contradiction. Thus the "if" part is proved. This completes the proof of Theorem 1.7.

In case of paracompactness, to assure the validity of Theorem 1.7 it is sufficient to assume the complete regularity of X instead of the normality.

A topological space X is called $\mathfrak{m}$-*compact* if every open covering of power $\leqslant \mathfrak{m}$ has a finite subcovering. In view of the condition (b) in Theorem 1.2 we obtain at once the following theorem.

THEOREM 1.8. *A normal space is $\mathfrak{m}$-compact if and only if it is $\mathfrak{m}$-paracompact and countably compact* (*or pseudo-compact*).

§ 2. Product spaces. We shall first prove two theorems as in Dowker [4].

THEOREM 2.1. *Let X be an $\mathfrak{m}$-paracompact space and Y a compact space.* Then *$X \times Y$ is $\mathfrak{m}$-paracompact.*

Proof. Let $\{U_\lambda \mid \lambda \in \Lambda\}$ be an open covering of $X \times Y$ with $|\Lambda| \leqslant \mathfrak{m}$. Denote by Γ the totality of all the finite subsets of Λ. Let us put for any $\gamma \in \Gamma$

$$V_\gamma = \{x \mid x \times Y \subset \cup \{U_\lambda \mid \lambda \in \gamma\}\}.$$

Then V_γ is open since Y is compact. On the other hand, for any point x of X there exists $\gamma \in \Gamma$ such that $x \times Y \subset \bigcup \{U_\lambda \mid \lambda \in \gamma\}$. Hence $\{V_\gamma \mid \gamma \in \Gamma\}$ is an open covering of X and $|\Gamma| \leqslant \mathfrak{m}$. Therefore, since X is $\mathfrak{m}$-paracompact, there exists a locally finite open covering $\{G_\gamma \mid \gamma \in \Gamma\}$ of X such that $G_\gamma \subset V_\gamma$ for each γ. Now let us put $G_{\gamma\lambda} = (G_\gamma \times Y) \cap U_\lambda$, for $\lambda \in \gamma$. Then $\{G_{\gamma\lambda} \mid \gamma \in \Gamma, \lambda \in \gamma\}$ is a locally finite open covering of $X \times Y$ and is a refinement of $\{U_\lambda \mid \lambda \in \Lambda\}$.

Remark. Similarly we can prove that the product of an $\mathfrak{m}$-compact space with a compact space is $\mathfrak{m}$-compact.

THEOREM 2.2. *Let X be an $\mathfrak{m}$-paracompact normal space and Y a compact normal space with an open base of power $\leqslant \mathfrak{m}$. Then $X \times Y$ is normal.*

Proof. Let A and B be two disjoint closed sets of $X \times Y$, and let $\{G_\lambda \mid \lambda \in \Lambda\}$ be an open base of Y with power $\leqslant \mathfrak{m}$ where $|\Lambda| \leqslant \mathfrak{m}$. Denote by Γ the totality of all the finite subsets of Λ. Let us put

$$H_\gamma = \bigcup \{G_\lambda \mid \lambda \in \gamma\} \quad \text{for} \quad \gamma \in \Gamma,$$
$$A[x] = \{y \mid (x, y) \in A\}, \quad B[x] = \{y \mid (x, y) \in B\}$$

for $x \in X$. If we put further

$$U_\gamma = \{x \mid A[x] \subset H_\gamma,\ \overline{H}_\gamma \subset Y - B[x]\},$$

then U_γ is open since Y is compact. On the other hand, since $A[x]$ and $B[x]$ are disjoint closed sets of the compact normal space Y, there exists for any point x of X an H_γ such that $A[x] \subset H_\gamma$, $\overline{H}_\gamma \cap B[x] = \emptyset$. Therefore $\{U_\gamma\}$ is an open covering of X with power $\leqslant \mathfrak{m}$.

