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GENERAL SOLUTIONS OF SOME TYPES OF RICCATI EQUATIONS

1. Paper [1] devoted to the so-called generalized Riccati equations has proved to be of use for the solution of a number of Riccati equations in a relatively easy manner. A collection of these solutions is given below.

For scientific workers interested in applied mathematics a readily-obtained accurate solution has not lost its value even in the age of a rapid development of various computers making it easy to obtain numerical solutions with the required accuracy. It suffices to mention the popularity of E. Kamke's book [2], containing the most complete of the existing collections of solutions of differential equations, or the excellent collection of functions, integrals, series and integral transforms by [3].

Our solutions may also prove useful, because they concern non-linear equations whose applicability increases as a result of the tendency to employ more accurate schemes of physical phenomena involving the necessity of considering non-linear relations.

In our list we give equations which are not contained in E. Kamke's book mentioned above. We use the same principle of arranging the material, that is we first give simpler equations and then pass to more difficult and more general ones, in which the coefficients depend on an arbitrary function.

2. List of solutions

(1) $y' = ay^2+by+c$

Under the term of characteristic equation of (1) we shall understand the algebraic equation

$$r^2+br+ac = 0.$$

Let us denote its roots by r_1 and r_2. If $r_1 \neq r_2$, we have

$$y = \frac{1}{a}\cdot\frac{C_1 r_1 e^{-r_1 x}+C_2 r_2 e^{-r_2 x}}{C_1 e^{-r_1 x}+C_2 e^{-r_2 x}}$$

if $r_1 = r_2 = r$, we have

$$y = \frac{1}{a}\cdot\frac{C_1 r + C_2(1-rx)}{C_1 - C_2 x}.$$

If the characteristic equation has complex roots $r_1 = a+i\beta$ and $r_2 = a-i\beta$, then

$$y = \frac{1}{a}\cdot\frac{C_1(a\cos\beta x+\beta\sin\beta x)+C_2(a\sin\beta x-\beta\cos\beta x)}{C_1\cos\beta x+C_2\sin\beta x}.$$

(2) $\quad y' = y^2+a^2.$

This is a particular case of equation (1),

$$y = \frac{a(C_1\sin ax - C_2\cos ax)}{C_1\cos ax+C_2\sin ax}.$$

(3) $\quad y' = y^2-a^2.$

This is also a particular case of equation (1),

$$y = -\frac{a(C_1\operatorname{sh} ax+C_2\operatorname{ch} ax)}{C_1\operatorname{ch} ax+C_2\operatorname{sh} ax}.$$

(4) $\quad y' = y^2-a^2x^2-a;$

$$y = -\frac{C_1 ax+C_2(e^{-ax^2}+ax\int e^{-ax^2}dx)}{C_1+C_2\int e^{-ax^2}dx}.$$

(5) $\quad y' = y^2-[a^2e^{2x}+a(2b+1)e^x+b^2];$

$$y = -\frac{C_1(ae^x+b)+C_2[(ae^x+b)\int E^{-2}dx+E^{-2}]}{C_1+C_2\int E^{-2}dx},$$

where $E = \exp(ae^x+bx)$.

(6) $\quad y' = y^2-ab\sin 2x-\frac{a^2-b^2}{2}\cos 2x-(2bc-a)\sin x-(2ac+b)\cos x-$

$$-\frac{a^2+b^2}{2}-c^2;$$

$$y = -\frac{C_1A+C_2(A\int E^{-2}dx+E^{-2})}{C_1+C_2\int E^{-2}dx},$$

where $E = \exp(a\sin x-b\cos x+cx)$, $A = a\cos x+b\sin x+c$.

(7) $\quad y' = y^2-2\operatorname{tg}^2x-1;$

$$y = -\frac{C_1\sin x+C_2(\cos^3x+x\sin x+\cos x)}{[C_1+C_2(\sin x\cos x+x)]\cos x}.$$

(8) $\quad y' = y^2-f^2-f' \quad (f = f(x));$

$$y = -\frac{C_1f+C_2[f\int\exp(-2F)dx+\exp(-2F)]}{C_1+C_2\int\exp(-2F)dx},$$

where $F = \int f(x)\,dx$.

