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ON THE LOCAL OPTIMALITY OF SOME REGULAR SAMPLING PATTERNS IN THE PLANE

Introduction. This note is an extension of the discussion of optimality properties of regular sampling patterns in the plane, as presented by Dalenius, Hájek and Zubrzycki in [1]. The problem considered is the estimation of the expected value of an isotropic stationary stochastic process in the plane. Since in practice the correlation function is not exactly known, it seems interesting to find a sampling pattern which would prove the optimum one for a possibly large class of correlation functions. Unfortunately the promising conjecture that among all regular nets with a given density the net of regular triangles is the optimum one in the sense of yielding the least limiting variance of the mean of observations for all isotropic stationary processes with a convex correlation function has been disproved in [1]. It has been shown there for a fairly special correlation function with discontinuities of its second derivative that the optimum regular net may depend strongly on the density of the net. The problem whether the net of regular triangles is still the optimum one for isotropic stationary processes with the correlation function as regular as the exponential correlation function has been left open. In this note we prove that for the class of isotropic stationary processes with an exponential correlation function the net of regular triangles is locally the optimum one with respect to infinitesimal affine deformations preserving the density of the net if the density of the net is small enough. To be more specific, we consider two such affine deformations, called extending and shearing, and also their compositions, we represent the limiting variance as a function of the parameters of the deformation, and we investigate the minimum property of the resulting function of the parameters in the sense of the differential calculus.

Besides the net of regular triangles, we investigate in this manner also the net of squares. We show that the net of squares may usually be improved by shearing though it is sometimes the optimum one with respect to extending, which transforms squares into rectangles. This completes in a sense the examples given by Matérn ([2], p. 83), who compared numerically the limiting variances for some nets of regular triangles, of squares and of rectangles in the case of an isotropic stationary stochastic process with an exponential correlation function to the effect that triangles turned out to be better than squares, and squares better than rectangles. There is some empirical evidence that the nets used in geological investigations are to be qualified as nets of small density. E.g. Zubrzycki ([3], p. 130), has shown that the exponential correlation function $e^{-\lambda d}$ with relatively large values of the parameter λ is the best description of some zinc deposits. Therefore the nets of regular triangles may be preferable to other nets.

Definitions and results. The family of random variables $\eta(p)$ assigned to the points p of a plane will be called a *plane stationary isotropic stochastic process* if the random variables $\eta(p)$ have a common expected value μ and a common variance σ^2, and if the correlation coefficient between the random variable $\eta(p)$ and $\eta(q)$ depends only upon the distances $|p-q|$ of the points p and q. In symbols:

$$R[\eta(p), \eta(q)] = f(|p-q|). \tag{1}$$

The function $f(d)$, $0 \leqslant d < \infty$, will be called the *correlation function* of the process $\eta(p)$. The process $\eta(p)$ is called *continuous* if

$$\lim_{d\to 0} f(d) = 1. \tag{2}$$

In the sequel we shall consider only the plane stationary isotropic and continuous stochastic process $\eta(p)$. Every countable set N of points in the plane which has no concentration points will be called a *net*. A net will be called *regular* if for every two of its points p' and p'' there exists an isometric transformation of the plane which transforms p' into p'' and N onto N. If the limit

$$g(N) = \lim_{R\to\infty} \frac{1}{\pi R^2} \operatorname{Card}\{N \cap K(0, R)\}, \tag{3}$$

exists, we call $g(N)$ the *density* of the net; here $K(0, R)$ denotes a circle with centre 0 and radius R. As is easily seen, the limit, if it exists, does not depend on the choice of the centre 0. The applicability of the given regular net N to the estimation of the expected value μ of the process $\eta(p)$ may be characterized by means of the limiting variance $s_\infty^2 = s_\infty^2(N)$ defined by the relation

$$s_\infty^2(N) = \lim_{R\to\infty} n_R D^2\bar{\eta}_R, \tag{4}$$

where n_R is the number of the points of the net contained in the circle $K(0, R)$ with centre 0 and radius R, and

$$(5)\qquad \bar{\eta}_R = \frac{1}{n_R}[\eta(p_{1,R})+\ldots+\eta(p_{n_R,R})],$$

where $p_{1,R}, \ldots, p_{n_R,R}$ are all the points of the nets N contained in the circle $K(0, R)$. Obviously the value s^2_∞ does not depend on the choice of centre 0.