Since X is $\mathfrak{m}$-paracompact and normal, there exists a locally finite open covering $\{V_\gamma \mid \gamma \in \Gamma\}$ of X such that $\overline{V}_\gamma \subset U_\gamma$ for each γ. We put

$$V = \bigcup \{V_\gamma \times H_\gamma \mid \gamma \in \Gamma\}.$$

Then $\overline{V} = \bigcup \{\overline{V}_\gamma \times \overline{H}_\gamma\}$ since $\{V_\gamma \times H_\gamma\}$ is locally finite in $X \times Y$, and we can easily prove that $A \subset V$, $\overline{V} \cap B = \emptyset$. This proves Theorem 2.2.

COROLLARY 2.3. *Under the same assumption as in Theorem 2.2, if X is collectionwise normal, then $X \times Y$ is also collectionwise normal.*

Proof. As is proved by M. Katětov [6], a normal space is collectionwise normal and countably paracompact if and only if for every locally finite collection $\{F_\lambda\}$ of closed subsets there exists a locally finite collection $\{G_\lambda\}$ of open subsets such that $F_\lambda \subset G_\lambda$ for each λ. To apply this theorem to our case, let $\{F_\lambda\}$ be a locally finite collection of closed subsets of $X \times Y$.

The projection f defined by $f(x, y) = x$ is a closed continuous mapping of $X \times Y$ onto X since Y is compact. Then $\{f(F_\lambda)\}$ is a locally finite collection of closed sets of X; since Y is compact, there exists for each point x of X an open set G of $X \times Y$ such that $x \times Y \subset G$ and G meets only a finite number of sets of $\{F_\lambda\}$, and hence an open neighborhood U of x such that $U \times Y \subset G$ meets only a finite number of sets of $\{f(F_\lambda)\}$. Hence there exists a locally finite collection $\{G_\lambda\}$ of open sets of X such that $f(F_\lambda) \subset G_\lambda$. Therefore $\{G_\lambda \times Y\}$ is a locally finite collection of open sets of $X \times Y$ such that $F_\lambda \subset G_\lambda \times Y$ for each λ. This proves the collectionwise normality of $X \times Y$ by the theorem mentioned in the beginning of the proof.

We shall now establish our main theorem.

THEOREM 2.4. *A topological space X is $\mathfrak{m}$-paracompact and normal if and only if $X \times I^{\mathfrak{m}}$ is normal.*

The "only if" part is obvious by Theorem 2.2, since $I^{\mathfrak{m}}$ has an open base of power $\leqslant \mathfrak{m}$. Before proving the "if" part we shall first prove a lemma by generalizing Dowker's argument in [4].

LEMMA 2.5. *Let X be a topological space such that $X \times W(\omega_\alpha + 1)$ is normal. Then every covering of X with power $\leqslant \aleph_\alpha$ admits as a refinement a closed covering of power $\leqslant \aleph_\alpha$. Here $W(\omega_\alpha + 1)$ is the linearly ordered space consisting of all ordinals $\leqslant \omega_\alpha$.*

Proof. From the assumption X is clearly normal. Assume that the lemma is true for any ordinal β less than α. We shall prove the validity of the lemma for α. Let $\{U_\lambda \mid \lambda < \omega_\alpha\}$ be any open covering of X with power $\leqslant \aleph_\alpha$. We put $V_\lambda = \bigcup \{U_\mu \mid \mu \leqslant \lambda\}$ and consider two sets:

$$A = X \times W - \bigcup \{V_\lambda \times (\lambda, \omega_\alpha] \mid \lambda < \omega_\alpha\},$$

$$B = X \times \omega_\alpha;$$

where W stands for $W(\omega_\alpha + 1)$. Then for any point x of X there exists some V_λ such that $x \in V_\lambda$, and hence $x \times \omega_\alpha \in V_\lambda \times (\lambda, \omega_\alpha]$, and consequently $x \times \omega_\alpha \notin A$. This shows that $A \cap B = \emptyset$. Since A, B are closed and $X \times W$ is normal, there exist open sets G, H of $X \times W$ such that $A \subset G$, $B \subset H$, $G \cap H = \emptyset$. We put

$$F_\lambda = \{x \mid x \times \lambda \notin G\} \quad \text{for} \quad \lambda < \omega_\alpha.$$