(9) $y' = y^2 - ay + be^x + c$;

$$y = -\frac{d}{dx}\ln\left|\exp\left(\frac{-ax}{2}\right)Z_\nu\left[2\sqrt{b}\exp\left(\frac{x}{2}\right)\right]\right|,$$

where $\nu = \sqrt{a^2 - 4c}$ and Z_ν is a cylindrical function.

(10) $y' = y^2 - xy - 1$;

$$y = -\frac{C_1 + C_2\int\exp(-x^2/2)\,dx}{C_1x + C_2[\exp(-x^2/2) + x\int\exp(-x^2/2)\,dx]}.$$

(11) $y' = y^2 - xy + n + 1, \quad n = 0, 1, 2, \ldots$;

$$y = -\frac{d}{dx}\ln\left|\frac{d^n}{dx^n}\left\{\exp\left(-\frac{x^2}{2}\right)\left[C_1 + C_2\int\exp\left(\frac{x^2}{2}\right)dx\right]\right\}\right|.$$

(12) $y' = y^2 - xy + \frac{1}{4}x^2 + a$.

If $a \neq \frac{1}{2}$ and with $\lambda = \sqrt{\frac{1}{2} - a}$, we have

$$y = \frac{C_1(\lambda + \frac{1}{2}x)e^{-\lambda x} - C_2(\lambda - \frac{1}{2}x)e^{\lambda x}}{C_2e^{\lambda x} + C_1e^{-\lambda x}}.$$

If $a = \frac{1}{2}$ we have

$$y = \frac{C_1x + C_2(x^2 - 2)}{2(C_1 + C_2x)}.$$

(13) $y' = y^2 + xy + 2$;

$$y = -\frac{2C_1x(x^2 - 1) + C_2[2x(x^2 - 1)I + \exp(x^2/2)]}{(x^2 - 1)^2(C_1 + C_2I)},$$

where $I = \int\frac{\exp(x^2/2)\,dx}{(x^2 - 1)^2}$.

(14) $y' = y^2 + xy + x - 1$;

$$y = -\frac{C_1 + C_2(\int E\,dx + E)}{C_1 + C_2\int E\,dx},$$

where $E = \exp(\frac{1}{2}x^2 - 2x)$.

(15) $y' = y^2 - 4xy + 4x^2 + 2$;

$$y = \frac{2C_1x + C_2(2x^2 - 1)}{C_1 + C_2x}.$$

(16) $y' = y^2 + 4xy + 4x^2 - 2$;

$$y = -\frac{2C_1x + C_2(1 + 2x^2)}{C_1 + C_2x}.$$

(17) $y' = y^2+4xy+4x^2-3;$

$$y = -\frac{C_1(2x+1)e^x+C_2(2x-1)e^{-x}}{C_1e^x+C_2e^{-x}}.$$

(18) $y' = y^2-2axy+a^2x^2.$

If $a>0$ and $\lambda = \sqrt{a}$, we have

$$y = \frac{C_1(\lambda^2x\operatorname{ch}\lambda x-\lambda\operatorname{sh}\lambda x)+C_2(\lambda^2x\operatorname{sh}\lambda x-\lambda\operatorname{ch}\lambda x)}{C_1\operatorname{ch}\lambda x+C_2\operatorname{sh}\lambda x}.$$

If $a<0$ and $\lambda = \sqrt{-a}$, we have

$$y = -\frac{C_1(\lambda^2x\cos\lambda x-\lambda\sin\lambda x)+C_2(\lambda^2x\sin\lambda x+\lambda\cos\lambda x)}{C_1\cos\lambda x+C_2\sin\lambda x}.$$

(19) $y' = y^2+2(ax+b)y+(ax+b)^2-a;$

$$y = -\frac{C_1(ax+b)+C_2[a+(ax+b)^2]}{C_1+C_2(ax+b)}.$$

(20) $y' = y^2+x^2y+x;$

$$y = \frac{C_2I-C_1}{C_1x+C_2[\exp(x^3/3)-xI]},$$

where $I = \int x\exp(x^3/3)\,dx$.