Consider now three affine transformations of the plane which preserve the area and thus do not change the density of a regular net of points. The first of them is the extension of r_δ transforming the point (x, y) into $(\delta x, y/\delta)$; in symbols:

$$(6)\qquad r_\delta:\quad (x, y) \to (\delta x, y/\delta).$$

Further two transformations are: shearing s'_ε which transforms the point (x, y) into $(x, y+\varepsilon x)$; in symbols:

$$(7)\qquad s'_\varepsilon:\quad (x, y) \to (x, y+\varepsilon x),$$

and transformation s''_ε which transforms the point (x, y) into

$$\left(\frac{x+\varepsilon y}{\sqrt{1-\varepsilon^2}}, \frac{y+\varepsilon x}{\sqrt{1-\varepsilon^2}}\right);$$

in symbols:

$$(8)\qquad s''_\varepsilon:\quad (x, y) \to \left(\frac{x+\varepsilon y}{\sqrt{1-\varepsilon^2}}, \frac{y+\varepsilon x}{\sqrt{1-\varepsilon^2}}\right).$$

This transformation is a composition of extension shearing and rotation of the plane. This composition has a simple geometrical interpretation: it transforms squares into rhombes composed of two regular triangles.

It is clear that if we suitably select the values of the parameters ε and δ in the composition of the deformations r_δ and s'_ε defined by the formula

$$(9)\qquad r_\delta s'_\varepsilon:\quad (x, y) \to \big(\delta x, (y+\varepsilon x)/\delta\big),$$

we can obtain any affine transformation which preserves the area, does not change the orientation of the axes and transforms the y axis into itself.

Denote by $N'_{\varepsilon,\delta}$ the image of the net N by the transformation $r_\delta s'_\varepsilon$. For a given process $\eta(p)$ and a given net N the limiting variance $s^2_\infty(N'_{\varepsilon,\delta})$ of the net $N'_{\varepsilon,\delta}$ becomes a function of two variables ε and δ:

$$(10)\qquad s^2_\infty(N'_{\varepsilon,\delta}) = n(\varepsilon, \delta).$$

The net is called the optimum one *locally* if the function $n(\varepsilon, \delta)$ has a local minimum at the point $(\varepsilon, \delta) = (0, 1)$. The net N is called the optimum one *locally with respect to extension* if the function $n(\varepsilon, \delta)$ with $\varepsilon = 0$ has a local minimum with respect to δ at the point $\delta = 1$. The net N is called the optimum one *locally with respect to shearing* if the function $n(\varepsilon, \delta)$ with $\delta = 1$ has a local minimum with respect to ε at the point $\varepsilon = 0$.

We shall consider two regular nets with density $g = 1$, namely the net of squares composed of all points with integer coordinates:

$$S = \{(i, j) \colon i, j \text{ integer}\}, \tag{11}$$

and the net of regular triangles defined by

$$T = \left\{\left(\frac{i}{\sqrt[4]{12}}, j\sqrt[4]{3/4}\right) \colon i, j \text{ integer}, \ i+j = 0 (\mathrm{mod}\, 2)\right\}, \tag{12}$$

(see fig. 1).

These are our results:

THEOREM 1. *If $\eta(p)$ is an isotropic stationary stochastic process with an exponential correlation function*

(13)

$$f(d) = e^{-\lambda d}, \quad 0 \leqslant d < \infty, \ \lambda > 0,$$

then

(a) *if* $\lambda \geqslant \dfrac{3\sqrt[4]{12}}{2}$, *then the net* (12) *of regular triangles is the optimum one locally*

(b) *if the λ is large enough, then the net* (11) *of squares is not the optimum one, though,*

(c) *if $\lambda \geqslant 2$, then the net* (11) *of squares is the optimum one locally with respect to extension.*

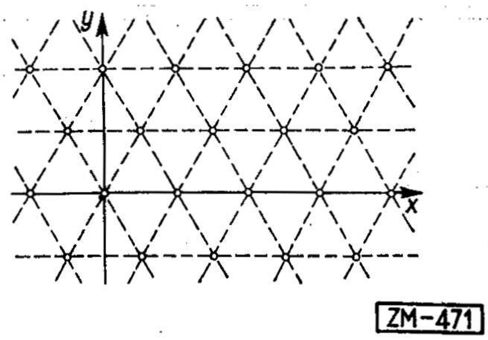

Fig. 1

Case (a) is a partial answer to the problem 4.2 presented in [1]. Cases (b) and (c) supplement the numerical examples given by Matérn ([2], p. 83).