Then F_λ is closed, since $X - F_\lambda = \{x \mid x \times \lambda \in G\}$ and $\{x \mid x \times \lambda \in G\}$ is clearly open. We shall now prove

$$X = \bigcup \{F_\lambda \mid \lambda < \omega_\alpha\}, \tag{1}$$

$$F_\lambda \subset V_\lambda \quad \text{for} \quad \lambda < \omega_\alpha. \tag{2}$$

To prove (1), let x be any point of X. Then $x \times \omega_\alpha \in B \subset H$. Hence there exists some $\lambda < \omega_\alpha$ such that $x \times (\lambda, \omega_\alpha] \subset H$. For this λ we have $x \times (\lambda + 1) \in H$, and hence $x \times (\lambda + 1) \notin G$. Therefore $x \in F_{\lambda+1}$. Thus (1) is proved.

To prove (2), suppose that $x \in X - V_\lambda$. Then $x \in X - V_\mu$ for every $\mu \leqslant \lambda$, since $V_\mu \subset V_\lambda$. Hence $x \times \lambda$ is not contained in $V_\mu \times (\mu, \omega_\alpha]$ for any $\mu \leqslant \lambda$. For $\mu > \lambda$ we have $\lambda \notin (\mu, \omega_\alpha]$ and hence $x \times \lambda \notin V_\mu \times (\mu, \omega_\alpha]$ for $\mu > \lambda$. Thus $x \times \lambda \in A \subset G$. Hence we have $x \notin F_\lambda$. This shows that $X - V_\lambda \subset X - F_\lambda$, and (2) is proved hereby.

The power $\aleph_\beta$ of the set $\{\mu \mid \mu \leqslant \lambda\}$ is smaller than $\aleph_\alpha$, since ω_α is the initial ordinal of the power $\aleph_\alpha$. Hence $\beta < \alpha$. Now $W(\omega_\beta + 1)$ is a closed set of $W(\omega_\alpha + 1)$, and hence $F_\lambda \times W(\omega_\beta + 1)$ is normal as a closed subset of the normal space $X \times W(\omega_\alpha + 1)$. Therefore an open covering $\{U_\mu \cap F_\lambda \mid \mu \leqslant \lambda\}$ of F_λ with power $\leqslant \aleph_\beta$ is refined, by the assumption of transfinite induction, by a closed covering of F_λ with power $\leqslant \aleph_\beta$, which we denote by $\{F_{\lambda\mu} \mid \mu \leqslant \lambda\}$. Then the collection $\{F_{\lambda\mu} \mid \lambda < \omega_\alpha, \mu \leqslant \lambda\}$ is a closed covering of X with power $\leqslant \aleph_\alpha$ since $\aleph_\alpha^2 = \aleph_\alpha$. Thus the lemma is true for α. The above argument applies to $\alpha = 0$ without induction hypothesis. Thus our proof by transfinite induction is completed.

Now we proceed to the proof of the "if" part of Theorem 2.4.

Suppose that $X \times I^{\mathfrak{m}}$ is normal and that $\mathfrak{m} = \aleph_\alpha$. Since $W(\omega_\alpha + 1)$ is a compact Hausdorff space with an open base of power $\leqslant \aleph_\alpha$, $W(\omega_\alpha + 1)$

is homeomorphic to a closed subspace of $I^{\mathfrak{m}}$. Hence $X \times W(\omega_\alpha + 1)$ is normal.

