

E. SENEITA (Canberra)

## THE RANDOM WALK AND BACTERIAL GROWTH

A model frequently used to describe the growth of a bacterial population is the linear birth-and-death process  $\{\xi_t\}$ , with birth intensity  $\lambda_n = n\lambda$ , and death-intensity  $\mu_n = n\mu$  ( $\lambda, \mu > 0$ ) when there are  $n$  individuals present in the population (see Kendall [5], p. 76). Certain aspects of this problem have been studied by Urbanik [8], who has found the distribution of the maximum number of individuals in the population; and the mean time to achieve this maximum, conditional on extinction eventually occurring. His method has been a direct analysis of the continuous-time situation.

Several further questions of interest about the process may be answered by utilizing an imbedded chain of the process, which reduces these problems to equivalent ones about the simple discrete random walk with absorbing barrier at the origin. We may thus obtain the mean number of birth-death events to achieve the maximum; a measure of the mean population size at a birth-death event; and several distributions describing the transient behaviour of the process, all conditional on extinction eventually occurring. The results concerning these "quasi-stationary" distributions are somewhat paradoxical, in a different sense to the apparent paradox pointed out by Urbanik [8].

**1. Preliminary remarks.** The solution to the linear birth-and-death process described above is known, in the sense that the generating function of the transition probabilities  $p_{ij}(t)$  is known viz. for  $i, j = 0, 1, 2, \dots$

$$(1.1) \quad \sum_{j=0}^{\infty} p_{ij}(t) s^j = [F(s, t)]^i = \left[ \frac{\alpha(t) + [1 - \alpha(t) - \beta(t)]s}{1 - \beta(t)s} \right]^i,$$

where

$$\alpha(t) = \frac{\mu(e^{(\lambda-\mu)t} - 1)}{\lambda e^{(\lambda-\mu)t} - \mu}, \quad \beta(t) = \frac{\lambda(e^{(\lambda-\mu)t} - 1)}{\lambda e^{(\lambda-\mu)t} - \mu}, \quad \lambda \neq \mu$$

(see Bharucha-Reid [2], p. 86-91). The results for  $\lambda = \mu$  are obtained from the above (by L'Hôpital's rule) as  $\lambda \rightarrow \mu$ . However, if we are concerned with problems which essentially involve counting the individuals themselves, it is difficult or impossible to proceed by an analysis of the continuous-time situation. A possible alternative approach is to consider the system only at the instants at which transitions (i.e. birth-death events) actually occur, and the initial time point.

This results in the discrete-time random walk,  $\{X_k\}$ , described by the following scheme:

Transition	Probability
$i \rightarrow i+1$	$\left. \begin{array}{l} \lambda/\lambda+\mu \\ \mu/\lambda+\mu \end{array} \right\} i \geq 1$
$i \rightarrow i-1$	
$0 \rightarrow 0$	1

the theory of which is well-known (e.g. Feller [4]) or is readily deducible by fairly elementary means. The time-units between transitions, in this discrete situation are the intervals between birth-death events.

As an example of the use of this technique, we may immediately derive the well-known result for the distribution of the maximum number of bacteria in the population. For convenience, let us put

$$p = \frac{\lambda}{\lambda + \mu}, \quad q = \frac{\mu}{\lambda + \mu}$$

and consider a conceptual absorbing barrier at  $k \geq r$  where  $r$  is the initial point of the process  $\{\xi_t\}$ . Then, putting

$$S(\xi) = \max_{0 \leq t < \infty} \xi_t = \max_{0 \leq k < \infty} X_k$$

we have

$$P[S(\xi) \geq k \mid \xi_0 = r] = \theta_{rk},$$

where  $\theta_{rk}$  is the probability of absorption at the barrier  $k$  in the corresponding random walk, starting at  $X_0 = r \leq k$ . Now

$$(1.2) \quad \theta_{rk} = \begin{cases} \frac{1 - (q/p)^r}{1 - (q/p)^k} & \text{if } p \neq q \text{ (i.e. } \mu \neq \lambda), \\ \frac{r}{k} & \text{if } p = q \text{ (i.e. } \mu = \lambda) \end{cases}$$

for  $k \geq r$ . Thus an application of random walk theory immediately gives us one of Urbanik's results.