We will base our proof on some lemmas.

LEMMA 1. *If the net N is regular, then*

$$s_\infty^2(N) = \sigma^2 \sum_{p \in N} f(|p_0 - p|), \tag{14}$$

where p_0 is a fixed point of the net N, provided the series on the right-hand side of (14) *is absolutely convergent.*

Proof. We may assume without loss of generality that $p_0 = (0, 0) \in N$. It is easily seen that

$$n_R D^2 \bar{\eta}_R = \frac{1}{n_R} \sigma^2 \sum_{\substack{p \in N \cap K(O,R) \\ q \in N \cap K(O,R)}} f(|p-q|).$$

Therefore in order to prove (14) it is sufficient to show that

$$\lim_{R\to\infty} \frac{1}{n_R} \sum_{\substack{p \in N \cap K(O,R) \\ q \in N \cap K(O,R)}} f(|p-q|) = \sum_{p \in N} f(|p-p_0|). \tag{15}$$

Since of the assumed absolute convergence of the series on the right-hand side of (14), for every $\varepsilon > 0$ there is an $R_\varepsilon > 0$ such that

$$\sum_{p \in N - K(O,R_\varepsilon)} |f(|p-p_0|)| < \varepsilon. \tag{16}$$

Let us write

$$N_R = N \cap K(0, R), \quad u = \sum_{p \in N} f(|p-p_0|), \quad v = \sum_{p \in N} |f(|p-p_0|)|.$$

If $R > R_\varepsilon$, we can write

$$\frac{1}{n_R} \sum_{p \in N_R} \sum_{q \in N_R} f(|p-q|)$$

$$= \frac{1}{n_R} \Big[\sum_{p \in N_{R-R_\varepsilon}} \sum_{q \in N_R} f(|p-q|) + \sum_{p \in N_R \cap N_{R-R_\varepsilon}} \sum_{q \in N_R} f(|p-q|) \Big]$$

$$= \frac{n_{R-R_\varepsilon}}{n_R} \cdot \frac{1}{n_{R-R_\varepsilon}} \sum_{p \in N_{R-R_\varepsilon}} \sum_{q \in N_R} f(|p-q|) + \frac{1}{n_R} \sum_{p \in N_R - N_{R-R_\varepsilon}} \sum_{q \in N_R} f(|p-q|)$$

$$= \frac{n_{R-R_\varepsilon}}{n_R} U + V.$$

Now it is easily seen that for any regular net N

$$\lim_{R\to\infty} \frac{n_{R-R_\varepsilon}}{n_R} = 1, \quad \lim_{R\to\infty} \frac{n_R - n_{R-R_\varepsilon}}{n_R} = 0,$$

and also

$$|V| \leqslant \frac{1}{n_R} \sum_{p \in N_R - N_{R-R_\varepsilon}} \sum_{q \in N} |f(|p-q|)| = \frac{n_R - n_{R-R_\varepsilon}}{n_R} v.$$

Hence, by (16), we have

$$u-\varepsilon \leqslant U \leqslant u+\varepsilon,$$

which proves Lemma 1.

LEMMA 2. *For each function $h(x)$, if the series*

$$\sum_{(i,j)\in W} (i^2-3j^2)h(i^2+3j^2), \tag{17}$$

where

$$W = \{(i,j)\colon\ i,j \text{ integer},\ i+j = 0 (\operatorname{mod} 2)\}, \tag{18}$$

is absolutely convergent, then its sum equals 0.

