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OPTIMUM TRANSPORTATION BASES

1. Introduction. The transportation problem can be stated as follows. There is given a system (C, M), where $C = \{c_{ij}\}$ is a $m \times n$ rectangular matrix (cost matrix) with c_{ij} real, and $M = (a_1, \ldots, a_m;\ b_1, \ldots, b_n)$ is a system of $m+n$ positive numbers a_i, b_j with $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

The problem is to find a $m \times n$ matrix $X = \{x_{ij}\}$ whose elements x_{ij} satisfy for all $i = 1, \ldots, m\ j = 1, \ldots, n$ conditions

$$x_{ij} \geqslant 0, \tag{1}$$

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad \sum_{i=1}^{m} x_{ij} = b_j \tag{2}$$

and minimize a function (cost function)

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} = z_X. \tag{3}$$

In section 2 we recall such terms as graph, cycle, basis, zero matrix and basic matrix. It is known that to each basis B there exist:

a) exactly one zero matrix i. e., a matrix $C' = \{c'_{ij}\}$ with $c'_{ij} = c_{ij} + u_i + v_j$ (u_i, v_j — constants) satisfying condition $c'_{ij} = 0$ for all $(i, j) \epsilon B$.

We denote such a matrix by $C_B = \{c^B_{ij}\}$.

b) exactly one basic matrix, i. e. a matrix $Y = \{y_{ij}\}$ whose elements satisfy (2) and where $y_{ij} = 0$ for all $(i, j) \bar{\epsilon} B$. Such a matrix we denote by $Y(B) = \{y^B_{ij}\}$.

Matrix C is said to be of property A if to each basis B

$$c^B_{ij} \neq 0 \quad \text{for all} \quad (i, j) \bar{\epsilon} B.$$

Basis B is called an optimal basis of C if

$$c_{ij}^{B} \geqslant 0 \quad \text{for all} \quad (i,j) \bar{\in} B.$$

The paper presents a method of solving the transportation problem by means of optimal bases (OBM). An initial version of this method was published in [1]. The OBM is illustrated by a numerical example (section 7).

This method consists in finding a sequence

$$(*) \qquad Y(B_1),\ Y(B_2),\ \ldots,\ Y(B_k)$$

with

$$z_{Y(B_1)} \leqslant z_{Y(B_2)} \leqslant \ldots \leqslant z_{Y(B_k)},$$

where $B_1, B_2, \ldots, B_k$ are optimal bases and $Y(B_k)$ is an optimal solution of the transportation problem. Any two consecutive bases in $(*)$ differ by one element. In sections 6 and 7 it is shown how to find B_1 and $Y(B_1)$. In sections 3, 4 and 5 some special perturbation technique is developed which guarantees the finiteness of sequence $(*)$ (theorem 13). Applying this technique we are to solve instead of the original problem another problem (W, M) with $W = C+E$ and E defined by formula (6). Both E and W have property A (theorem 2 and 5) and each optimal basis of W is an optimal basis of C (theorem 6). Theorems 3, 4, 7, 8, 9, 10, 11 develop some other important properties of E and W which make it possible to simplify considerably the process of solving the (W, M) problem.

In sections 8 and 9 some properties of the set of optimal bases are examined. Thus, for instance, if C is of property A then for ach pair of its optimal bases B_0, B there exists a sequence of optimal bases of C

$$B_0, B_1, B_2, \ldots, B$$

where ony two consecutive bases differ by one element (theorem 14).

Theorems 15, 16, 17 and 18 imply the following result (corollary 7). The number of optimal bases of a $m \times n$ matrix which is of property A is equal $\binom{m+n-2}{m-1}$. Theorems 6 and 18 imply theorem 19: The number N_{opt} of optimal bases of an arbitrary $m \times n$ matrix satisfies the following relation

$$\binom{m+n-2}{m-1} \leqslant N_{\mathrm{opt}} \leqslant m^{n-1} n^{m-1}.$$

In the concluding section of the paper (section 10) some comparisons between the OBM and the classical primal transportation method are made.

2. Notation and definitions. Let Φ be the set of all nodes (i,j) $i = 1, \ldots, m$; $j = 1, \ldots, n$ of an $m \times n$ rectangular net. Two nodes

$a_1 = (i_1, j_1)$ and $a_2 = (i_2, j_2)$ are said *to lie on one* line if either $i_1 = i_2$ or $j_1 = j_2$. Two nodes of $\Omega \subset \Phi$ are *neighboring* if they lie on one line and no other node of Ω lies between them on the same line. Let a_1 and a_2 be two neighboring nodes of Ω. By a *link* a_1a_2 we mean a straight line segment of endpoints a_1 and a_2. We assume that $a_1a_2 \equiv a_2a_1$.

A set of nodes Ω, together with the set of all possible links in Ω, is called a *graph* G_Ω. Graph $G_{\Omega'}$, is a *subgraph* of G_Ω if $\Omega' \subset \Omega$. By a *route* $a_1 - a_k$ we mean a sequence of different links $a_1a_2, a_2a_3, \ldots, a_{k-1}a_k$, where every two consecutive links are perpendicular and at most two nodes of the route are on one line. For $k \geqslant 5$, if $a_1 = a_2$, then we call a *cycle* either the route $a_1 - a_k$, or the graph G_Ω where Ω is the set of all nodes in the route $a_1 - a_k$.

We say that G_Ω *contains a cycle* if there exists a subgraph of G_Ω which is a cycle. G_Ω is said to be *connected* if to any two nodes a_1 and a_k of Ω there exists a subgraph (of G_Ω) whose links all form a route $a_1 - a_k$. We say then that G_Ω *contains a route* $a_1 - a_k$.

Let B a subset of Φ consisting of $m+n-1$ nodes. B is called a *basis* if G_B contains no cycle. It is known (see e. q. [4]) that G_B is a connected graph.

Let there be given a matrix $C = \{c_{ij}\}$ and a basis B. A $m \times n$ matrix $C' = \{c'_{ij}\}$ of the form $c'_{ij} = c_{ij} + u_i + v_j$ where u_i and v_j are arbitrary constant and where

$$c'_{ij} = 0 \quad \text{for all} \quad (i,j) \in B$$

is called a *zero matrix*(1). It is known that to each basis B there exists exactly one zero matrix. Let us denote it by $C_B = \{c^B_{ij}\}$. A basis B is called an *optimal basis of C* or, more compactly, an *optimal basis*, if

$$c^B_{ij} \geqslant 0 \quad \text{for all} \quad (i,j) \bar{\in} B.$$

Matrix C is said to be *of property A* if for each optimal basis B

$$c^B_{ij} > 0 \quad \text{for all} \quad (i,j) \bar{\in} B.$$

Thus we can assert the following

COROLLARY 1. *If for each basis B*

$$c^B_{ij} \neq 0 \quad \text{for all} \quad (i,j) \bar{\in} B$$

then C is of property A.

We now introduce:

PROCEDURE 1. Consider an arbitrary cycle G_Γ. Divide Γ into two subsets say Γ_1, Γ_2 assigning a) neighboring nodes to different sets, b) a specified node of Γ to Γ_1.

(1) In [4] such a matrix was called a *zero cost matrix*.

Let us introduce the following notations

$$c(\Gamma_1, \Gamma_2) = \sum_{(i,j)\in\Gamma_1} c_{ij} - \sum_{(i,j)\in\Gamma_2} c_{ij}$$

and

$$c(\Gamma) = |c(\Gamma_1, \Gamma_2)|.$$

It may be observed that there are two possible partitions of Γ if only part a) of procedure 1 is used. Newertheless, the value of $c(\Gamma)$ does not depend on what of the two partitions we take. Bearing this in mind, we now assert and prove:

THEOREM 1. *A sufficient condition for C to be of property A is that $c(\Gamma) > 0$ for each cycle G_Γ.*

Proof. To prove the theorem (see corollary 1) we are required to show that if the condition is fulfilled then $c_{ij}^B \neq 0$ for each basis B and $(i,j)\bar{\in} B$. Thus, take an arbitrary basis B and an arbitrary node $(p,q)\bar{\in} B$. Graph $G_{B+(p,q)}$ contains exactly one cycle ([4]), say G_Γ. Using procedure 1 divide into Γ_1 and Γ_2 assigning (p,q) to Γ_1. From [4] (page 163) it follows that

$$c(\Gamma_1, \Gamma_2) = \sum_{\Gamma_1} c_{ij} - \sum_{\Gamma_2} c_{ij} = \sum_{\Gamma_1} c_{ij}^B - \sum_{\Gamma_2} c_{ij}^B = c_{pq}^B \tag{4}$$

and $c(\Gamma) = |c_{pq}^B|$. So if $c(\Gamma) > 0$ then $c_{pq}^B \neq 0$ and this completes the proof.

3. Matrix E and its properties. We introduce a number d as follows

$$d = \begin{cases} \min\left\{ \min\limits_{\Gamma:c(\Gamma)>0} c(\Gamma), \quad \min\limits_{\Gamma,\Gamma':c(\Gamma)\neq c(\Gamma')} c(\Gamma)-c(\Gamma'), \right. \\ \text{any positive number if there is no cycle } G(\Gamma), \\ \text{such that } c(\Gamma) \neq 0. \end{cases} \tag{5}$$

We also introduce

PROCEDURE 2. Establish a one-to-one correspondence between all nodes (i,j) of Φ and numbers $1, 2, \ldots, mn$ so that the nodes (i,j) can be arranged in a sequence

$$a_1, \ldots, a_i, \ldots, a_{mn}.$$

To illustrate this, suppose that (i,j) corresponds to number 1 under procedure 2. Then $a_1 = (i,j)$.

We finally introduce an $m\times n$ matrix $E = \{e_{ij}\}$ defined as follows,

$$e_{ij} = h^l \delta \quad \text{for all} \quad (i,j)\in\Phi, \tag{6}$$

where h is a positive integer and $\delta > 0$.

This permits us to state and prove a series of theorems and corollaries as follows:

THEOREM 2. *Given a matrix E where $h > 1$ and two different subsets of Φ: Ω_1 and Ω_2, then*

$$\sum_{\Omega_1} e_{ij} \neq \sum_{\Omega_2} e_{ij}.$$

Proof. Let $W_1 = \sum_{\Omega_1} e_{ij}$ and $W_2 = \sum_{\Omega_2} e_{ij}$. Suppose contrary to the hypothesis of the theorem,

$$W_1 = W_2.$$

Subtracting identical terms, if any, from both sides of the last equation and dividing it by δ we get

$$W_1' = W_2'.$$

There are two cases:

1° $W_1' = W_2' = 0$. Then we get a contradiction since, by hypothesis, Ω_1 and Ω_2 are different.