(21) $y' = y^2+x^2y-(x+1)^2;$

$$y = -\frac{C_1(x^2+1)+C_2[(x^2+1)I+\exp(-\frac{1}{3}x^3-2x)}{C_1+C_2I},$$

where $I = \int\exp(-\frac{1}{3}x^3-2x)\,dx$.

(22) $y' = y^2+x^2(x+1)y+x(x^4-2);$

$$y = -\frac{C_1x^2+C_2[x^2I+\exp(\frac{1}{4}x^4-\frac{1}{3}x^3)]}{C_1+C_2I},$$

where $I = \int\exp(\frac{1}{4}x^4-\frac{1}{3}x^3)\,dx$.

(23) $y' = y^2-x^4y-x^3;$

$$y = -\frac{C_1+C_2[I+x^{-1}\exp(-x^5/5)]}{C_1x+C_2xI},$$

where $I = \int x^{-2}\exp(-x^5/5)\,dx$.

(24) $y' = y^2 - x^{1/2}y + \frac{x^{-1/2}}{4} + \frac{x}{4} - 9;$

$$y = \frac{C_1\left(\frac{x^{1/2}}{2} - 3\right)e^{3x} + C_2\left(\frac{x^{1/2}}{2} + 3\right)e^{-3x}}{C_1e^{3x} + C_2e^{-3x}}.$$

(25) $y' = y^2 + x^{-1/2}y - \frac{x^{-1}}{4} - \frac{x^{-3/2}}{4} - 2x^{-2};$

$$y = -\frac{C_1(x^{7/2} + 4x^3) + C_2(x^{1/2} - 2)}{2(C_1x^4 + C_2x)}.$$

(26) $y' = y^2 - (ae^x + b)y + (Ae^{2x} + Be^x + C).$

In the particular case where $A = -\alpha(a+\alpha)$, $B = -(a\beta + b\alpha + 2\alpha\beta + \alpha)$, $C = -\beta(b+\beta)$, we have

$$y = -\frac{C_1(\alpha e^x + \beta) + C_2[(\alpha e^x + \beta)\int E\,dx + E]}{C_1 + C_2\int E\,dx},$$

where $E = \exp[-(a+2\alpha)e^x - (b+2\beta)x]$.

(27) $y' = y^2 - 2y\,\mathrm{th}\,x + b.$

If $b-1 = a^2$, we have

$$y = \frac{C_1(a\sin ax\,\mathrm{ch}\,x + \cos ax\,\mathrm{sh}\,x) - C_2(a\cos ax\,\mathrm{ch}\,x - \sin ax\,\mathrm{sh}\,x)}{(C_1\cos ax + C_2\sin ax)\,\mathrm{ch}\,x};$$

if $b-1 = -a^2$, we have

$$y = \frac{C_1(\mathrm{ch}\,ax\,\mathrm{sh}\,x - a\,\mathrm{sh}\,ax\,\mathrm{ch}\,x) + C_2(\mathrm{sh}\,x\,\mathrm{sh}\,ax - a\,\mathrm{ch}\,ax\,\mathrm{ch}\,x)}{(C_1\,\mathrm{ch}\,ax + C_2\,\mathrm{sh}\,ax)\,\mathrm{ch}\,x}.$$

(28) $y' = y^2 - 2ny\,\mathrm{cth}\,x + n^2 - a^2;$

$$y = -\frac{d}{dx}\ln\left|\left(\frac{1}{\mathrm{sh}\,x}\cdot\frac{d}{dx}\right)^n(C_1e^{ax} + C_2e^{-ax})\right|.$$

(29) $y' = y^2 - y\,\mathrm{tg}\,x + \cos^2 x;$

$$y = \frac{[C_1\sin(\sin x) - C_2\cos(\sin x)]\cos x}{C_1\cos(\sin x) + C_2\sin(\sin x)}.$$