Proof. Since the function $(i^2-3j^2)h(i^2+3j^2)$ is even with regard to i and j, and for $i = j = 0$ we have $(i^2-3j^2)h(i^2+3j^2) = 0$, it is sufficient to prove that

$$\sum_{(i,j)\in W_0} a_{i,j}(i^2-3j^2)h(i^2+3j^2) = 0, \tag{19}$$

where

$$a_{i,j} = \begin{cases} 2 & \text{if} \quad i.j = 0, \\ 4 & \text{if} \quad i.j \neq 0, \end{cases} \tag{20}$$

and

$$W_0 = \{(i,j)\colon\ i,j \text{ integer},\ i+j = 0 (\operatorname{mod} 2),\ i \geqslant 0,\ j \geqslant 0,\ i+j > 0\}.$$

Let us consider the division of the set W_0 into seven disjoint subsets

$$\begin{aligned}
A_1 &= \{(i,j)\colon\ (i\ j)\in W_0,\ j = 0,\ i > 0\}, \\
B_1 &= \{(i,j)\colon\ (i,j)\in W_0,\ 0 < j < \tfrac{1}{3}i\}, \\
A_2 &= \{(i,j)\colon\ (i,j)\in W_0,\ 0 < i = 3j\}, \\
B_2 &= \{(i,j)\colon\ (i,j)\in W_0,\ 0 < \tfrac{1}{3}i < j < i\}, \\
A_3 &= \{(i,j)\colon\ (i,j)\in W_0,\ 0 < i = j\}, \\
B_3 &= \{(i,j)\colon\ (i,j)\in W_0,\ 0 < i < j\}, \\
A_4 &= \{(i,j)\colon\ (i,j)\in W_0,\ 0 = i < j\},
\end{aligned}$$

(see fig. 2) and the one-to-one transformations

$$a_1\colon (i,j) \to \left(\frac{i+3j}{2}, \frac{i-j}{2}\right), \qquad a_2\colon (i,j) \to \left(\frac{i-3j}{2}, \frac{i+j}{2}\right), \tag{21}$$

Now it is easy to prove that

$$a_1(A_1) = A_3, \qquad a_1(B_1) = B_2;$$
$$a_2(A_2) = A_4, \qquad a_2(B_2) = B_3.$$

Let us consider the ellipses

$$i^2+3j^2 = \text{const}. \tag{22}$$

For $p = (i,j)$ we write $a_{i,j}(i^2-3j^2)h(i^2+3j^2) = h(p)$.
It is easy to prove that

(i) if ellipse (22) has a point $p = (i,j)\in A_1$ on it, it has also the point $a_1(p)\in A_3$ on it and we have $h(p)+h(a_1(p)) = 0$,

(ii) if ellipse (22) has a point $p\in B_1$ on it, then it has also the points $a_1(p)\in B_2$ and $a_2(p)\in B_3$ on it and we have $h(p)+h(a_1(p))+h(a_2(p)) = 0$,

(iii) if ellipse (22) has a point $p\in A_2$ on it, then it has also the point $a_2(p)\in A_4$ on it and we have $h(p)+h(a_2(p)) = 0$.

From (i), (ii), and (iii) follows (19), whence also Lemma 2.

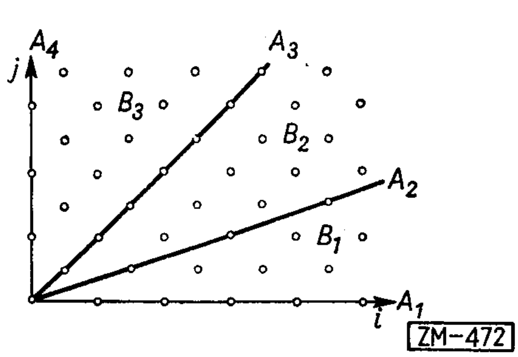

Fig. 2

Proof of Theorem 1. In view of Lemma 1 for the exponential correlation function (13) and the net of regular triangles given in (12) the function $t(\varepsilon, \delta)$ is given by

$$t(\varepsilon, \delta) = \sigma^2 \sum_{(x,y)\in T} \exp\left(-\lambda\sqrt{\delta^2x^2+(y+\varepsilon x)^2/\delta^2}\right)$$

$$= \sigma^2 \sum_{(i,j)\in W} \exp\left(-\lambda\sqrt{\delta^2 i^2/\sqrt{12}+(j\sqrt[4]{3/4}+\varepsilon i\sqrt[4]{1/12})^2/\delta^2}\right).$$