Before proceeding, we remark that if we are to be concerned only with results conditional on extinction occurring eventually, it is sufficient to treat only the case  $q \geq p$ , for which the condition is automatically satisfied. The reason is that if  $p > q$  in which case extinction is no longer certain, results conditional as extinction occurring may be obtained by substituting  $p$  for  $q$  in the results for  $q > p$  (see e.g. Waugh [9]). Thus if  $p > q$

$$P[S(\xi) \geq k \mid S(\xi) < \infty, \xi_0 = r] = \frac{1 - (p/q)^r}{1 - (p/q)^k}.$$

## 2. The mean number of birth-death events to attain a maximum.

Since we consider only the case  $q \geq p$ , this problem, in random-walk language, reduces to finding the mean number of steps for the corresponding random walk to attain its maximum position before absorption. This result does not appear to be known in the theory of random walks, so it is necessary to derive it. The other results of this paper are obtainable from known results on random walks, however.

In the derivation of the required quantity, it seems easiest to partially imitate for this case, the derivation for continuous time of the analogous quantity considered by Urbanik [8].

Let

$$\tau(X) = \min_{X_n = S(\xi)} n.$$

Then we have, for  $a = 1, 2, \dots, k > r \geq 1$ ,

$$\begin{aligned} P[\tau(X) \geq a, S(\xi) = k \mid X_0 = r] \\ &= P[\max_{0 \leq n < a} X_n < k, \max_{n \geq a} X_n = k \mid X_0 = r] \\ &= \sum_{s=1}^{k-1} P[\max_{0 \leq n < a} X_n < k, X_{a-1} = s, \max_{n \geq a} X_n = k \mid X_0 = r] \\ &= \sum_{s=1}^{k-1} P[\max_{n \geq a} X_n = k \mid X_{a-1} = s] P[X_{a-1} = s, \max_{n < a} X_n < k \mid X_0 = r] \\ &= \sum_{s=1}^{k-1} P[S(\xi) = k \mid X_0 = s] P[X_{a-1} = s, \max_{n < a} X_n < k \mid X_0 = r]. \end{aligned}$$

Now, for  $a = 0, 1, 2, \dots, n = r, r+1, \dots, k \geq r = 1, 2, \dots$

$$P[\tau(X) \geq a \mid S(\xi) \leq n, X_0 = r] = \sum_{k=r}^n \frac{P\{\tau(X) \geq a, S(\xi) = k \mid X_0 = r\}}{P\{S(\xi) \leq n \mid X_0 = r\}}.$$

Hence

$$(2.1) \quad P[\tau(X) \geq a \mid S(\xi) \leq n, X_0 = r] \\ = \begin{cases} 1 & \text{if } a = 0; n = r, r+1, \dots \\ \frac{\sum_{k=r+1}^n \sum_{s=1}^{k-1} P[S(\xi) = k \mid X_0 = s] P[X_{a-1} = s, \max_{n < a} X_n < k \mid X_0 = r]}{P[S(\xi) \leq n \mid X_0 = r]} & \text{if } a = 1, 2, \dots; n = r, \end{cases}$$

from the above. The only unknown quantity in this expression is, for  $k \geq r+1$ ,  $a \geq 1$ ,  $s \leq k-1$ ,

$$P[X_{a-1} = s, \max_{n < a} X_n < k \mid X_0 = r] = P[X_T = s, \max_{n \leq T} X_n < k \mid X_0 = r]$$

putting  $T = a-1$ . We may obtain this as follows:

$$\begin{aligned} P[X_T = s, X_h = k, \max_{h+1 \leq n \leq T} X_n < k \mid X_0 = r] \\ = P[X_{T-h-1} = s, \max_{0 \leq n \leq T-h-1} X_n < k \mid X_0 = k-1] \times \\ \times P[X_{h+1} = k-1 \mid X_h = k] P[X_h = k \mid X_0 = r] \\ = qp_{rk}^{(h)} P[X_{T-h-1} = s, \max_{0 \leq n \leq T-h-1} X_n < k \mid X_0 = k-1] \end{aligned}$$

for  $T \geq 1$ .