2° $W_1' = W_2' \neq 0$. Then $W_1' = h^{l_1} + \ldots + h^{l_u}$, $W_2' = h^{l_1'} + \ldots + h^{l_v'}$ where $u \geqslant 1$, $v \geqslant 1$ and all $l_1, \ldots, l_u, l_1', \ldots, l_v'$ are different integers. Without loss of generality $l_1 = \min(l_1, \ldots, l_u, l_1', \ldots, l_v')$. But then $l_1 < \min(l_2, \ldots$ $\ldots, l_n, l_1', \ldots, l_v')$. Dividing both sides of equation $W_1' = W_2'$ by h^{l_1} we get

$$1 + h^{l_2 - l_1} + \ldots + h^{l_u - l_1} = h^{l_1' - l_1} + \ldots + h^{l_v' - l_1}$$

which is impossible since the right side of the equation is divisible by $h > 1$ while the left side is not.

COROLLARY 2. *Given a matrix E, where $h > 1$, and an arbitrary cycle G_Γ then $e(\Gamma) > 0$.*

Proof. Divide Γ by procedure 1 into Γ_1 and Γ_2. For $\Gamma_1 \neq \Gamma_2$ we then have (theorem 2) $\sum_{\Gamma_1} e_{ij} \neq \sum_{\Gamma_2} e_{ij}$. Therefore $e(\Gamma) > 0$. Q. e. d.

REMARK. Theorem 1 and corollary 2 imply that if $h > 1$ then E is of property A.

THEOREM 3. *Given a matrix E where $h > 2$ a basis B and two different nodes $(p, q) \bar{\epsilon} B$, $(r, s) \bar{\epsilon} B$. Then*

$$e_{pq}^B \neq e_{rs}^B.$$

Proof. Let G_Γ and $G_{\Gamma'}$, be cycles contained in $G_{B+(p,q)}$, $G_{B+(r,s)}$ respectively. Divide Γ by procedure 1 into Γ_1 and Γ_2 assigning (p, q) to Γ_1, and Γ' into Γ_1' and Γ_2' assigning (r, s) to Γ_1'.

Suppose contrary to the hypothesis of the theorem

$$e_{pq}^B = e_{rs}^B.$$

According to (4) this is equivalent to

$$\sum_{\Gamma_1} e_{ij} - \sum_{\Gamma_2} e_{ij} = \sum_{\Gamma_1'} e_{ij} - \sum_{\Gamma_2'} e_{ij}$$

or to

$$\sum_{\Gamma_1} e_{ij} + \sum_{\Gamma_2'} e_{ij} = \sum_{\Gamma_1'} e_{ij} + \sum_{\Gamma_2} e_{ij}.$$

Subtracting identical terms from both sides of this equation and dividing these sides by δ we get

$$W_1 = \sum_{\Gamma_1 - \Gamma_1\Gamma_1'} e_{ij} + \sum_{\Gamma_2' - \Gamma_2\Gamma_2'} e_{ij} = \sum_{\Gamma_1' - \Gamma_1\Gamma_1'} e_{ij} + \sum_{\Gamma_2 - \Gamma_2\Gamma_2'} e_{ij} = W_2.$$

If $W_1 = 0$ then also $W_2 = 0$ and this implies $\Gamma - \Gamma'$ which is impossible. Since sets $\Gamma_1\Gamma_2'$ and $\Gamma_1'\Gamma_2$ may not be empty one can present W_1 and W_2 in the form

$$W_1 = k_1 h^{l_1} + \ldots + k_u h^{l_u}, \qquad W_2 = k_1' h^{l_1} + \ldots + k_v' g^{l_v'},$$

where numbers $k_1, \ldots, k_u, k_1', \ldots, k_v''$ are either 1 or 2. Suppose $l_1 = \min(l_1, \ldots, l_u, l_1', \ldots, l_v')$. Then $l_1 < \min(l_2, \ldots, l_u, l_1', \ldots, l_v')$. Divide both sides of the equation $W_1 = W_2$ by h^{l_1} to get

$$k_1 + k_2 h^{l_2 - l_1} + \ldots + k_u h^{l_u - l_1} = k_1' h^{l_1' - l_1} + \ldots + k_v' h^{l_v' - l_1}$$

which is impossible for the right side is divisible by $h > 2$ while the left side is not. This completes the proof.

THEOREM 4. *There are given a matrix C and a matrix E defined by* (6). *If*

$$h > 2 \quad \text{and} \quad 0 < \delta < \frac{d}{2}\,\frac{h-1}{h^{mn+1}-1} \tag{7}$$

for d definded by (5). *Then $e(\Gamma) < d/2$ for each cycle G_Γ.*

Proof.

$$e(\Gamma) = \Big|\sum_{\Gamma_1} e_{ij} - \sum_{\Gamma_2} e_{ij}\Big| < \sum_{\Gamma} e_{ij} < \sum_{\Phi} e_{ij} = \delta(h + h^2 + \ldots + h^{mn})$$
$$= \delta\,\frac{h^{mn+1} - h}{h-1}.$$

For h and δ satisfying (7) we get $e(\Gamma) < d/2$.

4. Matrix W and its properties. Let there be given a matrix C and a matrix E defined by (6). We introduce an $m \times n$ matrix $W = \{w_{ij}\}$ where

$$W = C + E. \tag{8}$$

Take an arbitrary cycle, say G_Γ. Divide Γ into Γ_1 and Γ_2 by procedure 1. Then

$$\begin{aligned} w(\Gamma_1, \Gamma_2) &= \sum_{\Gamma_1} w_{ij} - \sum_{\Gamma_2} w_{ij} = \sum_{\Gamma_1} (c_{ij} + e_{ij}) - \sum_{\Gamma_2} (c_{ij} + e_{ij}) \\ &= \sum_{\Gamma_1} c_{ij} - \sum_{\Gamma_2} c_{ij} + \sum_{\Gamma_1} e_{ij} - \sum_{\Gamma_2} e_{ij}. \end{aligned}$$

So

$$w(\Gamma_1, \Gamma_2) = c(\Gamma_1, \Gamma_2) + e(\Gamma_1, \Gamma_2). \tag{9}$$

Given a matrix W defined by (8) where d satisfies (5) and h and δ satisfy (7), the following theorems holds.

THEOREM 5. *For each basis B $w_{ij}^B \neq 0$ for all $(i, j) \bar{\epsilon} B$.*

THEOREM 6. *Each optimal basis of W is an optimal basis of C.*

THEOREM 7. *Relation $c_{ij}^B > 0$ implies $w_{ij}^B > 0$.*

THEOREM 8. *Relation $c_{pq}^B > c_{rs}^B$ implies $w_{pq}^B > w_{rs}^B$.*

Proof of theorems 5-8. Consider an arbitrary node $(p, q) \bar{\epsilon} B$. In the same way as in the proof of theorem 1 we get

$$w(\Gamma) = |w_{pq}^B|$$

and from (9)

$$|w_{pq}^B| = |c(\Gamma_1, \Gamma_2) + e(\Gamma_1, \Gamma_2)|.$$

If $c(\Gamma_1, \Gamma_2) = 0$ then

$$|w_{pq}^B| = |e(\Gamma_1, \Gamma_2)| = e(\Gamma).$$

By corollary 2, however, $e(\Gamma)$ is positive so $w_{pq}^B \neq 0$. If $c(\Gamma_1, \Gamma_2) \neq 0$ formula (5) implies

$$d \leqslant c(\Gamma).$$

On the other hand

$$\begin{aligned} |w_{pq}^B| &= |c(\Gamma_1, \Gamma_2) + e(\Gamma_1, \Gamma_2)| \geqslant |c(\Gamma_1, \Gamma_2)| - |e(\Gamma_1, \Gamma_2)| \\ &= c(\Gamma) - e(\Gamma) > d - \frac{d}{2} = \frac{d}{2} > 0. \end{aligned}$$

Thus $w_{pq}^B \neq 0$ and the proof of theorem 5 is complete.

To prove theorem 6 now we need only to show that $w_{ij}^B > 0$ implies $c_{ij}^B \geqslant 0$. Suppose, on the contrary, that for some (p, q) the inequality

$w^B_{pq} > 0$ implies $c^B_{pq} < 0$. From the definition of C_B it then follows that $(p, q) \bar{\epsilon} B$. Let G_Γ be the cycle contained in $G_{B+(p,q)}$. Divide Γ into Γ_1, Γ_2 by procedure 1 assigning (p, q) to Γ_1. Then (see proof of theorem 1)

$$c(\Gamma_1, \Gamma_2) = \sum_{\Gamma_1} c_{ij} - \sum_{\Gamma_2} c_{ij} = \sum_{\Gamma_1} c^B_{ij} - \sum_{\Gamma_2} c^B_{ij} = c^B_{pq} < 0.$$

But $c(\Gamma) = |c(\Gamma_1, \Gamma_2)| \geqslant d > 0$. If $c(\Gamma_1, \Gamma_2)$ is negative we have

$$c(\Gamma_1, \Gamma_2) \leqslant -d.$$

On the other hand $e(\Gamma) < d/2$ (theorem 4) so $e(\Gamma_1, \Gamma_2) < d/2$. From the last inequalities we get

$$w(\Gamma_1, \Gamma_2) = c(\Gamma_1, \Gamma_2) + e(\Gamma_1, \Gamma_2) < -d + \frac{d}{2} < 0.$$

But $w^B_{pq} = w(\Gamma_1, \Gamma_2) < 0$ where $(p, q) \epsilon \Gamma_1$, which contradicts the assumption that w^B_{pq} is positive. Thus $c^B_{pq} \geqslant 0$.

To prove theorem 7 take be an arbitrary node $(p, q) \bar{\epsilon} B$ for which $c^B_{pq} > 0$. Let G_Γ be the cycle contained in $G_{B+(p,q)}$. Divide Γ into Γ_1 and Γ_2 by procedure 1 assiging (p, q) to Γ_1. Since

$$c^B_{pq} = c(\Gamma_1, \Gamma_2) > 0$$

and therefore

$$c^B_{pq} \geqslant d.$$

From theorem 2 we have

$$e(\Gamma) < \frac{d}{2}, \quad \text{so} \quad e(\Gamma_1, \Gamma_2) > -\frac{d}{2}.$$

Using all these results we get

$$w^B_{pq} = w(\Gamma_1, \Gamma_2) = c(\Gamma_1, \Gamma_2) + e(\Gamma_1, \Gamma_2) > d - \frac{d}{2} = \frac{d}{2} > 0.$$

Consider a node (p, q). If $(p, q) \epsilon B$ then obviously $w^B_{pq} = c^B_{pq} = 0$. Suppose $(p, q) \bar{\epsilon} B$. From the proof of theorem 7 we got the following formula

$$(*) \qquad \begin{aligned} w^B_{pq} &= w(\Gamma_1, \Gamma_2) = c(\Gamma_1, \Gamma_2) + e(\Gamma_1, \Gamma_2), \\ c^B_{pq} &= c(\Gamma_1, \Gamma_2), \end{aligned}$$

where $\Gamma = \Gamma_1 + \Gamma_2$ is the cycle contained in the graph $G_{B+(pq)}$. Consider another node (r, s). Then

$$w^B_{rs} = c^B_{rs} = 0 \quad \text{if} \quad (r, s) \epsilon B$$

$$(**) \qquad \begin{aligned} w^B_{rs} &= w(\Gamma'_1, \Gamma'_2) = c(\Gamma'_1, \Gamma'_2) + e(\Gamma'_1, \Gamma'_2), \\ c^B_{rs} &= c(\Gamma'_1, \Gamma'_2), \end{aligned}$$

where $\Gamma' = \Gamma_1' + \Gamma_2'$ is the cycle contained in the graph $G_{B+(r,s)}$. Since $(p, q) \neq (r, s)$ therefore $\Gamma \neq \Gamma'$. It is known that $(p, q) \in \Gamma$ and $(r, s) \in \Gamma'$.