Differentiating under the sum sign we find

$$\frac{\partial}{\partial\varepsilon} t(\varepsilon, \delta)\Big|_{\substack{\varepsilon=0\\ \delta=1}} = -\sqrt{3}\,\sigma^2\lambda' \sum_{(i,j)\in W} ij/\sqrt{i^2+3j^2}\exp(\lambda'\sqrt{i^2+3j^2}), \tag{23}$$

$$\frac{\partial}{\partial\delta} t(\varepsilon, \delta)\Big|_{\substack{\varepsilon=0\\ \delta=1}} = -\sigma^2\lambda' \sum_{(i,j)\in W} (i^2-3j^2)/\sqrt{i^2+3j^2}\exp(\lambda'\sqrt{i^2+3j^2}), \tag{24}$$

where $\lambda' = \lambda/\sqrt[4]{12}$.

A necessary condition of the existence of an extremum at point $(\varepsilon, \delta) = (0,1)$ is fulfilled because these derivatives vanish for any λ. In the case of (23) this follows from the fact that the sum terms calculated for (i,j) and for $(-i,j)$ reduce, and in the case of (24) this follows from Lemma 2 if we assume $h(x) = -\sigma^2\lambda'/\sqrt{x}\exp(\lambda'\sqrt{x})$.

Let us calculated the second partial derivatives:

$$\frac{\partial^2}{\partial\varepsilon^2}t(\varepsilon,\delta)\Big|_{\substack{\varepsilon=0\\ \delta=1}} = \sigma^2\lambda' \sum_{(i,j)\in W} \frac{(3\lambda' j^2 - i^2/\sqrt{i^2+3j^2})\,i^2}{(i^2+3j^2)\exp(\lambda'\sqrt{i^2+3j^2})}, \tag{25}$$

$$\frac{\partial^2}{\partial\varepsilon\,\partial\delta}t(\varepsilon,\delta)\Big|_{\substack{\varepsilon=0\\ \delta=1}} = \sqrt{3}\sigma^2\lambda' \sum_{(i,j)\in W} \frac{ij\,[\lambda'(i^2-3j^2)+3(i^2+j^2)/\sqrt{i^2+3j^2}]}{(i^2+3j^2)\exp(\lambda'\sqrt{i^2+3j^2})}, \tag{26}$$

$$\frac{\partial^2}{\partial\delta^2}t(\varepsilon,\delta)\Big|_{\substack{\varepsilon=0\\ \delta=1}} = \sigma^2\lambda' \sum_{(i,j)\in W} \frac{[\lambda'(i^2-3j^2)^2-18j^2(i^2+j^2)/\sqrt{i^2+3j^2}]}{(i^2+3j^2)\exp(\lambda'\sqrt{i^2+3j^2})}. \tag{27}$$

The mixed partial derivative (26) vanishes for any λ for the same reason as does the derivative (23). Let us write, for $p=(i,j)$,

$$(3\lambda' j^2 - i^2/\sqrt{i^2+3j^2})/(i^2+3j^2)\exp(\lambda'\sqrt{i^2+3j^2}) = g(p). \tag{28}$$

Since the terms of series (25) are even, the sum equals

$$\sigma^2\lambda' \sum_{p=(i,j)\in W,\, i\geqslant 0,\, j\geqslant 0} a(p)g(p), \tag{29}$$

where

$$a(p) = a_{i,j} = \begin{cases} 2 & \text{if} \quad i\cdot j = 0, \\ 4 & \text{if} \quad i\cdot j > 0. \end{cases}$$

If we make use of the transformations α_1 and α_2 given by (21), then we can give to series (29) the form

$$\begin{aligned} \sigma^2\lambda'\Big[& \sum_{p=(i,j)\in W,\, j=0,\, i>0} a(p)g(p) + a(\alpha_1(p))g(\alpha_1(p)) + \\ + & \sum_{p=(i,j)\in W,\, i>0,\, i>0} a(p)g(p) + a(\alpha_1(p))g(\alpha_1(p)) + a(\alpha_2(p))g(\alpha_2(p)) + \\ + & \sum_{p=(i,j)\in W,\, i=3j>0} a(p)g(p) + a(\alpha_2(p))g(\alpha_2(p))\Big]. \end{aligned} \tag{30}$$