Thus

$$\begin{aligned} P[X_T = s, \max_{n \leq T} X_n < k \mid X_0 = r] \\ = p_{rs}^{(T)} - q \sum_{h=0}^{T-1} P[X_{T-h-1} = s, \max_{0 \leq n \leq T-h-1} X_n < k \mid X_0 = k-1] p_{rk}^{(h)}, \end{aligned}$$

where the sum is to be replaced by 0 if  $T = 0$ . Hence, for  $k \geq r+1$ ,  $a \geq 2$ ,  $s \leq k-1$ ,

$$(2.2) \quad P[X_{a-1} = s, \max_{n < a} X_n < k \mid X_0 = r] \\ = p_{rs}^{(a-1)} - q \sum_{h=0}^{a-2} P[X_{T-h-1} = s, \max_{0 \leq n \leq T-h-1} X_n < k \mid X_0 = k-1] p_{rk}^{(h)}.$$

Let us now sum this equation from  $a = 1$  to  $\infty$ ,

$$\begin{aligned} \sum_{a=1}^{\infty} P[X_{a-1} = s, \max_{n \leq a-1} X_n < k \mid X_0 = r] \\ = \sum_{a=1}^{\infty} p_{rs}^{(a-1)} - q \sum_{a=2}^{\infty} \sum_{h=0}^{a-2} P[X_{T-h-1} = s, \max_{0 \leq n \leq T-h-1} X_n < k \mid X_0 = k-1] p_{rk}^{(h)}, \end{aligned}$$

so that, on changing the order of summation, we get

$$\begin{aligned} \sum_{a=0}^{\infty} P[X_a = s, \max_{n \leq a} X_n < k \mid X_0 = r] \\ = \sum_{a=0}^{\infty} p_{rs}^{(a)} - q \left( \sum_{n=0}^{\infty} p_{rk}^{(n)} \right) \left( \sum_{a=0}^{\infty} P\{X_a = s, \max_{n \leq a} X_n < k \mid X_0 = k-1\} \right). \end{aligned}$$

Hence

$$\begin{aligned} (2.3) \quad \sum_{a=0}^{\infty} P[X_a = s, \max_{n \leq a} X_n < k \mid X_0 = r] \\ = \sum_{n=0}^{\infty} p_{rs}^{(n)} - \frac{q \sum_{n=0}^{\infty} p_{rk}^{(n)} \sum_{n=0}^{\infty} p_{k-1,s}^{(n)}}{1 + q \sum_{n=0}^{\infty} p_{k-1,k}^{(n)}}. \end{aligned}$$

Putting

$$\mu_{kj} = \sum_{n=0}^{\infty} p_{kj}^{(n)}$$

(which has the intuitive meaning of the mean time spent in state  $j$  before absorption, starting from state  $k$ ), and noting that

$$E[\tau(X) \mid S(\xi) \geq n, X_0 = r] = \sum_{a=1}^{\infty} P[\tau(X) \geq a \mid S(\xi) \leq n, X_0 = r],$$

it is obvious that an explicit expression for this quantity is obtainable, from (2.1), with the aid of (1.2) in terms of  $p$  and  $q$  (i.e.  $\mu$  and  $\lambda$ ), providing expressions for  $\mu_{kj}$  in terms of these quantities are known. These latter quantities are well known (see e.g. Barnett [1]); in fact