(30) $y' = y^2 - y\,\mathrm{tg}\,x - \cos^2 x;$

$$y = \frac{[C_1\exp(-\sin x) - C_2\exp(\sin x)]\cos x}{C_2\exp(\sin x) + C_1\exp(-\sin x)}.$$

(31) $y' = y^2 + y\operatorname{ctg}x + \sin^2 x;$

$$y = \frac{[C_1\cos(\cos x) - C_2\sin(\cos x)]\sin x}{C_2\cos(\cos x) + C_1\sin(\cos x)}.$$

(32) $y' = y^2 + 2y\operatorname{tg}x + b.$

If $b+1 = a^2$, we have

$$y = \frac{C_1(a\sin ax\cos x - \cos ax\sin x) - C_2(a\cos ax\cos x + \sin ax\sin x)}{(C_1\cos ax + C_2\sin ax)\cos x};$$

if $b+1 = -a^2$, we have

$$y = -\frac{C_1(a\operatorname{sh}ax\cos x + \operatorname{ch}ax\sin x) + C_2(a\operatorname{ch}ax\cos x + \operatorname{sh}ax\sin x)}{(C_1\cos ax + C_2\sin ax)\cos x}.$$

(33) $y' = y^2 - 2ay\operatorname{ctg}ax + b^2 - a^2 \quad (a \neq 0,\ b \neq 0);$

$$y = \frac{C_1(b\sin bx\sin ax + a\cos bx\cos ax) - C_2(b\cos bx\sin ax - a\sin bx\cos ax)}{(C_1\cos bx + C_2\sin bx)\sin ax}.$$

(34) $y' = y^2 - ay\operatorname{ctg}2x + 1 - a^2/4;$

$$y = \frac{-C_1\nu\sin^{\nu-1}x\cos x + C_2\nu\cos^{\nu-1}x\sin x}{C_1\sin^\nu x + C_2\cos^\nu x},$$

where $\nu = 1 - a/2$.

(35) $y' = y^2 - fy - a(a+1)f^2 - af' \quad (f = f(x));$

$$y = -\frac{C_1 af + C_2[af\int E\,dx + E]}{C_1 + C_2\int E\,dx},$$

where $E = \exp[-(2a+1)F]$, $F = \int f(x)\,dx$.

(36) $y' = y^2 - fy + f' \quad (f = f(x)).$

This is a particular case of equation (35), where $a = -1$.

$$y = \frac{C_1 f + C_2(f\int e^F dx - e^F)}{C_1 + C_2\int e^F dx},$$

where $F = \int f(x)\,dx$.

(37) $y' = y^2 - (af+b)y + cf + d \quad (f = f(x)).$

If $a^2d - abc + c^2 = 0$ and $a \neq 0$, we have

$$y = \frac{C_1 c + C_2(c\int E\,dx - aE)}{(C_1 + C_2\int E\,dx)a},$$

where $E = \exp\left[\left(\frac{2c}{a} - b\right)x - a\int f\,dx\right]$.

(38) $y' = y^2 + \left(\frac{f'}{f} + 2a\right)y + a\frac{f'}{f} + a^2 - b^2f^2$;

$$y = -\frac{C_1(a+bf)E + C_2E^{-1}(a-bf)}{C_1E + C_2E^{-1}},$$

where $E = \exp(b\int f\,dx)$.

(39) $4y' = 4y^2 - 4y\,\mathrm{tg}\,x - 5\,\mathrm{tg}^2x - 2$;

$$y = -\frac{C_1\,\mathrm{tg}\,x + C_2[\mathrm{tg}\,x(x+\sin x\cos x) + 4\cos^2 x]}{2C_1 + 2C_2(x+\sin x\cos x)}.$$

(40) $a^2y' = a^2y^2 - a(a^2 - 2be^{-ax})y + b^2e^{-2ax}$;

$$y = \frac{C_1ab + C_2b(be^{ax} - a^2)}{C_1a^2e^{-ax} + C_2ab}.$$

(41) $y' = fy^2 + gy \quad (f = f(x),\ g = g(x))$;

$$y = \frac{C_2e^G}{C_1 - C_2\int fe^G dx},$$

where $G = \int g(x)\,dx$.