By expanding this series according to (28) and reducing similar terms we get

$$\frac{\partial^2}{\partial\varepsilon^2}t(\varepsilon,\delta)\Big|_{\substack{\varepsilon=0\\ \delta=1}} = \sigma^2\lambda' \sum_{(i,j)\in W,\, 0\leqslant 3j\leqslant i} \frac{b_{ij}(\lambda' - 3/\sqrt{i^2+3j^2})(i^2+3j^2)}{\exp(\lambda'\sqrt{i^2+3j^2})}, \tag{31}$$

where

$$b_{ij} = \begin{cases} \frac{3}{4} & \text{if} \quad j = 0 \text{ or } i = 3j, \\ \frac{3}{2} & \text{if} \quad 0 < 3j < i. \end{cases}$$

If we transform (27) similarly, we get

$$\frac{\partial^2}{\partial\delta^2} t(\varepsilon, \delta)\Big|_{\substack{\varepsilon=0\\ \delta=1}} = 4\sigma^2\lambda' \sum_{(i,j)\in W, 0\leqslant 3j\leqslant i} \frac{b_{ij}(\lambda' - 3/\sqrt{i^2+3j^2})(i^2+3j^2)}{\exp(\lambda'\sqrt{i^2+3j^2})} = 4\frac{\partial^2}{\partial\varepsilon^2} t(\varepsilon, \delta)\Big|_{\substack{\varepsilon=0\\ \delta=1}}. \tag{32}$$

For $\lambda' \geqslant \frac{3}{2}$ the series (31) is positive since all its terms are non-negative and some are positive. Therefore for a fixed $\lambda \geqslant 3\sqrt[4]{12}/2$ the function $s_\infty^2(N) = t(\varepsilon, \delta)$ has a local minimum for $\varepsilon = 0$ and $\delta = 1$, which proves (a).

Let us now consider the net of sequares (11) and the composition $r_\delta s_\varepsilon''$ of transformations defined by (7) and (8). In view of Lemma 1, for an exponential correlation function we can express $s_\infty^2(s_{\varepsilon,\delta}'') = s(\varepsilon, \delta)$ by

$$s(\varepsilon, \delta) = \sigma^2 \sum_{(i,j)\in S} \exp\left[-\lambda\left(\frac{\delta^2(i+\varepsilon j)^2 + (j+\varepsilon i)^2/\delta^2}{1-\varepsilon^2}\right)^{1/2}\right]. \tag{33}$$

To prove (b) we shall show that the function $s(\varepsilon, \delta)$ has a local maximum with respect to ε at the point $(\varepsilon, \delta) = (0, 1)$. Differentiating under the sum sign, we obtain

$$\frac{d}{d\varepsilon} s(\varepsilon, 1)\Big|_{\varepsilon=0} = -2\sigma^2 \sum_{(i,j)\in S} ij/\sqrt{i^2+j^2} \exp(\lambda\sqrt{i^2+j^2}).$$

The sum of this series vanishes for any λ, because the terms corresponding to (i, j) and $(-i, j)$ reduce. In order to establish the extremum we compute the derivative

$$\frac{d^2}{d\varepsilon^2} s(\varepsilon, 1)\Big|_{\varepsilon=0} = 2\lambda\sigma^2 \sum_{(i,j)\in S} \frac{2\lambda i^2j^2 - (i^4+j^4)/\sqrt{i^2+j^2}}{(i^2+j^2)\exp(\lambda\sqrt{i^2+j^2})}.$$

Hence

$$\frac{d^2}{d\varepsilon^2} s(\varepsilon, 1)\Big|_{\varepsilon=0} = 2\lambda\sigma^2 e^{-\lambda}\left[-4 + e^{-\lambda\frac{\sqrt{2}-1}{2}} s_1(\lambda)\right], \tag{34}$$

where

$$s_1(\lambda) = \sum_{(i,j)\in S, i^2+j^2\geqslant 2} \frac{2\lambda i^2j^2 - (i^4+j^4)/\sqrt{i^2+j^2}}{(i^2+j^2)\exp\left[\lambda\left(\sqrt{i^2+j^2} - \frac{1}{2}(\sqrt{2}+1)\right)\right]}.$$