To prove theorem 8 first remark that both (p, q) and (r, s) cannot belong to B (for then $c_{pq}^B = c_{rs}^B$). Hence there are only the following three cases

$$\begin{array}{lll} 1^\circ & (p, q) \bar{\in} B, & (r, s) \bar{\in} B, \\ 2^\circ & (p, q) \bar{\in} B, & (r, s) \in B, \\ 3^\circ & (p, q) \in B, & (r, s) \bar{\in} B. \end{array}$$

Consider case 1°. Assumption $c_{pq}^B > c_{rs}^B$ implies that at least one of $c(\Gamma_1, \Gamma_2)$, $c(\Gamma_1', \Gamma_2')$ is different from zero. So (see (5))

$$c(\Gamma_1, \Gamma_2) - c(\Gamma_1', \Gamma_2') \geqslant d.$$

Theorem 4 implies

$$e(\Gamma_1, \Gamma_2) \geqslant -\frac{d}{2} \quad \text{and} \quad -e(\Gamma_1', \Gamma_2') > -\frac{d}{2}.$$

Using (*) and (**) we get

$$w_{pq}^B - w_{rs}^B = [c(\Gamma_1, \Gamma_2) - c(\Gamma_1', \Gamma_2')] + [e(\Gamma_1, \Gamma_2) - e(\Gamma_1', \Gamma_2')] > d - \\ -\frac{d}{2} - \frac{d}{2} = 0.$$

Consider case 2°. Here w_{pq}^B is given by (*) while $c_{rs}^B = w_{rs}^B = 0$ and the assumption $c_{pq}^B > c_{rs}^B$ implies $c_{pq}^B > 0$. Hence (see theorem 7) $w_{pq}^B > 0$. For $w_{rs}^B = 0$ then $w_{pq}^B > w_{rs}^B$.

Finally, we turn to case 3°. Here $c_{pq}^B = w_{pq}^B = 0$ white w_{rs}^B is defined by (**). For $c_{pq}^B > c_{rs}^B$ so $c_{rs}^B < 0$. From the definition of d and from theorem 4 we get

$$c_{rs}^B = c(\Gamma_1', \Gamma_2') \leqslant -d,$$

$$e(\Gamma_1', \Gamma_2') \leqslant \frac{d}{2}$$

and

$$w_{rs}^B = c(\Gamma_1', \Gamma_2') + e(\Gamma_1', \Gamma_2') < -d + \frac{d}{2} < 0.$$

For

$$w_{pq}^B = 0 \quad \text{then} \quad w_{pq}^B > w_{rs}^B.$$

Theorems 2, 5 and corollary 1 imply

COROLLARY 3. *If E satisfies (7) then both E and W are of property A.*

Theorem 8 implies the following.

COROLLARY 4. *There is given a matrix $W = C+E$ where d satisfies* (4) *and E satisfies* (7). *Let Ω be some subset of Φ. If (r, s) is the unique element of Ω satisfying*

$$c_{rs}^B = \min_{(i,j)\in\Omega} c_{ij}^B$$

then

$$w_{rs}^B = \min_{(i,j)\in\Omega} w_{ij}^B.$$

5. Further properties of matrix E. Let the following be given: a matrix $E = \{e_{ij}\}$ defined by (6), an arbitrary basis B and a node $(p, q)\bar{\epsilon} B$. By G_Γ denote a cycle contained in $G_{B+(p,q)}$. Divide Γ by procedure 1 into Γ_1 and Γ_2 assigning (p, q) to Γ_1. To each element (i, j) of Γ there corresponds exactly one natural number l such that $1 \leqslant l \leqslant mn$ and $(i, j) \equiv a_l$. We call l an *index* of (i, j).

Suppose that $(i, j) = a_l$ has the t-th biggest index among all nodes of Γ. Then we introduce the following notation

$$u_{B,t}^{pq} = \begin{cases} l & \text{if} \quad (i,j)\in\Gamma_1, \\ -l & \text{if} \quad (i,j)\in\Gamma_2. \end{cases} \tag{10}$$

THEOREM 9. *If E satisfies* (7) *then for each basis B and for each $(p, q)\bar{\epsilon} B$*

$$\operatorname{sgn} e_{pq}^B = \operatorname{sgn} u_{B,1}^{pq}.$$

Proof. Since $u_{B,1}^{pq} \neq 0$ it is enough to show that
1° $u_{B,1}^{pq} > 0$ implies $e_{pq}^B > 0$ and
2° $u_{B,1}^{pq} < 0$ implies $e_{pq}^B < 0$.

Because the arguments are similar we need only consider case 1° in detail. Thus, let $(r, s) = a_l$ be the greatest index among all elements of Γ. By assumption $u_{B,1}^{pq} > 0$ so $u_{B,1}^{pq} = l$ and $(r, s)\in\Gamma_1$. For (p, q) belongs also to Γ, and since all e_{ij} are positive, we have

$$e_{pq}^B = \sum_{\Gamma_1} e_{ij} - \sum_{\Gamma_2} e_{ij} \geqslant e_{rs} - \sum_{\Gamma_2} e_{ij}.$$

But $\Gamma_2 \subset \{a_1, \ldots, a_{l-1}\}$ (for $a_l\in\Gamma_1$). Therefore

$$\sum_{\Gamma_2} e_{ij} \leqslant h\delta + h^2\delta + \ldots + h^{l-1}\delta = \delta\frac{h^l - h}{h-1}$$

and $e_{rs} = h^l\delta$. Hence

$$e_{pq}^B \geqslant h^l\delta - \delta\frac{h^l-h}{h-1} = \frac{\delta}{h-1}(h^{l+1} - 2h^l + h)$$

$$\geqslant \frac{\delta}{h-1}(2h^l - 2h^l + h) = \frac{\delta h}{h-1} > 0 \qquad \text{Q. e. d.}$$

Let there be given a basis B, and two different nodes $(p, q)\bar{\in} B$ and $(r, s)\bar{\in} B$. By G_{Γ} and $G_{\Gamma'}$ denote cycles contained in $G_{B+(p,q)}$, $G_{B+(r,s)}$ respectively. Let k be the number of nodes in Γ and k' be the number of nodes in Γ'. We will prove the following.

THEOREM 10. *There exists a number* $t \leqslant \min(k, k')$ *satisfying*

$$u_{B,t}^{pq} \neq u_{B,t}^{rs}. \tag{11}$$

Proof. Let us assume $k \leqslant k'$. From the definition of $u_{B,t}^{pq}$ it follows that there exists a number say t', $t' \leqslant k$ such that $|u_{B,t'}^{pq}|$ is just the index of (p, q). From the definition of E it follows that to no node other than (p, q) does there correspond an index $|u_{B,t}^{pq}|$. Therefore $|u_{B,t}^{pq}| \neq |u_{B,t}^{rs}|$ which implies

$$u_{B,t'}^{pq} \neq u_{B,t}^{rs} \quad \text{Q. e. d.}$$

REMARK 1. The definition of $u_{B,t}^{pq}$ and properties of E imply that the equality

$$u_{B,t}^{pq} = u_{B,t}^{rs}$$

requires that (p, q) belongs either to $\Gamma_1\Gamma_1'$ or to $\Gamma_2\Gamma_2'$.

THEOREM 11. *There are given a matrix C and a matrix E defined by* (6). *If*

$$h \geqslant 3, \quad 0 < \delta < \frac{d}{2}\cdot\frac{h-1}{h^{mn+1}-h} \tag{12}$$

for d defined by (5) *then for each basis B and for each pair of different nodes* $(p, q)\bar{\in} B$, $(r, s)\bar{\in} B$

$$\operatorname{sgn}(e_{pq}^{B} - e_{rs}^{B}) = \operatorname{sgn}(u_{B,g}^{pq} - u_{B,g}^{rs}).$$

Proof. From the definition of $u_{B,g}^{pq}$ and $u_{B,g}^{rs}$ it follows that $(p, q)\in\Gamma_1$ and $(r, s)\in\Gamma_1'$. Then

(13)
$$\begin{aligned} e_{pq}^{B} - e_{rs}^{B} &= e(\Gamma_1, \Gamma_2) - e(\Gamma_1', \Gamma_2') = \sum_{\Gamma_1} e_{ij} - \sum_{\Gamma_2} e_{ij} - \sum_{\Gamma_1'} e_{ij} + \sum_{\Gamma_2'} e_{ij} \\ &= \sum_{\Gamma_1-\Gamma_1\Gamma_1'} e_{ij} - \sum_{\Gamma_2-\Gamma_2\Gamma_2'} e_{ij} - \sum_{\Gamma_1'-\Gamma_1\Gamma_1'} e_{ij} + \sum_{\Gamma_2'-\Gamma_2\Gamma_2'} e_{ij}. \end{aligned}$$

Let $\Pi = (\Gamma_1 - \Gamma_1\Gamma_1') + (\Gamma_2 - \Gamma_2\Gamma_2') + (\Gamma_1' - \Gamma_1\Gamma_1') + (\Gamma_2' - \Gamma_2\Gamma_2') = \Gamma + \Gamma' - (\Gamma_1\Gamma_1' + \Gamma_2\Gamma_2')$. Consider a node of Π with the biggest index, say l. We will show by a procedure 3 that

$$l = \max(|u_{B,g}^{pq}|, |u_{B,g}^{rs}|).$$

PROCEDURE 3.

Step 1. Let (i_1, j_1) be the node of $\Gamma+\Gamma'$ with the biggest index. There are two mutually exclusive cases as follows:

1° $(i_1, j_1) \epsilon \Pi$;

2° $(i_1, j_1) \epsilon \Gamma_1\Gamma_1'+\Gamma_2\Gamma_2'$.

In the first case $u_{B,1}^{pq} \neq u_{B,1}^{rs}$ (see remark 1). So $g = 1$ and the index of (i_1, j_1) is equal $\max(|u_{B,1}^{pq}|, |u_{B,1}^{rs}|) = 1$. Thus the statement has been proved. When case 2° holds turn to

Step 2. Consider a node of $\Gamma+\Gamma'$ with the second biggest index say (i_2, j_2). But then as was the case in step 1 we again have two cases:

1° $(i_2, j_2) \epsilon \Pi$;

2° $(i_2, j_2) \epsilon \Gamma_1\Gamma_1'+\Gamma_2\Gamma_2'$.