Now let $\mathfrak{G}$ be any open covering of X with power $\leqslant \mathfrak{m}$. Then by the remark above it follows from Lemma 2.5 that $\mathfrak{G}$ is refined by a closed covering $\mathfrak{F}$ of power $\leqslant \mathfrak{m}$. Hence we may assume that $\mathfrak{F} = \{F_\lambda \mid \lambda < \omega_\alpha\}$ and $\{G_\lambda \mid \lambda < \omega_\alpha\}$ is an open refinement of $\mathfrak{G}$ such that

$$F_\lambda \subset G_\lambda \quad \text{for each} \quad \lambda < \omega_\alpha.$$

Since X is normal, there exists for each λ a continuous mapping φ_λ: $X \to I$ such that $\varphi_\lambda(x)$ is 0 or 1 according as $x \in F_\lambda$ or $x \in X - G_\lambda$. Let us put

$$\varphi(x) = \{\varphi_\lambda(x) \mid \lambda < \omega_\alpha\} \quad \text{for} \quad x \in X.$$

Since $I^{\mathfrak{m}}$ is considered as the space consisting of the sets $\{y_\lambda \mid \lambda < \omega_\alpha\}$ with $0 \leqslant y_\lambda \leqslant 1$, φ defines a continuous mapping from X into $I^{\mathfrak{m}}$. We put

$$H = \bigcup V_\lambda; \quad V_\lambda = \big\{\{y_\mu\} \mid y_\lambda < 1,\ \{y_\mu\} \in I^{\mathfrak{m}}\big\},$$

for $\lambda < \omega_\alpha$. Then we have

$$\varphi(X) \subset H; \quad \varphi^{-1}(V_\lambda) \subset G_\lambda \quad \text{for} \quad \lambda < \omega_\alpha, \tag{3}$$

since $X = \bigcup F_\lambda$, $\varphi(F_\lambda) \subset V_\lambda$. We denote by D the graph of the mapping φ: $D = \big\{(x, \varphi(x)) \mid x \in X\big\}$. Then D is closed in $X \times I^{\mathfrak{m}}$. Since $D \subset X \times H$ by (3) and $X \times H$ is open in $X \times I^{\mathfrak{m}}$, it follows from the normality of $X \times I^{\mathfrak{m}}$ that there exists a continuous mapping Φ: $X \times I^{\mathfrak{m}} \to I$ such that

$$\Phi(x, y) = \begin{cases} 0, & \text{for} \quad (x, y) \in D, \\ 1, & \text{for} \quad (x, y) \notin X \times H. \end{cases}$$

We construct a function $\psi(x, x')$ on the product space $X \times X$ by

$$\psi(x, x') = \sup_{y \in I^{\mathfrak{m}}} |\Phi(x, y) - \Phi(x', y)|, \quad \text{for} \quad x, x' \text{ of } X\ ([4]).$$

Then $\psi(x, x')$ is continuous over $X \times X$, since $I^{\mathfrak{m}}$ is compact. Moreover, we have

$$\Phi\big(x, \varphi(x')\big) \leqslant \psi(x, x') \quad \text{for} \quad x, x' \in X,$$

since

$$\Phi\big(x, \varphi(x')\big) = \Phi\big(x, \varphi(x')\big) - \Phi\big(x', \varphi(x')\big) \leqslant \sup_{y \in I^{\mathfrak{m}}} |\Phi(x, y) - \Phi(x', y)|.$$

If we put

$$U_x = \{x' \mid \psi(x, x') < 2^{-1}\}, \quad \text{for} \quad x \in X,$$

([4]) The argument below is an elaboration of Tamano's in [16].

$\{U_x \mid x \in X\}$ is a normal open covering of X since $\psi(x, x')$ is a pseudo-metric of X (i.e. $\psi(x, x) = 0$; $\psi(x', x) = \psi(x, x') \geqslant 0$; $\psi(x, x') + \psi(x', x'') \geqslant \psi(x, x'')$ for $x, x', x'' \in X$). For any point x' of U_x, we have

$$\Phi\big(x, \varphi(x')\big) \leqslant \psi(x, x') < 2^{-1}$$

and hence

$$\varphi(U_x) \subset \{y \mid \Phi(x, y) < 2^{-1}\}$$

and consequently

$$\varphi(U_x) \subset \overline{\varphi(U_x)} \subset \{y \mid \Phi(x, y) \leqslant 2^{-1}\} \subset H,$$

since $\Phi(x, y) = 1$ for $y \notin H$. By definition, $H = \bigcup V_\lambda$ and $\overline{\varphi(U_x)}$ is compact. Hence there exists a finite subset γ_x of the set $\{\lambda \mid \lambda < \omega_a\}$ such that $\varphi(U_x) \subset \bigcup \{V_\lambda \mid \lambda \in \gamma_x\}$. Therefore we have by (3)