$$(2.4) \quad \mu_{kj} = \begin{cases} \frac{(p/q)^j - 1}{p - q}, & j \leq k, \\ \frac{(p/q)^j - (p/q)^{j-k}}{p - q}, & j \geq k, \end{cases}$$

when  $q > p$ , and

$$\mu_{kj} = \begin{cases} 2j, & j \leq k, \\ 2k, & j \geq k \end{cases}$$

for  $q = p = \frac{1}{2}$ . Thus we have from (2.3) for  $r \leq s \leq k-1$

$$(2.5) \quad \sum_{a=0}^{\infty} P[X_a = s, \max_{n \leq a} X_n < k | X_0 = r] \\ = \begin{cases} \frac{(\alpha^r - 1)(\alpha^{k-s} - 1)}{(q-p)(\alpha^k - 1)}, & a = \frac{q}{p} > 1, \\ \frac{r(k-s)}{qk}, & a = \frac{q}{p} = 1. \end{cases}$$

Hence, finally, taking  $r = 1$  for convenience, for  $n \geq 2$

$$(2.6) \quad E[\tau(X) | S(\xi) \leq n, X_0 = 1] \\ = \begin{cases} \frac{(1-\alpha)^2(1-\alpha^{n+1})}{(\alpha-\alpha^{n+1})(q-p)} \sum_{k=2}^n \sum_{s=1}^{k-1} \frac{\alpha^k(1-\alpha^s)(\alpha^{k-s}-1)}{(1-\alpha^{k+1})(1-\alpha^k)^2}, & a = \frac{q}{p} > 1, \\ \frac{(n+1)}{qn} \sum_{k=2}^n \frac{(k-1)}{6k}, & a = \frac{q}{p} = 1. \end{cases}$$

The last follows from the equation

$$\sum_{k=2}^n \sum_{s=1}^{k-1} \frac{s(k-s)}{(k+1)k^2} = \sum_{k=2}^n \frac{(k-1)}{6k}.$$

Comparing (2.6) for  $q \geq p$  ( $\mu \geq \lambda$ ) with the corresponding expressions obtained by Urbanik, we note that there are two essential differences. Firstly, since  $\mu$  and  $\lambda$  of the latter are replaced by  $p$  and  $q$  respectively in the present case, this has an overall effect of introducing an extra factor,  $\mu + \lambda$ , into the numerator. Secondly, in Urbanik's expressions, there is a factor  $1/s$  within the double sums which is not present here. Both these differences originate from the difference of expression (2.5) to the continuous time analogue. Thus we note that in the case  $p = q$  ( $\mu = \lambda$ ), we infer from above that

$$E[\tau(X) | S(\xi) \leq n, X_0 = 1] \approx \frac{1}{3}n$$

as  $n \rightarrow \infty$ , whereas in the continuous time case we have from Urbanik's result (which is misprinted in [8]) that

$$E[\tau(\xi) | S(\xi) \leq n, \xi_0 = 1] \approx \frac{1}{2\mu} \log n$$

as  $n \rightarrow \infty$ .

From (2.6), it immediately follows that

$$(2.7) \quad E[\tau(X) \mid X_0 = 1] = \begin{cases} \frac{(1-\alpha)^2(1-\alpha^{n+1})(\mu+\lambda)}{(\alpha-\alpha^{n+1})(\mu-\lambda)} \sum_{k=2}^{\infty} \sum_{s=1}^{k-1} \frac{\alpha^k(1-\alpha^s)(\alpha^{k-s}-1)}{(1-\alpha^{k+1})(1-\alpha^k)^2} & , \quad \alpha > 1, \\ \infty & , \quad \alpha = 1. \end{cases}$$

The finiteness of the first expression is a consequence of Urbanik's inequality

$$\left| \sum_{s=1}^{k-1} \frac{\alpha^k(1-\alpha^s)(1-\alpha^{k-s})}{(1-\alpha^{k+1})(1-\alpha^k)^2} \right| \leq \frac{k\alpha^{-k-1}}{(1-\alpha^{-1})^3}$$

since  $\alpha > 1$ .