For the first case $g = 2$ and the maximal index of nodes in 1° is equal to the index of (i_2, j_2) which is equal $\max(|u_{B,2}^{pq}|, |u_{B,2}^{rs}|) = 1$, and the statement has been proved. In case 2° turn to

Step 3. Repeat step 2 again and again considering nodes of $\Gamma+\Gamma'$ with third, fourth, and so on, biggest indices until case 1° from step 2 arises.

The latter happens after at most k steps where k is number of nodes in Γ. For suppose on the contrary that we have performed k steps where we considered k different nodes $(i_1, j_1)(i_2, j_2), \ldots, (i_k, j_k)$ and where case 2° always arises. Let us denote by Ω the set $\{(i_1, j_1), \ldots, (i_k, j_k)\}$. Then: $\Omega \subset \Gamma$ and $\Omega \subset \Gamma'$. For Γ consists of k nodes so $\Omega = \Gamma$. But then $\Gamma \subset \Gamma'$ which leads to a contradiction since $(p, q) \epsilon \Gamma$ and $(p, q) \bar{\epsilon} \Gamma'$. Thus sgn $(u_{B,g}^{pq} - u_{B,g}^{rs}) \neq 0$ so to prove the theorem it is sufficient to show that

$$u_{B,g}^{pq} > u_{B,g}^{rs} \quad \text{implies } e_{pq}^{B} > e_{rs}^{B}.$$

(The second case: $u_{B,g}^{pq} < u_{B,g}^{rs}$ implies $e_{pq}^{B} < e_{rs}^{B}$, follows immediately from that one with the reversed inequality signs if we put (p, q) there instead of (r, s) and vice versa.)

There are two cases:

1° $|u_{B,g}^{pq}| > |u_{B,g}^{rs}|$ and $u_{B,g}^{pq} > u_{B,g}^{rs}$;

2° $|u_{B,g}^{pq}| < |u_{B,g}^{rs}|$ and $u_{B,g}^{pq} > u_{B,g}^{rs}$.

In case 1° $u_{B,g}^{pq}$ is positive so $(p, q) \epsilon \Gamma_1$, $u_{B,g}^{pq} = l$ and $e_{pq} = h^l\delta$. In case 2° $u_{B,g}^{rs}$ is negative so $(p, q) \epsilon \Gamma_2$, $|u_{B,g}^{rs}| = l$ and $e_{rs} = h^l\delta$. In either case

$$\Gamma_1'-\Gamma_1\Gamma_1' \subset (a_1, a_2, \ldots, a_{l-1}), \qquad \Gamma_2-\Gamma_2\Gamma_2' \subset (a_1, a_2, \ldots, a_{l-1}).$$

Therefore

$$\sum_{\Gamma_1'-\Gamma_1\Gamma_1'} e_{ij} \leqslant h\delta+h^2\delta+\ldots+h^{l-1}\delta = \delta\frac{h^l-1}{h-1} \quad \text{and} \quad \sum_{\Gamma_2-\Gamma_2\Gamma_2'} e_{ij} \leqslant \delta\frac{h^l-1}{h-1}.$$

So (see (13))

$$e^B_{pq}-e^B_{rs}\geqslant \sum_{\Gamma_1-\Gamma_1\Gamma_1'} e_{ij}+\sum_{\Gamma_2'-\Gamma_2\Gamma_2'} e_{ij}-2\frac{h^l-1}{h-1}.$$

In case 1° $(p,q)\epsilon\Gamma_1$, $(p,q)\bar{\epsilon}\Gamma_1'$, so $(p,q)\epsilon\Gamma_1-\Gamma_1\Gamma_1'$ and

$$e^B_{pq}-e^B_{rs}\geqslant e_{pq}-2\delta\frac{h^l-1}{h-1}=h^l\delta-2\delta\frac{h^l-1}{h-1}.$$

In case 2° $(r,s)\epsilon\Gamma_2'$, $(r,s)\bar{\epsilon}\Gamma_2$ so $(r,s)\bar{\epsilon}\Gamma_2$ so $(r,s)\bar{\epsilon}\Gamma_2'-\Gamma_2\Gamma_2'$ and

$$e^B_{pq}-e^B_{rs}\geqslant e_{rs}-2\delta\frac{h^l-1}{h-1}=h^l\delta-2\delta\frac{h^l-1}{h-1}.$$

Thus in either case we have

$$e^B_{pq}-e^B_{rs}\geqslant\frac{\delta}{h-1}(h^{l+1}-3h^l+h).$$

But $\delta>0$ and $h\geqslant 3$. Therefore

$$e^B_{pq}-e^B_{rs}\geqslant\frac{\delta}{h-1}(h\cdot h^l-3h^l+h)\geqslant\frac{\delta}{h-1}(3h^l-3h^l+h)=\frac{\delta}{h-1}h>0.$$

This completes the proof of theorem 11.

To see the point of these developments suppose there is given a matrix E satisfying (12) a basis B and a set Ω disjoint with B. Then theorems 10 and 11 imply a procedure by means of which it is possible to find in Ω a cell $(\bar{i},\bar{j})$ which satisfies the condition

$$e^B_{\bar{i}\bar{j}}=\min_{(i,j)\epsilon\Omega} e^B_{ij}. \tag{14}$$

Moreover this may be done while omitting the construction of $E_B=\{e^B_{ij}\}$.

The procedure is described in steps as follows:

PROCEDURE 4.

Step 1. Find

$$\min_{(p,q)\epsilon\Omega} u^{pq}_{B,1}. \tag{15}$$

Let Ω_1 be the set of (p,q) satisfying (15). There are two cases: a) Ω_1 consists of one element, b) Ω_1 consists of more than one element. Let $|u^{pq}_{B,1}|=l$. Consider case a. Then (theorem 11) $a_{i1}=(\bar{i},\bar{j})$ satisfies (14) and the procedure terminates. In case b turn to

Step 2. Repeat step 1 again and again where instead of Ω we have to consider sets $\Omega_1,\Omega_2,\ldots,$ and instead of $u^{pq}_{B,1}$-numbers $u^{pq}_{B,2},u^{pq}_{B,3},\ldots,$ until case a from step 1 arises.

Theorem 10 guarantees the finiteness of procedure 4.

6. A method of finding an optimal basis. One can give many ways to construct an optimal basis given a cost matrix $C = \{c_{ij}\}$. One possible method is as follows:

a. Choose any row of C, say row $\bar{i}$.

b. Construct a new matrix $C' = \{c'_{ij}\} = \{c_{ij}+v_j\}$ where $v_j = -c_{\bar{i}j}$ so that

$$c'_{\bar{i}j} = c'_{\bar{i}j}+v_j = 0 \quad \text{for all} \quad j = 1, \ldots, n.$$

c. Find

$$c'_i = \min_j c'_{ij}, \qquad i = 1, 2, \ldots, \bar{i}-1, \ \bar{i}+1, \ldots, m,$$

and for each $i, i \neq \bar{i}$ one node where c'_i is attained. These $n-1$ nodes together with n nodes of row i form a basis B which is optimal. For only single nodes of B appear in $m-1$ rows of C so G_B contains no cycle. B consists of $m+n-1$ elements and therefore ([4]) B is a basis. To prove optimality of B construct a new matrix $C'' = \{c''_{ij}\} = \{c'_{ij}+u_i\}$ where

$$u_{\bar{i}} = 0, \quad u_i = -c'_i \quad \text{for} \quad i \neq \bar{i}.$$

It is easy to see that C'' has the following properties

$$c''_{ij} = 0 \quad \text{for all} \quad (i, j) \in B,$$

$$c''_{ij} \geqslant 0 \quad \text{for all} \quad (i, j) \bar{\in} B.$$

Two bases B_1 and B_2 are neighboring if they differ by one element: so $B_2 = B_1-(k, l)+(r, s)$. Graph $G_{B_1-(k,l)}$ consists of two connected subgraphs G_{Ω_1} and G_{Ω_2} where Ω_1 and Ω_2 are two disjoint sets (one of these sets may be empty). By Ω_1 we mean either an empty set if (k, l) was the only node of B in column l or a set which contains another a node of B in column l. By I_1 and I_2 denote sets of the numbers of rows occupied by nodes of Ω_1 and Ω_2, respectively, and by J_1 and J_2 denote sets of the numbers of columns occupied by nodes of Ω_1 and Ω_2.

To conclude this section let $I = (1, \ldots, m)$, $J = (1, \ldots, n)$ and introduce sets ψ and ψ' as follows

$$\psi = (I-I_1)\times(J-J_1)-(k, l), \qquad \psi' = I_1\times J_2.$$

Now suppose a basis B_1 containing the node (k, l) is optimal. We define a procedure of finding another optimal basis B_2 which is neighboring to B_1 and does not contain the node (k, l).

PROCEDURE 5. Find a node $(r, s)\in\psi'$ for which $c^{B_1}_{rs} = \min_{(i,j)\in\psi'} c^{B_1}_{ij}$. Then $B_2 = B_1-(k, l)+(r, s)$.

Theorem 5.3 of [4] implies that B_2 is a basis. The optimality of B_2 follows from a) the definition of (r, s), b) formulas (6.3) of [4] which when applying theorems Λ and Λ' of [5] can be written in the following form

$$c_{ij}^{B_2} = c_{ij}^{B_1} - c_{rs}^{B_1} \quad \text{for} \quad (i,j)\in\psi,$$
$$c_{ij}^{B_2} = c_{ij}^{B_1} + c_{rs}^{B_1} \quad \text{for} \quad (i,j)\in\psi', \tag{16}$$
$$c_{ij}^{B_2} = c_{ij}^{B_1} \quad \text{for} \quad (i,j)\in\Phi-(\psi+\psi').$$

7. Solution of the transportation problem by the optimal bases method (OBM). There is given a transportation problem (C, M) and a basis B. By a basic matrix $Y = \{y_{ij}\}$ we mean any $m\times n$ matrix satisfying conditions

$$\sum_{j=1}^{n} y_{ij} = a_i, \quad i = 1,\dots,m,$$
$$\sum_{i=1}^{m} y_{ij} = b_j, \quad j = 1,\dots,n,$$
$$y_{ij} = 0 \quad \text{for all} \quad (i,j)\bar{\in} B.$$

As may be verified (see [3] page 159), to each basis B there exists exactly one basic matrix. Such a matrix we denote by $Y(B) = \{y_{ij}^{B}\}$. We give a procedure of finding $Y(b)$ (see proof of theorem 5.5 in [3]).