$$U_x \subset \bigcup \{G_\lambda \mid \lambda \in \gamma_x\}.$$

This shows that each set U_x of the normal open covering $\{U_x \mid x \in X\}$ of X is contained in a sum of a finite number of sets belonging to the covering $\{G_\lambda\}$ of X. Hence by Corollary 1.3 the covering $\{G_\lambda\}$ of X is normal. This completes our proof of Theorem 2.4.

COROLLARY 2.6. *Let X be a topological space with an open base of power $\leqslant \mathfrak{m}$. Then X is paracompact and normal if and only if $X \times I^{\mathfrak{m}}$ is normal.*

Proof. Any open covering of X has a subcovering of power $\leqslant \mathfrak{m}$, and hence Corollary 2.6 follows immediately from Theorem 2.4.

THEOREM 2.7. *Let X be a completely regular Hausdorff space. If $X \times T$ is normal for a compact Hausdorff space T which contains a subspace Y homeomorphic to X, then X is paracompact and normal.*

Proof. In the proof of Theorem 2.4, we replace $I^{\mathfrak{m}}$ by T and $\varphi\colon X \to I^{\mathfrak{m}}$ by a homeomorphism φ_0 of X onto Y. Then the proof of Theorem 2.4 is valid with V_λ replaced by any open sets W_λ of T such that $\varphi_0(G_\lambda) = W_\lambda \cap Y$ (5).

§ 3. Spaces having the weak topology. Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a closed covering of a topological space X. Then X is said to have the weak topology with respect to $\{A_\lambda\}$ if for any subset Λ' of Λ every subset C of $\bigcup\{A_\lambda \mid \lambda \in \Lambda'\}$ for which $C \cap A_\lambda$ is closed for each λ of Λ' is necessarily closed in X. (cf. Morita [13] and Michael [9].) Every topological space has always the weak topology with respect to any locally finite closed covering.

(5) It should be noted that Lemma 2.5 is dispensable in this proof of Theorem 2.7. This remark is also applicable to a direct proof (along the same line as in the proof of Theorem 2.4) of Corollary 2.6 if we assume X is a completely regular Hausdorff space; in this case there is a homeomorphism of X onto a subspace of $I^{\mathfrak{m}}$.

THEOREM 3.1. *If a topological space X has the weak topology with respect to a closed covering $\{A_\lambda\}$ such that each set A_λ is $\mathfrak{m}$-paracompact and normal, then X is $\mathfrak{m}$-paracompact and normal.*

Our proof is based on the following two lemmas which are proved simply in Morita [13].

LEMMA 3.2. *If a topological space X has the weak topology with respect to a closed covering $\{A_\lambda\}$ such that each A_λ is normal, then X is normal.*

LEMMA 3.3. *If a topological space X has the weak topology with respect to a closed covering $\{A_\lambda\}$, then $X \times Y$ has the weak topology with respect to the closed covering $\{A_\lambda \times Y\}$ for any compact Hausdorff space Y.*

Proof of Theorem 3.1. Suppose that X has the weak topology with respect to a closed covering $\{A_\lambda\}$ such that each A_λ is $\mathfrak{m}$-paracompact normal. Then, by Theorem 2.4, $A_\lambda \times I^{\mathfrak{m}}$ is normal. By Lemma 3.3, $X \times I^{\mathfrak{m}}$ has the weak topology with respect to the closed covering $\{A_\lambda \times I^{\mathfrak{m}}\}$. Hence $X \times I^{\mathfrak{m}}$ is normal by Lemma 3.2. Therefore X is $\mathfrak{m}$-paracompact normal again by Theorem 2.4.

Similarly we obtain the following theorem.