If $\lambda > \lambda_0 > 0$, then $s_1(\lambda)$ is bounded. Since $\lim\limits_{\lambda\to\infty} \exp\left(-\lambda\frac{\sqrt{2}-1}{2}\right) = 0$, we see from (34) that this derivative is negative for λ large enough. This proves that the function $s(\varepsilon, 1)$ has then a local maximum for $\varepsilon = 0$.

Therefore the nets of large squares may be improved by shearing. This proves (b).

We are going to prove (c). Let us fix $\varepsilon = 0$ in formula (33) and calculate the derivatives

$$\frac{d}{d\delta}s(0,\delta)\Big|_{\delta=1} = -\sigma^2\lambda\sum_{(i,j)\in S}\frac{i^2-j^2}{\sqrt{i^2+j^2}\exp(\lambda\sqrt{i^2+j^2})},\tag{35}$$

$$\frac{d^2}{d\delta^2}s(0,\delta)\Big|_{\delta=1} = \sigma^2\lambda\sum_{(i,j)\in S}\frac{(i^2-j^2)^2-(2j^4+6i^2j^2)/\sqrt{i^2+j^2}}{(i^2+j^2)\exp(\lambda\sqrt{i^2+j^2})}.\tag{36}$$

Derivative (35) vanishes for any λ, because the terms of the series calculated for (i,j) and (j,i) reduce. If we make use of the fact that the terms of series (36) are even and if we reduce the terms calculated for (i,j) and (j,i) we easily get

$$\frac{d^2}{d\delta^2}s(0,\delta)\Big|_{\delta=1} = \sigma^2\lambda\sum_{(i,j)\in S_0}\frac{c_{ij}[\lambda(i^2-j^2)-(i^4+6i^2j^2+j^4)/\sqrt{i^2+j^2}]}{(i^2+j^2)\exp(\lambda\sqrt{i^2+j^2})},\tag{37}$$

where

$$c_{ij} = \begin{cases} 4 & \text{if} \quad j=0 \text{ or } j=i, \\ 8 & \text{if} \quad 0<j<i, \end{cases}$$

and

$$S_0 = \{(i,j):\ i,j \text{ integer},\ 0<j<i\}.$$

Write

$$w_{ij} = \frac{c_{ij}[\lambda(i^2-j^2)^2-(i^4+6i^2j^2+j^4)/\sqrt{i^2+j^2}]}{(i^2+j^2)\exp(\lambda\sqrt{i^2+j^2})};$$

then it is easily seen that for $\lambda \geqslant 2$ we have the inequality

$$w_{10}+w_{11}+w_{20}+w_{21}+w_{22} > 0,$$

and if $\lambda \geqslant 2$, $i \geqslant 3$, $w_{i,(i/2)+k} < 0$, then for $|k| < i/2$ we have

$$\operatorname{sgn}[w_{i,(i/2)-k}+w_{i,(i/2)+k}]$$
$$= \operatorname{sgn}\left\{\sum_{a=-1,1}\frac{\lambda\big(i^2-(\frac{1}{2}i+ak)^2\big)^2-\big(i^4+6i^2(\frac{1}{2}i+ak)^2+(\frac{1}{2}i+ak)^4\big)/\sqrt{i^2+(\frac{1}{2}i+ak)^2}}{i^2+(\frac{1}{2}i+ak)^2}\right\}.$$

This proves that the derivative (36) is positive for $\lambda \geqslant 2$, and thus (c) follows. This completes the proof of Theorem 1.

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O LOKALNEJ OPTYMALNOŚCI REGULARNYCH SIECI PRÓB NA PŁASZCZYŹNIE

STRESZCZENIE

Praca ta jest kontynuacją badań własności regularnych sieci prób na płaszczyźnie przedstawionych przez Daleniusa, Hájka i Zubrzyckiego ([1]).