PROCEDURE 6. Take a node of B, say (i_1, j_1), which is the only element of B on some line, say on the i-th row (case 1) or on the j_1-th column (case 2). Set

$$y_{i_1j_1} = \begin{cases} a_{i_1} & \text{in case } 1, \\ b_{j_1} & \text{in case } 2. \end{cases}$$

Repeat the same procedure for the set $B-(i_1, j_1)$ and for the system

$$M^1 = \begin{cases} (a_1,\dots,a_{i_1-1}, 0, a_{i_1+1},\dots,a_m; b_1,\dots,b_{j_1-1}, b_{j_1}-a_{i_1}, b_{j+1},\dots,b_n) \\ \hfill \text{in case } 1, \\ (a_1,\dots,a_{i_1-1}, a_{i1}-b_{j_1}, a_{i_1+1},\dots,a_m; b_1,\dots,b_{j_1-1}, 0, b_{j_1+1},\dots,b_n) \\ \hfill \text{in case } 2 \end{cases}$$

and determine $y_{i_2j_2}$. Repeating this $m+n-2$ times find y_{ij} for all $(i,j)\in B$. The remaining y_{ij}, $(i,j)\bar{\in}B$, are set equal to zero.

Suppose we are given an optimal basis B_1 and, neighboring to B_1, an optimal basis $B_2 = B_1-(k,l)+(r,s)$ found by procedure 5. We now provide a procedure for finding $Y(B_2)$ provided $Y(B_1)$ is known.

PROCEDURE 7. Consider $G_{B_1+(r,s)}$. This graph contains exactly one cycle, say G_Γ, and $(r,s)\in\Gamma$. Divide Γ by procedure 1 into Γ_1 and Γ_2 assigning (r,s) to Γ_1 (then also $(k,l)\in\Gamma_1$). Define $Y(B_2)$ as follows:

$$y_{ij}^{B_2} = y_{ij}^{B_1} - y_{kl}^{B_1}, \quad (i,j)\in\Gamma_1,$$
$$y_{ij}^{B_2} = y_{ij}^{B_1} + y_{kl}^{B_1}, \quad (i,j)\in\Gamma_2, \tag{17}$$
$$y_{ij}^{B_2} = y_{ij}^{B_1}, \quad (i,j)\in\Gamma.$$

There are given two basic matrices $Y(B_1)$ and $Y(B_2)$ where B_1 and $B_2 = B_1-(k,l)+(r,s)$ are optimal bases. Hence the following theorem is pertinent:

THEOREM 12. *If* $y_{kl}^{B_1} < 0$ *then* $z_{Y(B_2)} \geqslant z_{Y(B_1)}$.

Proof. Procedure 7 and the proof of theorem 6.3 from [3] imply

$$z_{Y(B_2)} - z_{Y(B_1)} = c_{rs}^{B_1}(-y_{kl}^{B_1}).$$

For $c_{rs}^{B_1} \geqslant 0$ and $y_{kl}^{B_1} < 0$ so $z_{Y(B_2)} \geqslant z_{Y(B_1)}$.

We also have

COROLLARY 5. *If C is of property A and $y_{kl}^{B_1} < 0$ then $z_{Y(B_2)} > z_{Y(B_1)}$. The corollary assumption implies $c_{ij}^{B_1} > 0$ for all $(i,j)\bar{\epsilon} B_1$. Hence also $c_{rs}^{B_1} > 0$ and $z_{Y(B_2)} - z_{Y(B_1)} = c_{rs}^{B_1}(-y_{kl}^{B_1}) > 0$.*

In preparation for the illustrative example discussed in the next section we present the following method of solving the transportation problem (C, M)(2).

Step 1. Find an optimal basis B_1 (for example by the method given in section (6). Using procedure 6 find $Y(B_1)$.

Step 2. Find a node (k,l) such that

$$y_{kl}^{B_1} = \min_{(i,j)\epsilon\Phi} y_{ij}^{B_1}.$$

If $y_{kl}^{B_1} \geqslant 0$ then, $Y(B_1)$ is the optimal solution of the transportation problem. If $y_{kl}^{B_1} < 0$, then proceed to.

Step 3. By procedure 5 find an optimal basis B_2 neighboring to B_1 where $(k,l)\bar{\epsilon} B_2$. Find $Y(B_2)$ by procedure 7. Repeat step 2 again and again until a basic solution is obtained, say $Y(B_u)$, which is the optimal solution of the problem.

Evidently the above solution procedure reduces to finding a sequence of basic matrices

$$Y(B_1),\ Y(B_2),\ \ldots,\ Y(B_t),\ \ldots, \tag{18}$$

with $z_{Y(B_{t+1})} \geqslant z_{Y(B_t)}$, for $t = 1, 2, \ldots$

The procedure terminates as soon as in the sequence (18) a matrix is obtained with all elements nonnegative.

If some basic matrix containing negative elements say $Y(\bar{B})$ appears in (18) twice, then apply

(2) It should be pointed out that this method can be considered as an adaption of the well known G. E. Lemke's dual method [2] of solving the linear programming problem for the transportation case.

Step 4. Consider a transportation problem (W, M) with $W = C+E$ and matrix E defined by (6). We set $h \geqslant 3$ and

$$0 < \delta < \frac{d}{2} \frac{h-1}{h^{mn-1}-h}.$$

Using procedure 2 arrange the elements of Φ in a sequence

$$a_1, \ldots, a_i, \ldots, a_{mn}$$

so that the first $m+n-1$ nodes belong to $\bar{B}$. Perform steps 1, 2 and 3 starting with $Y(\bar{B})$ as the initial basic matrix until the optimal solution of the problem (W, M) is obtained.

The solution obtained in step 4 is also an optimal solution for the problem (C, M) (theorem 6).

REMARK 2. One can considerably simplify the step 4 of the procedure by applying the theory developed in sections 4 and 5.

In each iteration of the above solution procedure we go from $Y(B_t)$ which is already known to $Y(B_{t+1})$. In the process we are required to determine

$$\min_{(i,j)\in\Psi'} w_{ij}^{B_t}.$$

To find the node for which this minimum is attained proceed as follows. Find

$$\min_{(i,j)\in\Psi'} c_{ij}^{B_t}.$$

There are two cases

$$\text{either} \quad \min_{(i,j)\in\Psi'} c_{ij}^{B_t} = c_{rs}^{B_t} > 0, \quad \text{or} \quad \min_{(i,j)\in\Psi'} c_{ij}^{B_t} = 0.$$

In the first case we need only introduce node (r, s) into the new basis B_{t+1}. Therefore consider the second case. Let

$$\min_{(i,j)\in\Psi'} c_{ij}^{B_t} = 0$$

and $\Omega \subset \Psi'$ be the set of nodes (i, j) for which $c_{ij}^{B_t} = 0$. Then we are to find (since $w_{ij}^{B_t} = c_{ij}^{B_t} + e_{ij}^{B_t}$)

$$\min_{(i,j)\in\Omega} e_{ij}^{B_t}.$$

To do this use procedure 4 (see section 5).

To conclude the present discussion we now state and prove

THEOREM 13. *The number of iterations leading from any basic matrix $Y(B)$ where B is an optimal basis to the optimal solution of the transportation problem (C, M) is finite.*

Proof. If in sequence (18) no matrix appears twice then no basis (see [4] p. 159) appears in (18) twice. Thus this sequence is finite because the number of all optimal bases is finite (less then $m^{n-1}\, n^{m-1}$ (see [3]). Suppose now that some basic matrix appears twice. We then use step 4 considering the problem (W, M) where W is of property A for $h \geqslant 3$ and

$$0 < \delta < \frac{d}{2} \cdot \frac{h-1}{h^{mn+1}-h}$$

(theorem 5). Suppose we get the following sequence of basic matrices of the (W, M) problem.

$$(*) \qquad Y(\bar{B}_1),\ Y(\bar{B}_2),\ \ldots,\ Y(\bar{B}_t),\ \ldots,.$$

Then (corollary 5)

$$(**) \qquad z_{Y(\bar{B}_1)} < z_{Y(\bar{B}_2)} < \ldots < z_{Y(\bar{B}_t)} < \ldots$$

From (**), and also from the fact that to each basic matrix there corresponds exactly one value of z (see [4]), it follows that no matrix and therefore no basis appears in (*) twice. For the same reasons as before we conclude that (*) is a finite sequence.

8. An Example. Consider a 3×5 transportation problem with the following cost matrix

8	13	13	9	14	10
12	17	16	12	11	3
9	16	18	13	7	7
4	3	5	6	2	

The numbers a_i and b_j are on the right and below the cost matrix. To find an initial optimal basis we use the method from section 6.

Choose the first column, circle all the elements in that column and add numbers -8, -12, -9 to the first, second and third row, respectively. This produces the following matrix

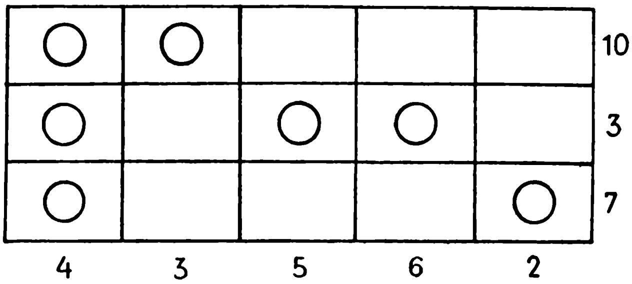
0	5	5	1	6	10
0	5	4	0	-1	3
0	7	9	4	-2	7
4	3	5	6	2	

Minimal elements in the remaining columns are 5, 4, 0, -2. We therefore circle one minimal element in each column. Substracting them from the corresponding columns we get the following matrix

0	0	1	1	8	10
0	0	0	0	1	3
0	2	5	4	0	7
4	3	5	6	2	

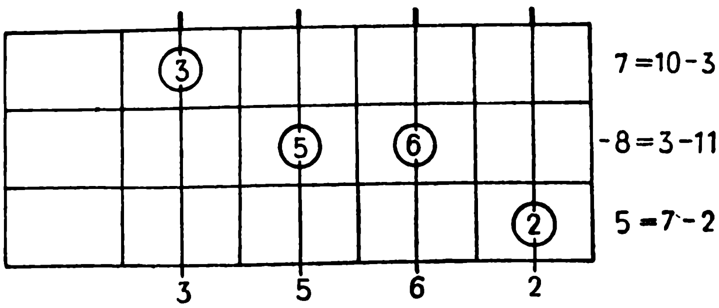
This gives $B_1 = \{(1,1),\ (1,2),\ (2,1),\ (2,3),\ (2,4),\ (3,1),\ (3,5)\}$. It is easy to see that the above matrix is c_{B_1}.

Find $Y(B_1)$ using some modification of procedure 6.