THEOREM 3.4. *If a topological space X has a countable closed covering $\{A_i \mid i = 1, 2, \ldots\}$ such that any subset C for which $C \cap A_i$ is closed for each i is necessarily closed in X, and if each A_i is $\mathfrak{m}$-paracompact and normal, then X is $\mathfrak{m}$-paracompact and normal.*

THEOREM 3.5. *If a topological space X has a countable closed covering $\{A_i \mid i = 1, 2, \ldots\}$ such that $X = \bigcup \{\operatorname{Int} A_i \mid i = 1, 2, \ldots\}$, and if each A_i s $\mathfrak{m}$-paracompact and normal, then X is $\mathfrak{m}$-paracompact and normal.*

Proof. In this case $\{A_i\}$ has the property described in Theorem 3.4. Another proof is obtained directly from Theorem 3.1, if we note that $\{C_i\}$ is a locally finite closed covering of X where $C_i = A_i - \bigcup\{\operatorname{Int} A_j \mid j < i\}$ for $i > 1$ and $C_1 = A_1$.

In connection with Theorem 3.4 we mention the following theorem.

THEOREM 3.6. *Let $\{A_i\}$ be a countable closed covering of a topological space X.*

(i) *If X is normal and each A_i is countably paracompact, then X is countably paracompact.*

(ii) *If X is collectionwise normal and each A_i is $\mathfrak{m}$-paracompact, then X is $\mathfrak{m}$-paracompact.*

Proof. (i) follows immediately from the fact that a normal space is countably paracompact if and only if any countable open covering admits a countable closed refinement (cf. Morita [11], Dowker [4]).

(ii) By (i) X is countably paracompact and hence the condition (e) of Theorem 1.1 holds in view of a theorem of Dowker [5]. Hence (ii) holds.

As is well known, a topological space which is a union of a countable number of closed metrizable subspaces is not always normal.

§ 4. The $\mathfrak{m}$-paracompactness of some topological spaces. As a generalization of a theorem of Dieudonné [3] we shall prove the following theorem.

THEOREM 4.1. *Let X be a paracompact normal space such that each point has a neighborhood base of power $\leqslant \mathfrak{m}$, and let Y be an $\mathfrak{m}$-compact normal space. Then $X \times Y$ is $\mathfrak{m}$-paracompact and normal.*

Proof. Let $\{U_\lambda(x) \mid \lambda \in \Lambda\}$ with $|\Lambda| \leqslant \mathfrak{m}$ be an open neighborhood base of power $\leqslant \mathfrak{m}$ at a point x of X. For any subset C of $X \times Y$ we can easily prove the relation

$$\bar{C}[x] = \bigcap_{\lambda} \overline{C[U_\lambda(x)]}$$

where $C[U_\lambda(x)] = \{y \mid (x', y) \in C \text{ for some } x' \in U_\lambda(x)\}$ and $\bar{C}[x] = \{y \mid (x, y) \in \bar{C}\}$. Suppose that $\bar{C}[x] \subset H$ for some open set H of Y. Then we have $\overline{C[U_\lambda(x)]} \subset H$ for some $\lambda \in \Lambda$; otherwise $\{\overline{C(U_\lambda(x)]} - H \mid \lambda \in \Lambda\}$ would have the finite intersection property and hence we would have $\bigcap_\lambda \overline{C[U_\lambda(x)]} - H = \bar{C}[x] - H \neq \emptyset$ because of $\mathfrak{m}$-compactness of Y, but this contradicts the assumption that $\bar{C}[x] \subset H$. Therefore $\{x \mid \bar{C}[x] \subset H\}$ is an open set of X (6).

To prove the normality of $X \times Y$, let A, B be two disjoint closed subsets of $X \times Y$. Then the set

$$U_H = \{x \mid A[x] \subset H,\ \bar{H} \subset X - B[x]\}$$

is open for each open set H of Y. On the other hand, for each point x of X the closed sets $A[x]$ and $B[x]$ are disjoint in Y. Since Y is normal, there exists an open set H of Y such that $A[x] \subset H$, $\bar{H} \subset Y - B[x]$. Thus $\{U_H \mid H$ ranges over all the open sets of $Y\}$ is an open covering of X. Now we apply the argument in the proof of Theorem 2.2 and we conclude that $X \times Y$ is normal.