### 3. A measure of the average population size at a birth-death event.

The problem of finding some measure of the mean population size, taken over all birth-death events from time zero, until the moment of extinction, is conceptual rather than mathematical. A reasonable quantity for this purpose when  $\mu > \lambda$  is the quotient

$$(3.1) \quad A_r = E[\theta_r] / E[T_r],$$

where  $\theta_r$  is the total accumulated number of individuals, summed over time 0 and at all birth-death events (starting from  $r$  individuals) and  $T_r$  is the number of birth-death events before extinction (starting from  $r$  individuals).

The problem may be reinterpreted in random-walk terminology (where  $q > p$ ) for  $\theta_r$  is the average position of the imbedded process before absorption, and  $T_r$  the time to extinction, starting from point  $r$  initially. Let us introduce the auxiliary variable

$\beta_{rk}$  = number of visits to position  $k$  before absorption,  
starting from  $r$ .

Then

$$E\beta_{rk} = \sum_{n=0}^{\infty} p_{rk}^{(n)} = \mu_{rk} < \infty$$

(see (2.4)). Now since, from (2.4) we have

$$\sum_{k=1}^{\infty} E[\beta_{rk}] < \infty, \quad \sum_{k=1}^{\infty} kE[\beta_{rk}] < \infty,$$

it follows from e.g. Fubini's theorem that

$$(3.2) \quad \begin{aligned} E[\theta_r] &= E\left(\sum_{k=1}^{\infty} k\beta_{rk}\right) = \sum_{k=1}^{\infty} k\mu_{rk}, \\ E[T_r] &= E\left(\sum_{k=1}^{\infty} \beta_{rk}\right) = \sum_{k=1}^{\infty} \mu_{rk} \end{aligned}$$

so that

$$(3.3) \quad A_r = \frac{\sum_{k=1}^{\infty} k\mu_{rk}}{\sum_{k=1}^{\infty} \mu_{rk}} \geq 1$$

and substituting for  $\mu_{rk}$ , we have

$$(3.4) \quad A_r = \frac{p}{q-p} + \frac{r+1}{2} = \frac{1}{a-1} + \frac{r+1}{2}$$

for  $a = q/p = \mu/\lambda > 1$ . Note that as  $q-p \rightarrow 0$ ,  $A_r \rightarrow \infty$ . It is clear that the ratio is not defined for  $p = q$ , from (3.1).

Before proceeding, it is relevant to note that  $A_r$  is just the mean of the distribution

$$\left\{ \frac{\mu_{rk}}{\sum_{k=1}^{\infty} \mu_{rk}} \right\}, \quad k = 1, 2, \dots,$$

which depends on the value of  $r$ ,  $r = 1, 2, 3, \dots$ . This distribution has been variously called the *ratio of means* distribution, or the *pseudo-transient* distribution (see [3]), and its dependence on the initial point has been regarded as making it unfavourable for describing (in some sense) a stationary situation before absorption, in contrast to the other distributions of [3]. However, from the above, we see that it describes the "average" rather than the „stationary" behaviour of the process before absorption, and in this context, dependence on the initial point is desirable.