First determine elements $y_{ij}^{B_1}$ for the circled nodes which are on separate lines. Thus, by following this instruction we get

$$y_{12}^{B_1} = 3, \quad y_{23}^{B_1} = 5, \quad y_{24}^{B_1} = 6, \quad y_{35}^{B_1} = 2.$$



With these values specified we then readily find the remaining $y_{ij}^{B_1}$ in the circled nodes to be $y_{11}^{B_1} = 7$, $y_{21}^{B_1} = -8$, $y_{31}^{B_1} = 5$. All remaining $y_{ij}^{B_1}$ are zeros. This yields

7	3			
-8		5	6	
5				2

For convenience we now write C_{B_1} and $Y(B_1)$ in the same table, as below

0^{7}	0^{3}	$\boxed{1}$	1	8
v 0^{-8}	0	0^{5}	0^{6}	1
0^{5}	2	5	4	0^{2}

Using step 2 we find $\min\limits_{(i,j)\in\Phi} y_{ij}^{B_1} = y_{21}^{B_1} = -8 < 0$, and therefore proceed to step 3. Consider graph $G_{B_1-(2,1)}$. This graph consists of two subgraphs G_{Ω_1} and G_{Ω_1} where $\Omega_1 = \{(1,1), (1,2), (3,1), (3,5)\}$ and $\Omega_2 = \{(2,3), (2,4)\}$. Therefore $\Psi' = I_1 \times J_2 = \{(1,3), (1,4), (3,3), (3,4)\}$ (cells which are on the intersection obtained by extending the "arrowed" lines). Thus we find $\min\limits_{(i,j)\in\Psi'} c_{ij}^{B_1} = c_{13}^{B_1} = 1$ and so $B_2 = B_1-(2,1)+(1,3)$.

Consider $G_{B_1+(1,3)}$. This graph contains a cycle G_Γ where

0^{7}	0	$\boxed{1}$	1	8
v 0^{-8}	0	0^{5}	0^{6}	1
0^{5}	2	5	4	0^{2}

$\Gamma = \{(1,1), (1,3), (2,1), (2,3)\}$. Using procedure 7 divide Γ into Γ_1 and Γ_2 assigning $(1,3)$ to Γ_1. So $\Gamma_1 = \{(1,3), (2,1)\}$, $\Gamma_2 = \{(1,1), (2,3)\}$. Applying (17) find $Y(B_2)$

$$
\begin{aligned}
y_{ij}^{B_2} &= y_{ij}^{B_1}+8, && (i,j)\in\Gamma_1,\\
y_{ij}^{B_2} &= y_{ij}^{B_1}-8, && (i,j)\in\Gamma_2,\\
y_{ij}^{B_2} &= y_{ij}^{B_1}, && (i,j)\bar{\in}\Gamma.
\end{aligned}
$$

			-1	-1	
	0^{-1}	0^{3}	$\boxed{1}^{8}$	1	8
1	0	0	V 0^{-3}	0^{6}	1
	0^{2}	2	5	4	0^{2}

To find C_{B_2} apply formula (6.2) from [4]. Adding: -1 to the third column, 1 to the second row and -1 to the fourth column we get C_{B_2}

				↓	
→	0^{-1}	0^{3}	0^{8}	$\boxed{0}$	8
	1	1	V 0^{-3}	0^{6}	2
→	0^{5}	2	4	3	0^{2}

REMARK 3. To find C_{B_2} one can also apply formula (16). In any event $Y(B_2)$ is not the optimal solution of the problem for

$$y_{23}^{B_2} \min_{(i,j)\in\Phi} y_{ij}^{B_2} = -3 .$$

We therefore have to consider $G_{B_2-(2,3)}$. Here $\Psi' = \{(1,4),(3,4)\}$ and $\min_{(i,j)\in\Psi'} c_{ij}^{B_2} = c_{14}^{B_2} = 0$. Therefore $B_3 = B_2-(2,3)+(1,4)$. Graph $G_{B_2+(1,4)}$ contains a cycle G_Γ where $\Gamma = \{(1,3),(1,4),(2,3),(2,4)\}$. So, as before, divide Γ into Γ_1 and Γ_2. Here $\Gamma_1 = \{(1,4),(2,3)\}$, $\Gamma_2 = \{(1,3),(2,4)\}$ and $Y(B_3)$ is defined as follows: $y_{14}^{B_3} = 0+3 = 3$, $y_{23}^{B_3} = -3+3 = 0$, $y_{13}^{B_3} = 8-3 = 5$, $y_{24}^{B_3} = 6-3 = 3$, $y_{ij}^{B_3} = y_{ij}^{B_2}$ for all remaining (i,j).

		↓	↓	↓	
	V 0^{-1}	0^{3}	0^{5}	0^{3}	8
	1	1	0	0^{3}	2
→	0^{5}	$\boxed{2}$	4	3	0^{2}

Here $C_{B_2} = C_{B_3}$. For $\min_{(i,j)\in\Phi} y_{ij}^{B_3} = y_{11}^{B_3} = -1$ so $\Psi' = \{(3,2),(3,3),(4,3)\}$ and $\min_{(i,j)\in\Psi} c_{ij}^{B_3} = c_{32}^{B_3} = 2$. Therefore $B_4 = B_3-(1,1)+(3,2)$.

	2				2
	0	0^2	0^5	0^3	8
	1	1	0	0^3	2
−2	0^4	$\boxed{2}^1$	4	3	0^2

Graph $G_{B_3+(3,2)}$ contains a cycle G_Γ where $\Gamma = \{(1,1), (1,2), (3,1), (3,2)\}$. Here $\Gamma_1 = \{(1,1), (3,2)\}$, $\Gamma_2 = \{(1,2), (3,1)\}$ so $y_{11}^{B_4} = -1+1 = 0$, $y_{32}^{B_4} = 0+1 = 1$, $y_{12}^{B_4} = 3-1 = 2$, $y_{31}^{B_4} = 5-1 = 4$, $y_{ij}^{B_4} = y_{ij}^{B_3}$ for all remaining (i, j). Adding -2 to the third row and 2 to the first and fifth columns we get C_{B_4}

2	0^2	0^5	0^3	10
3	1	0	0^3	4
0^4	0^1	2	1	0^2

Here

$Y(B_4) =$

	2	5	3	
			3	
4	1			2

As can be readily observed $Y(B_4)$ has no negative element. Hence this matrix is an optimal solution of the transportation problem.

It is perhaps worthwhile to observe also,

REMARK 4. Consider the following situation. There are given two transportation problems (C, M) and (C, M') where the elements of M differ only slightly from the corresponding elements of M'. Suppose that an optimal solution of (C, M), say $X(B)$, is available. Then to solve (C, M') it is especially useful to apply the OBM, starting with B as an initial optimal basis.

9. Sets of optimal bases. If an $m \times n$ matrix C with property A is given, we can then prove

THEOREM 14. *For each pair of optimal bases B_0 and B of C there exists a sequence*

$$B_0, B_1, B_2, \ldots, B$$

where consecutive elements are neighboring optimal bases.

Proof. Construct a matrix $\bar{X} = \{\bar{x}_{ij}\}$ as follows

$$\begin{aligned} \bar{x}_{ij} &= a > 0 \quad \text{for all} \quad (i,j)\in B, \\ x_{ij} &= 0 \quad \text{for all} \quad (i,j)\bar{\in} B, \end{aligned}$$

where a is some positive number. Consider a transportation problem (C, M) where the elements a_i and b_j of M are found from formulas

$$\begin{aligned} a_i &= \sum_{j=1}^{m} \bar{x}_{ij}, \quad i = 1, \ldots, m, \\ b_j &= \sum_{i=1}^{n} \bar{x}_{ij}, \quad j = 1, \ldots, n. \end{aligned}$$

Since all $\bar{x}_{ij} \geqslant 0$ and B is an optimal basis it follows that $\bar{X}$ is an optimal solution of the transportation problem (C, M). Now solve the transportation problem (C, M) by the OBM starting with B_0 as an initial optimal basis. According to corollary 5 and theorem 13 we will get the optimal solution after a finite number of iterations. For $\bar{X}$ is the only (because C has property A) optimal solution of (C, M) so the iterative procedure produces a sequence

$$Y(B_0), Y(B_1), \ldots, Y(B) = \bar{X}.$$

This completes the proof.

It is now convenient to introduce some definitions. Use $N(C)$ to denote the number of all optimal bases of C. Let $n_k(C)$ be the number of optimal bases of C, having at least two nodes in k rows of matrix C.

If C is a $m\times 1$ matrix then

$$n_k(C) = \begin{cases} 1 & \text{for} \quad k = 0, \\ 0 & \text{for} \quad k \geqslant 1. \end{cases}$$

If C is a $m\times n$ matrix with $n \geqslant 2$ then

$$n_k(C) = \begin{cases} 0 & \text{for} \quad k = 0, \\ 0 & \text{for} \quad k > \min(m, n-1). \end{cases}$$

Therefore the following formula is true

$$N(C) = \sum_{k=0}^{\min(m,n-1)} n_k(C). \tag{19}$$

Now let C_1 and C_2 be two arbitrary $m \times n$ matrices of property A where m and n are arbitrary integers. The following theorem holds.

THEOREM 15. $n_k(C_1) = n_k(C_2)$ *for all* $k = 0, 1, 2, \ldots$

Proof. We prove the theorem by induction. Since the theorem is true for $k = 0$ and $k > \min(m, n-1)$, we need only prove it for $0 < k \leqslant \min(m, n-1)$. Suppose that the theorem holds for all $k, k < l$ where $1 \leqslant l \leqslant \min(m, n-1)$. Let $I = (1, \ldots, m)$. By I_l denote a subset of I consisting of l elements and by $C(I_l)$ and $l \times n$ submatrix of C consisting of rows whose indices belong to I_l. Let Π be an arbitrary optimal basis of $C(I_l)$ containing in each row of $C(I_l)$ at least two nodes. Then there always exists exactly one optimal basis B of C with the following properties. 1° $\Pi \subset B$. 2° in each row $i \in I - I_l$ there is exactly one node of B. To show this let us determine $C_\Pi(I_l)$ (zero matrix for Π). Using formulas (6.2) from [4] find numbers u_i and v_j, $i \in I_l$, $j = 1, \ldots, n$. Add these numbers to the corresponding lines of C. We get a matrix $C' = c'_{ij}$ where

$$c'_{ij} = c_{ij} + u_i + v_j \quad \text{for} \quad i \in I_l, \quad j = 1, 2, \ldots, n;$$

$$c'_{ij} = c_{ij} + v_j \quad \text{for} \quad i \in I_l, \quad j = 1, 2, \ldots, n$$

and, furthermore

$$c'_{ij} = 0 \quad \text{for all} \quad (i, j) \in \Omega.$$

Find for each $i \in I - I_l$

$$\min_{1 \leqslant j \leqslant n} c'_{ij} = c'_i$$

and choose one node (it is unique as we later show) where this minimum is attained. Thus we have chosen $m - l$ cells which form a set, say Π'. It is easy to see that $\Pi + \Pi' = B$ is a basis. To show that B is an optimal basis form a matrix C'' as follows

$$c''_{ij} = c'_{ij} - c'_i \quad \text{for} \quad i \in I - I_l,$$

$$c''_{ij} = c'_{ij} \quad \text{for} \quad i \in I_l.$$

Obviously C'' is a zero matrix for B. For C is of property A so C'' has exactly $m+n-1$ elements equal zero. From this we conclude that for each $i \in I - I_l$ there exists exactly one node where $\min\limits_{1 \leqslant j \leqslant n} c'_{ij}$ is attained. Now we are justified in writing

$$n_l(C) = \sum_{I_l \subset I} n_l[C(I_l)].$$

Applying (19) find $N[C(I_l)]$ using the fact that $C(I_l)$ is an $l\times n$ matrix with $l<\min(m, n-1)$. We thus have

$$N[C(I_l)] = \sum_{k=0}^{l} n_k[C(I_l)].$$

Therefore

$$n_l[C(I_l)] = N[C(I_l)] - \sum_{k=0}^{l-1} n_k[C(I_l)].$$

Let D be a rectangular matrix and D^T its transpose. Obviously $N(D) = N(D^T)$. Therefore

$$N[C(I_l)] = N[C^T(I_l)] = \sum_{k=0}^{\min(n,l-1)} n_k[C^T(I_l)].$$

Applying this we find that

$$n_l[C(I_l)] = \sum_{k=0}^{\min(n,l-1)} n_k[C^T(I_l)] - \sum_{k=0}^{l-1} n_k[C(I_l)].$$