Rozważanym problemem jest estymacja wartości oczekiwanej izotropowego procesu stochastycznego na płaszczyźnie. Ponieważ w praktyce funkcja korelacyjna nie jest znana, interesujące jest określenie optymalnej sieci dla możliwie dużej klasy funkcji korelacyjnych. Niestety okazało się, że w klasie sieci regularnych nie jest prawdziwe dość intuicyjne przypuszczenie, że sieci trójkątów równobocznych są optymalne w sensie minimizowania granicznej wariancji średniej obserwacji dla wypukłych funkcji korelacyjnych. W pracy [1] postawiono zagadnienie, czy sieć trójkątów równobocznych jest optymalna dla izotropowych i stacjonarnych procesów z tak regularną funkcją korelacyjną jak wykładnicza. W tej pracy pokazaliśmy, że dla klasy wykładniczych funkcji korelacyjnych sieć trójkątów równobocznych jest optymalna lokalnie ze względu na infinitezymalne deformacje afiniczne zachowujące gęstość sieci, jeśli gęstość sieci jest dostatecznie mała. Ściślej mówiąc wprowadzamy afiniczne deformacje nazwane rozciąganiem i skręcaniem oraz ich kompozycje, przedstawiamy graniczną wariancję jako funkcję parametrów tej deformacji i badamy minima tak określonej funkcji.

Oprócz sieci trójkątów badamy również sieci kwadratów. W pracy pokazaliśmy, że sieci kwadratów można polepszyć przez skręcanie, choć czasem są one lokalnie optymalne ze względu na rozciąganie, które przekształca kwadraty w trójkąty. Uzupełnia to wynik Matérna ([2]), który numerycznie wykazał, że w przypadku izotropowych i stacjonarnych procesów stochastycznych z wykładniczą funkcją korelacyjną pewne sieci trójkątów są efektywniejsze od sieci kwadratów i sieci kwadratów są efektywniejsze od sieci prostokątów.

W praktyce, np. przy szacowaniu złóż geologicznych (por. [3]), sieci nie są gęste a nasz wynik stanowi pewną choć ograniczoną wskazówkę przemawiającą za siecią trójkątów.

Б. КОПОЦИНЬСКИ (Вроцлав)

О ЛОКАЛЬНОЙ ОПТИМАЛЬНОСТИ РЕГУЛЯРНЫХ СЕТЕЙ ПРОБ НА ПЛОСКОСТИ

РЕЗЮМЕ

Эта работа является продолжением исследований свойств регулярных сетей проб на плоскости, проведенных Далениусом, Гайком и Зубжицким ([1]). Здесь рассматривается распределение ожидаемой величины изотропного стохастического процесса на плоскости. Так как на практике корреляционная функция не известна, то интересно определить оптимальную сеть для возможно широкого класса корреляционных функций. К сожалению оказалось, что в классе регулярных сетей не верно интуитивное предположение о том, что сети равносторонних треугольников являются оптимальными в смысле минимазирования предельной дисперсии среднего наблюдения для выпуклых корреляционных функций. В [1] поставлен вопрос, является ли сеть равносторонних треугольников оптимальной для изотропных стационарных процессов с такой регулярной корреляционной функцией как экспоненциальная. В настоящей работе мы показали, что для класса экспоненциальных корреляционных функций сеть равносторонних треугольников является локально оптимальной относительно инфинитезимальных афинных деформаций не меняющих плотности сети, если эта плотность достаточно мала. Точнее, вводим афинные деформации, называемые растяжением и кручением, а также их композиции, представляем предельную дисперсию в виде функции параметров этой деформации и исследуем минимумы определенной таким образом функции.

Кроме сети треугольников исследуем также сети квадратов. В работе показано, что сети квадратов можно улучшить кручением, хотя иногда они являются локально оптимальными относительно растяжения, превращающего квадраты в треугольники. Сказанное пополняет результат Матерна ([2]), который показал численным путем, что в случае изотропных и стационарных стохастических процессов с экспоненциальной корреляционной функцией некоторые сети треугольников являются более эффективными, чем сети квадратов, а сети квадратов — более эффективны, чем сети прямоугольников.

На практике, например при оценке геологических месторождений (срав. с [3]), применяются сети не большой плотности, а наш результат является некоторым хотя и ограниченным указанием за сетью треугольников.