**4. Quasi-stationary distributions.** In attempting to describe the transient behaviour of a process where extinction will eventually occur, one may make use of "quasi-stationary" distributions which in a sense give a description of unstable equilibrium in such a process (see Darroch and Seneta [3], who discuss the problem when the number of states is discrete and finite). For the linear birth and death process, when  $\mu \geq \lambda$ , we may arrive at these distributions by considering the quantities

$$(4.1) \quad P[\xi_t = j | \xi_t > 0, \xi_c = r] = \frac{p_{rj}(t)}{1 - p_{r0}(t)}$$



as  $t \rightarrow \infty$ , and

$$(4.2) \quad P[\xi_t = j | \xi_\tau > 0, \xi_0 = r] = \frac{p_{rj}(t)(1 - p_{j0}(\tau - t))}{1 - p_{r0}(\tau)},$$

where  $\tau > t$ , as  $\tau \rightarrow \infty$  and  $t \rightarrow \infty$ ; for  $r, j = 1, 2, \dots$

In order to do this, it is easiest to consider the related probability generating functions. In the first case, from (1.1),

$$\sum_{j=1}^{\infty} s^j P[\xi_t = j | \xi_t > 0, \xi_0 = r] = \frac{[F(s, t)]^r - [F(0, t)]^r}{1 - [F(0, t)]^r}$$

for  $s \in [0, 1]$ . Clearly

$$\begin{aligned} \frac{[F(s, t)]^r - [F(0, t)]^r}{1 - [F(0, t)]^r} &= \left\{ \frac{F(s, t) - F(0, t)}{1 - F(0, t)} \right\} \times \\ &\times \left\{ \frac{[F(s, t)]^{r-1} + F(s, t)^{r-1} F(0, t) + \dots + F(0, t)^{r-1}}{1 + F(0, t) + \dots + [F(0, t)]^{r-1}} \right\} \end{aligned}$$

and since

$$\begin{aligned} \lim_{t \rightarrow \infty} F(s, t) &= 1, \quad s \in [0, 1], \\ \lim_{t \rightarrow \infty} \frac{F(s, t) - F(0, t)}{1 - F(0, t)} &= \frac{s(1 - \lambda | \mu)}{1 - \lambda s | \mu}, \end{aligned}$$

we see that

$$(4.3) \quad \lim_{t \rightarrow \infty} P[\xi_t = j | \xi_t > 0, \xi_0 = r] = \begin{cases} (1 - \lambda | \mu)(\lambda | \mu)^{j-1} & \text{if } \mu > \lambda, \\ 0 & \text{if } \mu = \lambda \end{cases}$$

for  $j = 1, 2, \dots$  independently of  $r = 1, 2, \dots$

To treat the quantity (4.2), where we need to consider a double limit, we begin in a slightly different way. Note first that for  $j, r = 1, 2, \dots$  and  $\tau > t$ , for  $\mu \geq \lambda$

$$\frac{1 - p_{j0}(\tau - t)}{1 - p_{r0}(\tau)} = \frac{1 - [\alpha(\tau - t)]^j}{1 - [\alpha(\tau)]^r},$$

so that

$$\begin{aligned} \lim_{\tau \rightarrow \infty} p_{rj}(t) \frac{1 - p_{j0}(\tau - t)}{1 - p_{r0}(\tau)} &= \frac{j p_{rj}(t)}{r} \left\{ \lim_{\tau \rightarrow \infty} \left\{ \frac{1 - \alpha(\tau - t)}{1 - \alpha(\tau)} \right\} \right\} = \frac{j}{r} p_{rj}(t) e^{(\mu - \lambda)t}. \end{aligned}$$



This eigenvalue plays the role of the unique largest eigenvalue in finite state-space theory for an indecomposable set of states, for we see from the above discussion that, for  $\mu > \lambda$ ,

$$(4.5) \quad p_{rj}(t) \approx r \left(1 - \frac{\lambda}{\mu}\right)^2 (\lambda/\mu)^{j-1} e^{-(\mu-\lambda)t}.$$

The quantity  $\lambda = (\mu - \lambda)$  is known as the *convergence parameter* of the set of states  $\{1, 2, 3, \dots\}$  and since

$$p_{rj}(t) e^{(\mu-\lambda)t} \rightarrow r(1 - \lambda/\mu)^2 (\lambda/\mu)^{j-1} = w_r v_j > 0$$

when  $\mu > \lambda$ , the set is said to be  $\lambda$ -positive.