But $l\leqslant n-1$ so

$$n_l[C(I_l)] = \sum_{k=0}^{l-1} n_k[C^T(I_l)] - \sum_{k=0}^{l-1} n_k[C(I_l)].$$

From the inductive assumption we have

$$n_k[C_1^T(I_l)] = n_k[C_2^T(I_l)]$$

and

$$n_k[C_1(I_l)] = n_k[C_2(I_l)]$$

for all $k = 0, 1, \ldots, l-1$. Therefore

$$n_l[C_1(I_l)] = n_l[C_2(I_l)].$$

Theorem 15 and formula (19) imply

COROLLARY 6. $N(C_1) = N(C_2)$.

10. Number of optimal bases. There is given an $m\times n$ matrix C of property A. We are to find $N(C)$. Suppose we know the number, say t, of optimal bases of some other $m\times n$ matrix of property A. Then applying corollary 6 we have

$$N(C) = t.$$

First introduce the following definition. By a Dantzig basis we mean a basis B such that the links in G_B form a nonincreasing step line conecting the nodes $(1, 1)$ and (m, n).

Now consider a matrix $D = d_{ij}$ defined as follows

$$d_{ij} = 2^k, \quad k = 1, 2, \ldots, mn$$

for nodes (i, j) ordered diagonally beginning from the southwest corner. In the illustration below this convention has been followed for a 4×6 matrix.

2^7	2^{11}	2^{15}	2^{19}	2^{22}	2^{24}
2^4	2^8	2^{12}	2^{16}	2^{20}	2^{23}
2^2	2^5	2^9	2^{13}	2^{17}	2^{21}
2^1	2^3	2^6	2^{10}	2^{14}	2^{18}

For any such matrix the following theorem holds

THEOREM 16. *Matrix D is of property A.*

Proof. Consider an arbitrary cycle, say G_Γ, and divide Γ be procedure 1 into Γ_1 and Γ_2 assigning to Γ_1 the greatest element $d_{\bar{i}\bar{j}} = 2^k$ of Γ. Since

$$\sum_{(i,j)\epsilon\Gamma_1} d_{ij} > d_{\bar{i}\bar{j}} = 2^k > \sum_{p=1}^{k-1} 2^p > \sum_{(i,j)\epsilon\Gamma_2} d_{ij}$$

so

$$d(\Gamma) > 0$$

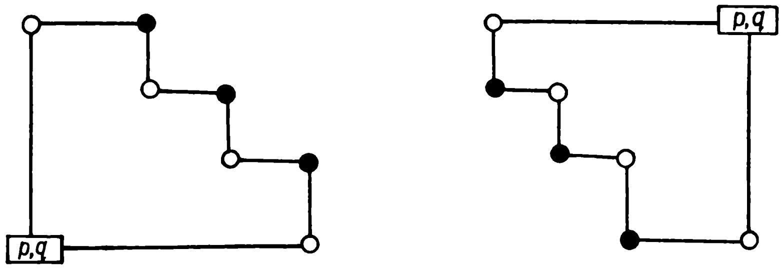
and due to theorem 1 matrix D is of property A. Next we prove the following.

THEOREM 17. *B is an optimal basis of D if and only if it is a Dantzig basis.*

Proof. Let B a Dantzig basis. To prove its optimality we will show that $d_{ij}^B > 0$ for all $(i, j)\bar{\epsilon}B$. Consider an arbitrary node $(p, q)\bar{\epsilon}B$. Graph $G_{B+(p+q)}$ contains exactly one cycle, say G_Γ. Divide Γ by procedure 1 into Γ_1 and Γ_2 assigning (p, q) to Γ_1. Then ([4] page 163)

$$d_{pq}^B = \sum_{\Gamma_1} d_{ij} - \sum_{\Gamma_2} d_{ij}.$$

Cycle G_Γ may be of two forms which may be schematically depicted as follows



(a) (b)

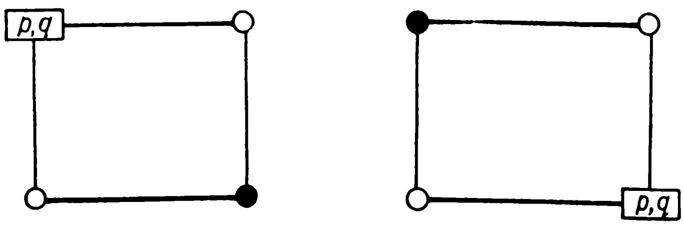
In either case cell (p, q) and all black cells belong to Γ_1. From the definition of D it follows that the greatest element d_{ij} in Γ belong to Γ_1 (in case (a) some element in the black cells; in case (b) the element d_{pq}). Therefore

$$\sum_{\Gamma_1} d_{ij} > \sum_{\Gamma_2} d_{ij} \quad \text{which implies} \quad d^B_{pq} > 0 .$$

Let us suppose now that we could have an optimal basis B which is not a Dantzig basis. Such a B must then contain three nodes located as follows

(a) or (b)

Consider the node, say (p, q), which is on the fourth corner of the "rectangle"



Obviously $(p, q) \bar{\in} B$ (otherwise G_B contains a cycle) and $G_{B+(p,q)}$ contains exactly one cycle G_Γ where Γ consists of corners of the "rectangle". Divide Γ by procedure 1 into Γ_1 and Γ_2 assigning (p, q) to Γ_1. From the definition of D it follows that in either case the greatest element d_{ij} in Γ is on the upper right corner of the rectangle. Therefore

$$\sum_{\Gamma_1} d_{ij} < \sum_{\Gamma_2} d_{ij} \quad \text{which implies} \quad d^B_{pq} < 0 .$$

This contradicts the assumption that B is an optimal basis. Therefore B is a Dantzig basis, and the proof of the theorem is complete.

We next also prove the following.

THEOREM 18. *The number of Dantzig bases for a* $m\times n$ *matrix is equal to* $\binom{m+n-2}{m-1}$.

Proof. Let B be any arbitrary Dantzig basis. Links in G_B form a step line linking points $(1,1)$ and (m,n). This can be uniquely described by an ordered sequence of $m+n-2$ numbers consisting of $n-1$ numbers 0 and $m-1$ numbers 1. Here 0 corresponds to a horizontal and 1 to a vertical segment of the step line. But the number of different sequences of $m+n-2$ elements with $n-1$ zeros and $m-1$ one is $\binom{m+n-2}{m-1}$. Hence the number of different Dantzig bases as it was to be proved.

Theorems 15, 16, 17 and 18 imply the following.

COROLLARY 7. *Let* C *be an arbitrary* $m\times n$ *matrix of property* A. *Then the number of optimal bases of* C *equals*

$$\binom{m+n-2}{m-1}.$$

We prove the following.

THEOREM 19. *Let* N_{opt} *be the number of optimal bases of an arbitrary* $m\times n$ *matrix. Then*

$$\binom{m+n-2}{m-1}\leqslant N_{\text{opt}}\leqslant m^{n-1}n^{m-1}.$$

Proof. The left inequality follows from corollary 7 and theorem 6. The right inequality follows from the fact that the number of all transportation bases is equal to $m^{n-1}n^{m-1}$. (see [3]).

We also insert

REMARK 5. For $m\geqslant 3$, consider a $m\times n$ matrix C defined as follows

$$c_{ij}=0;$$
$$c_{ij}=\begin{cases} c_{i-1,j}+c_{i,j+1}+1 & \text{for} \quad i>j,\\ c_{i,j-1}+c_{i+1,j}+1 & \text{for} \quad 1<j.\end{cases}$$

This matrix does not have the property A. Take, for example, the cycle G_Γ where $\Gamma=\{(1,2),\ (1,3),\ (2,3),\ (2,1),\ (3,1),\ (3,2)\}$. Divide Γ be procedure 1 into Γ_1 and Γ_2, assigning (1,2) to Γ_1. Then $\sum_{\Gamma_1} c_{ij}=1+1+3$ $=\sum_{\Gamma_2} c_{ij}=3+1+1$. One can easily show (the proof is quite similar to the proof of theorem 17) that each optimal basis of C is a Dantzig basis and *vice versa.* Therefore

$$N_{\mathrm{opt}} = \binom{m+m-2}{m-1}$$

which shows that property A is not a necessary condition for a matrix C to have the minimal number of optimal bases.

On the other hand take a matrix $C' = \{c'_{ij}\}$ with $c'_{ij} = u_i + v_j + p$ where u_i, v_j and p are arbitrary constants for $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then each basis of C' is an optimal basis and therefore

$$N_{\mathrm{opt}} = m^{n-1} n^{m-1}.$$

It is interesting to note that for any transportation problem (C, M) the number of feasible bases(³), say N_{feas}, satisfies the following relation(⁴)

$$N_{\mathrm{feas}} \geqslant \min(m, n)^{\max(m,n)-1}.$$

11. The primal transportation method and the OBM. There is given a transportation problem (C, M). By β_f denote the set of feasible bases of (C, M) and by β_0 the set of optimal bases of C.

The primal method of solving the problem (C, M) consists in finding a sequence of basic feasible solutions

$$X(B'_1), X(B'_2), \ldots, X(B'_k)$$

where $B'_1, \ldots, B'_k$ are elements of β_f and only B'_k belongs also to β_0. Corresponding values of the cost function satisfy the following relation

$$z_{X(B'_1)} \geqslant z_{X(B'_2)} \geqslant \ldots \geqslant z_{X(B'_k)}.$$

The OBM consists in constructing a sequence of basic matrices

$$Y(B_1), Y(B_2), \ldots, Y(B_l)$$

with

$$z_{Y(B_1)} \leqslant z_{Y(B_2)} \leqslant \ldots \leqslant z_{Y(B_l)}.$$

Here $B_1, B_2, \ldots, B_l$ are elements of β_0 and only $B_l \in \beta_f$.