The $\mathfrak{m}$-paracompactness of $X \times Y$ follows immediately from Theorem 2.4, since $Y \times I^{\mathfrak{m}}$ is $\mathfrak{m}$-compact and normal by Theorems 1.8, 2.2 and Remark to Theorem 2.1, and hence $X \times (Y \times I^{\mathfrak{m}})$, which is homeomorphic to $(X \times Y) \times I^{\mathfrak{m}}$, is normal by the proof stated above.

Recently, Mansfield [7] introduced the notation of almost-$\mathfrak{m}$-full normality. A topological space X is said to be *almost-$\mathfrak{m}$-fully normal* if for each open covering $\mathfrak{G}$ of X there exists an open covering $\mathfrak{H}$ of X such that (i) $\mathfrak{H}$ is a refinement of $\mathfrak{G}$ and (ii) for each set M with $|M| \leqslant \mathfrak{m}$

(6) Similarly it can be shown that the projection from $X \times Y$ onto X is a closed mapping.

and $M \subset \mathrm{St}(x, \mathfrak{H})$ (7) for some $x \in X$ there exists a set $G \in \mathfrak{G}$ with $M \subset G$. Here $\mathfrak{m}$ is any cardinal number $\geqslant 2$. It is proved by H. J. Cohen [1] that a topological space X has the property that all neighborhoods of the diagonal in $X \times X$ form a uniformity in the sense of Weil if and only if X is almost-2-fully normal. We shall prove

THEOREM 4.2. *Every almost-$\mathfrak{m}$-fully normal space is $\mathfrak{m}$-paracompact for an infinite cardinal number*. $\mathfrak{m}$.

This theorem is an immediate consequence of Lemma 4.3 below.

LEMMA 4.3. *A topological space X is almost-$\mathfrak{m}$-fully normal if and only if for each open covering $\mathfrak{G}$ of X there exists a normal open covering $\mathfrak{V}$ of X such that if $M \subset V$ for some set V of $\mathfrak{V}$ and $|M| \leqslant \mathfrak{m}$ then there is a set G of $\mathfrak{G}$ containing M.*

In fact, let X be almost-$\mathfrak{m}$-fully normal, and let $\mathfrak{G} = \{G_\lambda \mid \lambda \in \Lambda\}$ with $|\Lambda| \leqslant \mathfrak{m}$ be any open covering of X with power $\leqslant \mathfrak{m}$. If we admit the validity of Lemma 4.3, there exists a normal open covering $\mathfrak{V}$ with the property stated in Lemma 4.3. Then $\mathfrak{V}$ is a refinement of $\mathfrak{G}$; otherwise there would exist a set V of $\mathfrak{V}$ such that $V \not\subset G_\lambda$ for every $\lambda \in \Lambda$, and hence the set $M = \{x_\lambda \mid \lambda \in \Lambda\}$ obtained by taking a point x_λ from each set $V - G_\lambda$ would satisfy the condition that $|M| \leqslant \mathfrak{m}$, $M \subset V$, but $M \not\subset G_\lambda$ for any λ. Therefore X is $\mathfrak{m}$-paracompact.

Proof of Lemma 4.3. Assume that X is almost-$\mathfrak{m}$-fully normal. Let $\mathfrak{G}$ be any open covering of X. Then there exists an open covering $\mathfrak{H}$ of X with the property described in the definition of almost-$\mathfrak{m}$-full normality. Then the covering $\mathfrak{V} = \{\mathrm{St}(x, \mathfrak{H}) \mid x \in X\}$ is a normal open covering with the property stated in Lemma 4.3 (8). Conversely, assume that for any open covering $\mathfrak{G}$ of X there exists a normal open covering $\mathfrak{V}$ of X with the property stated in Lemma 4.3. Let $\mathfrak{W}$ be any open covering which is a Δ-refinement of $\mathfrak{V}$ in the sense of Tukey [18]. Then the covering $\mathfrak{H} = \{G \cap W \mid G \in \mathfrak{G},\ W \in \mathfrak{W}\}$ possesses the property stated in the definition of almost-$\mathfrak{m}$-full normality; indeed, $\mathfrak{H}$ is a refinement of $\mathfrak{G}$, and if $M \subset \mathrm{St}(x, \mathfrak{H})$, $|M| \leqslant \mathfrak{m}$, then $M \subset V$ for a set V of $\mathfrak{V}$ such that $\mathrm{St}(x, \mathfrak{H}) \subset V$ (such V certainly exists since $\mathfrak{H}$ is a Δ-refinement of $\mathfrak{V}$), and hence there is a $G \in \mathfrak{G}$ such that $M \subset G$. This completes the proof of Lemma 4.3.