(42) $xy' = xy^2 - y$.

This is a particular case of equation (41).

$$y = \frac{C_1}{C_1x - C_2x\ln|x|}.$$

(43) $xy' = xy^2 + y - ax^3$.

If $a = 4\alpha^2$, we have

$$y = -\frac{2\alpha x(C_1\,\mathrm{sh}\,\alpha x^2 + C_2\,\mathrm{ch}\,\alpha x^2)}{C_1\,\mathrm{ch}\,\alpha x^2 + C_2\,\mathrm{sh}\,\alpha x^2};$$

if $a = -4\alpha^2$, we have

$$y = \frac{2\alpha x(C_1\sin\alpha x^2 - C_2\cos\alpha x^2)}{C_1\cos\alpha x^2 + C_2\sin\alpha x^2}.$$

(44) $xy' = xy^2 - 2y - x$,

$$y = \frac{C_1(1-x)e^x + C_2(1+x)e^{-x}}{(C_1e^x + C_2e^{-x})x}.$$

(45) $y' = y^2 - a\frac{f'}{f}y - b\frac{f''}{f} \quad (f = f(x))$.

In the case, where $a+b = 1$

$$y = -\frac{C_1 bf' + C_2(f^{-b} + bf' I)}{C_1 f + C_2 f I},$$

where $I = \int f^{-(b+1)}(x)\,dx$.

(46) $\quad y' = y^2 - ay\operatorname{ctg}x + b$.

This is an example of equation (45). If, therefore, $a+b = 1$, we have

$$y = -\frac{C_1 b\cos x + C_2(\sin^{-b}x + b\cos x I)}{C_1 \sin x + C_2 \sin x I},$$

where $I = \int \sin^{-(b+1)} x\,dx$.

(47) $\quad y' = y^2 - (ae^{-bF} + bf + b^{-1})y + f \quad (b \neq 0)$,

$f = f(x)$ and $F = \int f(x)\,dx$,

$$y = \frac{C_1(ae^{-bF} + b^{-1}) + C_2[(ae^{-bF} + b^{-1})\int E\,dx - E]}{C_1 + C_2\int E\,dx},$$

where $E = \exp(a\int e^{-bF}dx + b^{-1}x - bF)$.

(48) $\quad y' = y^2 - (f+1)y + f \quad (f = f(x))$.

This is a particular case of equation (47), where $a = 0$, $b = 1$;

$$y = \frac{C_1 + C_2(\int E\,dx - E)}{C_1 + C_2\int E\,dx},$$

where $E = \exp\big(x - \int f(x)\,dx\big)$.

(49) $\quad y' = y^2 - y\cos^2 x - \sin^2 x$.

This is a particular case of equation (48), where $f = -\sin^2 x$;

$$y = \frac{C_1 + C_2(\int E\,dx - E)}{C_1 + C_2\int E\,dx},$$

where $E = \exp\left(\dfrac{3x}{2} - \dfrac{\sin 2x}{4}\right)$.

(50) $\quad y' = y^2 + \left(\dfrac{f''}{f'} - \dfrac{1}{a}\right)y - \dfrac{1}{a}\left(\dfrac{f''}{f'} - \dfrac{f'}{f+b}\right)$;

where $f = f(x)$;

$$y = \frac{1}{a}\cdot\frac{C_1(f - af' + b) + C_2[(f - af' + b)I - a^2 f' e^{x/a}(f+b)^{-1}]}{(C_1 + C_2 I)(f+b)},$$

where $I = \int f' e^{x/a}(f+b)^{-2}dx$.

(51) $$y' = y^2+\left(\frac{f''}{f'}-1\right)y-\frac{f''}{f'}+\frac{f'}{f}.$$

This is a particular case of equation (50): $b = 0$ and $a = 1$.

$$y = \frac{C_1 f(f-f')+C_2[f(f-f')I-f'e^x]}{(C_1+C_2)f^2},$$

where $I = \int f' e^x f^{-2}dx$.