Obviously $z_{X(B'_k)} = z_{Y(B_l)}$. So we can state the following theorem: *For each transportation problem* (C, M) *the following equation holds*

$$\min_{B \in \beta_f} z_{X(B')} = \max_{B \in \beta_0} z_{Y(B)}.$$

This theorem is reminiscent of the well known duality theorem in linear programming. It also implies that both sets β_f and β_0 have at least

(³) B is a feasible basis if all elements y^B_{ij} of $Y(G)$ are nonnegative.

(⁴) F. Nožička in a private communication to the second author, October 1964, has indicated that this result was found by M. Fiedler and others.

one common element, i. e. a basis which is feasible and optimal. So the primal method enables to choose among elements of a finite set β_f an element which belongs to β_0. The OBM consists in choosing from β_0 an element belonging to β_f.

Of course, we do not, in general, know the number of elements in β_0 and β_f but only lower bounds for N_{opt} and N_{feas}. It is easy to show that the lower bound of N_{opt} is considerably smaller than that of N_{feas} for arbitrary values of m and n. This, however, does not imply that the number of iterations involved in using OBM is smaller than the number of iterations when solving the (C, M) problem by the primal method. Nevertheless there exists several cases (see remark 4, section 7) when recourse to OBM is especially useful.

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OPTYMALNE BAZY TRANSPORTOWE

STRESZCZENIE

Zagadnienie transportowe da się sformułować następująco: Dany jest układ (C, M), gdzie $C = \{c_{ij}\}$ jest prostokątną macierzą typu $m \times n$, zwaną macierzą kosztów o elementach rzeczywistych, a $M = (a_1, \ldots, a_m; b_1, \ldots, b_n)$ jest układem $m+n$ liczb dodatnich a_i, b_j, przy czym $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Należy znaleźć macierz $X = \{x_{ij}\}$ typu $m \times n$, której elementy spełniają dla wszystkich $i = 1, \ldots, m$, $j = 1, \ldots, n$ warunki

$$x_{ij} \geqslant 0, \tag{1}$$

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad \sum_{i=1}^{m} x_{ij} = b_j \tag{2}$$

i dla których funkcja

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} = z_X \tag{3}$$

osiąga wartość najmniejszą.

W rozdziale 2 zdefiniowane są takie pojęcia, jak graf, cykl, baza, macierz zerowa i macierz bazowa. Wiadomo, że do każdej bazy B istnieje:

a) dokładnie jedna macierz zerowa, tzn. taka macierz $C' = \{c'_{ij}\}$, gdzie $c'_{ij} = c_{ij}+u_i+v_j (u_i, v_j$ — stałe) i spełniają warunki $c'_{ij} = 0$ dla wszystkich $(i, j) \in B$. Taką macierz będziemy oznaczać przez $C_B = \{c^B_{ij}\}$;

b) dokładnie jedna macierz bazowa, tzn. macierz $Y = \{y_{ij}\}$, której elementy spełniają (2) i ponadto $y_{ij} = 0$ dla wszystkich $(i, j) \notin G$. Taką macierz oznaczymy symbolem $Y_B = \{y^B_{ij}\}$.

Mówimy, że C ma własność A, jeśli do każdej bazy B

$$c^B_{ij} \neq 0 \quad \text{dla wszystkich} \quad (i, j) \notin B.$$

Baza B jest bazą optymalną dla C, jeśli

$$c^B_{ij} > 0 \quad \text{dla wszystkich} \quad (i, j) \notin B.$$

W pracy podana jest metoda rozwiązania zagadnienia transportowego oparta na bazach optymalnych, zwana dalej metodą baz optymalnych (OBM). Jest ona adaptacją metody Lemkego [2]. Początkowa wersja tej metody została opublikowana w [1]. Posługując się tą metodą w pracy rozwiązano przykład liczbowy (rozdział 7). Metoda baz optymalnych polega na wyznaczeniu ciągu macierzy bazowych

$$Y(B_1), (Y(B_2), \ldots, Y(B_k), \tag{$*$}$$

dla których

$$z_{Y(B_1)} \leqslant z_{Y(B_2)} \leqslant \ldots \leqslant z_{Y(B_k)}$$

i gdzie $B_1, B_2, \ldots, B_k$ są bazami optymalnymi, a $Y(B_k)$ jest optymalnym rozwiązaniem zagadnienia transportowego. Każde dwie kolejne bazy występujące w ciągu (*) różnią się jednym elementem. W rozdziałach 6 i 7 podana jest metoda wyznaczania B_1 i $Y(B_1)$. Rozdziały 3, 4, i 5 zawierają teorię pewnej techniki perturbacyjnej, której zastosowanie gwarantuje, że rozwiązując OBM-em, uzyskamy rozwiązanie optymalne zagadnienia transportowego po skończonej ilości kroków. Rozwiązujemy zamiast problemu (C, M) inne zagadnienie transportowe (W, M), gdzie $W = C+E$, a macierz E jest określona wzorem (6). Obie macierze E i W mają własność A (twierdzenia 2 i 5) i każda optymalna baza dla W jest optymalną bazą dla C (twierdzenie 6). Twierdzenia

3, 4, 7, 8, 9, 10 i 11 podają inne ważne własności macierzy E i W, umożliwiające znaczne uproszczenia przy rozwiązywaniu zagadnienia (W, M) metodą baz optymalnych.

W rozdziałach 8 i 9 podano pewne własności zbioru baz optymalnych. Do każdych dwóch baz optymalnych B_0 i B macierzy C, która ma własność A, istnieje ciąg baz optymalnych

$$B_0, B_1, B_2, \ldots, B,$$

gdzie każde dwie kolejne bazy różnią się jednym elementem (twierdzenie 14).

Twierdzenia 15, 16, 17 i 18 implikują następujący wniosek (wniosek 7): Ilość baz optymalnych dla macierzy typu $m\times n$ o własności A jest równa $\binom{m+n-2}{m-1}$.

Z twierdzeń 6 i 18 wynika następujące twierdzenie 19: Liczba N_{opt} baz optymalnych dla dowolnej macierzy typu $m\times n$ spełnia warunek

$$\binom{m+n-2}{m-1} \leqslant N_{\text{opt}} \leqslant m^{n-1}n^{m-1}.$$

W końcowym, dziesiątym, rozdziale pracy omówiono pewne różnice między klasyczną metodą transportową a metodą baz optymalnych.

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ОПТИМАЛЬНЫЕ ТРАНСПОРТНЫЕ БАЗИСЫ

РЕЗЮМЕ

Транспортная задача формулируется следующим образом: Задана система (C, M), где $C = \{c_{ij}\}$, матрица порядка $m\times n$, элементы которой действительны, а $M = (a_1, \ldots, a_m; b_1, \ldots, b_n)$ есть система $m+n$ положительных чисел a_i, b_j, причем $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. Следует определить матрицу $X = \{x_{ij}\}$ порядка $m\times n$ элементы которой удовлетворяют для $i = 1, \ldots, m; j = 1, \ldots, n$ условиям

(1) $$x_{ij} \geqslant 0,$$

(2) $$\sum_{j=1}^{n} x_{ij} = a_i,$$

(2) $$\sum_{i=1}^{m} x_{ij} = b_j$$

и минимизируют функцию

(3) $$z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} = z_X.$$

Во второй главе введены такие понятия, как граф, цикл, базис, нулевая матрица, базисная матрица. Известно, что для всякого базиса B существует

а) точно одна нулевая матрица, т.е. такая матрица $C' = \{c'_{ij}\}$, где $c'_{ij} = c_{ij} + u_i + v_j$ (u_i, v_j — константы) и выполняются условия $c'_{ij} = 0$ для всех $(i, j) \epsilon B$. Такую матрицу обозначим через $C_B = \{c^B_{ij}\}$;

б) точно одна базисная матрица, т.е. такая матрица $Y = \{y_{ij}\}$, элементы которой выполняют (2), а также $y_{ij} = 0$ для всех $(i, j) \bar{\in} B$. Такую матрицу обозначим $Y_B = \{y_{ij}^B\}$.

Скажем, что C обладает свойством A, если для каждого базиса B

$$c_{ij}^B \neq 0 \quad \text{для всех} \quad (i, j) \bar{\in} B.$$

Базис B есть оптимальный базис для C, если

$$c_{ij}^B > 0 \quad \text{для всех} \quad (i, j) \bar{\in} B.$$

В работе приведен метод решения транспортной задачи опирающийся на оптимальные базисы. Этот метод будем в дальнейшем называть методом оптимальных базисов (OBM). Начальная версия этого метода опубликована в [1]. В работе решен этим методом численный пример (глава 7). Метод оптимальных базисов (являющийся адаптацией метода Лемке [2]) состоит в определении ряда базисных матриц

$$(*) \qquad Y(B_1),\ Y(B_2),\ \ldots,\ Y(B_k)$$

выполняющих условия

$$z_{Y(B_1)} \leqslant z_{Y(B_2)} \leqslant \ldots \leqslant z_{Y(B_k)},$$

где $B_1, B_2, \ldots, B_k$ суть оптимальные базисы, причем $Y(B_t), t = 1, \ldots, k-1$ имеет по крайней мере один отрицательный элемент, а $Y(B_k)$ — оптимальное решение транспортной задачи. Каждые два очередные базисы ряда (*) отличаются одним элементом. В шестой и седьмой главах приведен метод определения B_1 и $Y(B_1)$. Третья, четвертая и пятая главы заключают в себе теорию одной пертурбационной техники, применение которой гварантирует получение методом оптимальных базисов оптимального решения транспортной задачи после конечного числа шагов. Вместо проблемы (C, M) решаем другую проблему (W, M), причем $W = C+E$, а E выполняет (6). Матрицы E и W обладают свойством A (теоремы 2 и 5) и каждый оптимальный базис для W является оптимальным базисом для C (теорема 6). Теоремы 3, 4, 7, 8, 9, 10 и 11 содержат другие важные свойства матриц E и W, позволяющие значительно упростить решение проблемы (W, M) методом оптимальных базисов.

В восьмой и девятой главах приведены некоторые свойства множества оптимальных базисов. Для всяких двух оптимальных базисов B_0 и B матрицы C обладающей свойством A существует ряд оптимальных базисов для C

$$B_0, B_1, B_2, \ldots, B,$$

причем каждые два очередные базисы отличаются одним элементом (теорема 14).

Из теорем 15, 16, 17, и 18 вытекает следствие 7: число оптимальных базисов для матрицы порядка $m \times n$ обладающей свойством A равно $\binom{m+n-2}{m-1}$.

Из теорем 6 и 18 следует теорема 19. Число N_{opt} оптимальных базисов для всякой матрицы порядка $m \times n$ выполняет условие

$$\binom{m+n-2}{m-1} \leqslant N_{\mathrm{opt}} \leqslant m^{n-1} n^{m-1}.$$

В последней, десятой главе обсуждаются некоторые различия между прямым методом решения транспортной задачи а методом оптимальных базисов.