It should be noted that some of the properties of almost-$\mathfrak{m}$-fully normal spaces obtained by Mansfield ([7], § 5) are essentially those of $\mathfrak{m}$-paracompact, almost-2-fully normal spaces.

(7) $\mathrm{St}(x, \mathfrak{H}) = \bigcup \{H \mid x \in H,\ H \in \mathfrak{H}\}$.

(8) Generally, $\{\mathrm{St}(x, \mathfrak{U}) \mid x \in X\}$ is a normal open covering for any open covering $\mathfrak{U}$ of an almost-2-fully normal space X.

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A fixed point theorem for the hyperspace of a snake-like continuum

by

J. Segal (Seattle, Wash.)

Introduction. If X is a metric continuum, $C(X)$ denotes the space of subcontinua of X with the finite topology. As a partial answer to question 186 (due to B. Knaster 4/29/52) of the New Scottish Book it is shown that $C(X)$ has fixed point property if X is a snake-like continuum. This is done by showing that $C(X)$ is a quasi-complex and since $C(X)$ is acyclic (see [9]) it has fixed point property by the Lefschetz Fixed Point Theorem.

DEFINITION 1. If G is a finite collection of open sets of X let $\Omega(G)$ denote $\{K \in C(X) \mid K \cap g \neq \emptyset$ for each $g \in G$ and $K \subset \bigcup_{g \in G}(g)\}$. The finite topology on $C(X)$ is the one generated by open sets of the form $\Omega(G.)$ (See [8], pp. 153.) If U is a finite open covering of X define U^* to be $\{\Omega(G) \mid G$ is a finite subset of $U\}$.

LEMMA 1. *If U is a finite open covering of X, then U^* is a finite open covering of $C(X)$.*

Proof. The elements of U^* are open by the definition of the finite topology, and since U is finite, so is U^*. If $A \in C(X)$, there is a subcollection G of U which irreducibly covers A, so $A \in \Omega(G)$. Hence U^* covers $C(X)$.

LEMMA 2. *If U is a finite collection of open sets, then* $\operatorname{mesh} U^* \leqslant \operatorname{mesh} U$.

Proof. Suppose that G is a subcollection of U and K and L are elements of $\Omega(G)$. If $x \in K$, there is an element g_x of G containing x. Given $L \cap g_x \neq \emptyset$ and $\operatorname{diam} g_x \leqslant \operatorname{mesh} U$, there is a point y of L such that $d(x, L) \leqslant \operatorname{mesh} U$. Hence for each x in K, $d(x, L) \leqslant \operatorname{mesh} U$. Therefore since $d'(K, L) = \max\big(\max_{x \in K} d(x, L), \max_{y \in L} d(y, K)\big)$, $d'(K, L) \leqslant \operatorname{mesh} U$, and hence $\operatorname{diam} G \leqslant \operatorname{mesh} U$.

LEMMA 3. *If $\{U_a\}$ is a cofinal sequence of open coverings of X, then $\{U_a^*\}$ is a cofinal sequence of open coverings of $C(X)$.*

Proof. A sequence $\{U_a\}$ of open coverings of a compact space X is cofinal (in the set of all open coverings of X) if and only if $\operatorname{mesh} U_a \to 0$. By Lemma 2 if $\operatorname{mesh} U_a \to 0$ then $\operatorname{mesh} U_a^* \to 0$.