(52) $$y' = y^2+\left(\frac{f''}{f'}-\frac{1}{a}\right)y-\frac{1}{a}\cdot\frac{bf'}{bf+c};$$

where $f = f(x)$ and $a \neq 0$;

$$y = -\frac{C_1 bf' + C_2[bf'I+e^{-x/a}(bf+c)^{-1}]}{(C_1+C_2 I)(bf+c)},$$

where $I = \int e^{-x/a}(bf+c)^{-2}dx$.

(53) $$y' = y^2+\left(\frac{f''}{f'}-1\right)y-\frac{f'}{f}.$$

This is a particular case of equation (52): $c = 0$ and $a = 1$.

$$y = -\frac{C_1 ff' + C_2(f'fI+e^{-x})}{(C_1+C_2 I)f^2},$$

where $I = \int e^{-x}f^{-2}dx$.

(54) $$y' = y^2-(\operatorname{tg}x+1)y-\operatorname{ctg}x.$$

Example of equation (53) ($f = \sin x$).

$$y = -\frac{C_1\cos x+C_2(I\cos x-e^{-x}\sin^{-1}x)}{(C_1+C_2 I)\sin x},$$

where $I = \int e^{-x}\sin^{-2}dx$.

(55) $$y' = y^2-\left(af-\frac{f'}{f}+\frac{1}{a}\right)y+f \quad (f = f(x));$$

$$y = \frac{C_1 af+C_2\left[afI-f\exp\left(aF-\frac{x}{a}\right)\right]}{C_1+C_2 I},$$

where $I = \int f \exp\left(aF - \frac{x}{a}\right) dx$, $F = \int f(x) dx$.

(56) $y' = y^2 + fy + a(f-a) \quad (f = f(x))$;

$$y = -\frac{C_1 a + C_2[aI + \exp(F - 2ax)]}{C_1 + C_2 I},$$

where $I = \int \exp(F - 2ax) dx$ and $F = \int f dx$.

Solution of the Riccati equation

(57) $y' = fy^2 + gy + h \quad (f = f(x),\ g = g(x),\ h = h(x))$

is reduced to the solution of the following linear equation of the second order

(a) $$fu'' - (fg + f')u' + hf^2 u = 0,$$

and, if the equation (a) has a general solution $u = C_1 u_1 + C_2 u_2$, the corresponding solution of the Riccati equation (57) has the solution

(b) $$y = -\frac{1}{f} \cdot \frac{C_1 u_1' + C_2 u_2'}{C_1 u_1 + C_2 u_2}.$$

References

[1] T. Iwiński, *On the n-th order Riccati equation of the second kind*, Colloq. Math. 1 (1963), pp. 173-182.

[2] E. Kamke, *Differentialgleichungen, Lösungsmethoden und Lösungen*, Leipzig 1959.

[3] I. M. Ryżyk i I. S. Gradsztejn, *Tablice całek, sum, szeregów i iloczynów* (*Tables of Integrals, Sums, Series and Products*), Warszawa 1964.

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CAŁKI OGÓLNE PEWNYCH SZCZEGÓLNYCH PRZYPADKÓW RÓWNANIA RICCATI'EGO

STRESZCZENIE

Opierając się na wynikach wcześniejszej swej pracy [1] autor podał nie publikowane dotychczas rozwiązania 57 równań różniczkowych typu Riccati'ego, w tym pewną liczbę rozwiązań typu dość ogólnego o współczynnikach zależnych od jednej funkcji dowolnej.

Т. ИВИНЬСКИ (Варшава)

ОБЩИЕ ИНТЕГРАЛЫ НЕКОТОРЫХ ЧАСТНЫХ СЛУЧАЕВ УРАВНЕНИЯ РИККАТИ

РЕЗЮМЕ

Основываясь на результатах одной из своих предыдущих работ [1], автор приводит не опубликованные до сих пор решения 57 дифференциальных уравнений типа Риккати, в том числе некоторое количество решений уравнений довольного общего типа, косающихся коэффициентов, зависящих от одной произвольной функции.