The paradox arises in the present context when we consider quantities analogous to (4.1) and (4.2) for the imbedded random-walk chain. Intuitively one might expect the limiting results to be substantially the same, and bear the same kind of stationary interpretation. This is in fact not so. The transition submatrix corresponding to states  $\{1, 2, 3, \dots\}$  becomes

$$\begin{bmatrix} 0 & p & . & . & . & . & . & . \\ q & 0 & p & . & . & . & . & . \\ 0 & q & 0 & p & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix} \quad (q \geq p),$$

where  $q = \mu/\mu + \lambda$ ,  $p = \lambda/\mu + \lambda$ . It is clear that this matrix is cyclic, so that limits must be taken through appropriate subsets of states, or as Césaro limits. A study of this random walk process has been made by Seneta and Vere-Jones [7], who obtain the results that

$$(4.6) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n P[X_k = j \mid X_k > 0, X_0 = r] = \begin{cases} 0 & \text{if } \mu = \lambda, \\ c_j j (\sqrt{\lambda/\mu})^{j-1} & \text{if } \mu > \lambda, \end{cases}$$

• where  $c_j > 0$  are certain constants such that

$$\sum_{j=1}^{\infty} c_j j (\sqrt{\lambda/\mu})^{j-1} = 1.$$

Thus we note that we obtain as the limit here

$$c_j \left(1 - \frac{\lambda}{\mu}\right)^{-3/2} w_j \cdot v_j^{1/2}$$

(where the  $w_j$  and  $v_j$  are defined above) in contrast to (4.3). Moreover,

$$(4.7) \quad \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} P[X_m = j \mid X_n > 0, X_0 = r] = 0 \quad (\mu \geq \lambda)$$

in contrast to (4.4), for all  $r, j = 1, 2, \dots$

The reason for the differences of the two sets of limiting results is not obvious intuitively. A partial explanation appears to lie in the fact that since the birth-death intensities are linearly dependent on position, as the population increases in size, changes will occur more rapidly in time; and conversely if the population decreases. However, the imbedded random-walk process does not take this into account, giving equal weight to all time intervals between birth-death events. This reasoning is substantiated by considering the *simple* homogeneous birth-death process, which has the same imbedded chain as in the present linear-birth-death process. As expected, the paradox does not arise (see Seneta [6]).

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AUSTRALIAN NATIONAL UNIVERSITY  
CANBERRA, AUSTRALIA

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E. SENEТА (Canberra)

**BŁĄDZENIE PRZYPADKOWE I ROZWÓJ BAKTERII****STRESZCZENIE**

Teorii prostego błądzenia przypadkowego można użyć do rozwiązywania pewnych problemów powstających w modelu stochastycznym rozwoju populacji bakterii. W tym celu należy rozważyć odpowiednio włożony łańcuch tego procesu narodzin i śmierci. W pracy wyprowadza się wzór na średnią ilość urodzin i śmierci potrzebną dla osiągnięcia maksymalnego rozwoju populacji, ocenę średniej wielkości populacji w momencie urodzin lub śmierci, oraz pewne warunkowe rozkłady opisujące zachowanie się procesu. Te ostatnie wyniki powodują istnienie interesującego paradoksu.

Е. СЕНЕТА (Канберра)

**СЛУЧАЙНОЕ БЛУЖДЕНИЕ И РАЗВИТИЕ БАКТЕРИИ****РЕЗЮМЕ**

Теорию простого случайного блуждания можно использовать для решения некоторых проблем, возникающих в стохастической модели развития популяции бактерий. С этой целью следует рассмотреть соответствующую вложенную цепь этого процесса рождения и смерти. В работе выводится формула среднего числа рождений и смертей, необходимого для достижения максимального развития популяции, оценка средней величины популяций в момент рождения или смерти и некоторые условные распределения, описывающие поведение этого процесса. Эти последние результаты приводят к интересному парадоксу.