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CYCLIC RANDOM INEQUALITIES

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables and let

(1) $$\xi_i = P(X_i < X_{i+1} < X_{i+2}) \quad (i = 1, 2, \ldots, n)$$

where indices in (1) are considered modulo n. Denote

(2) $$\varkappa_n = \sup\min(\xi_1, \xi_2, \ldots, \xi_n).$$

The supremum in (2) is taken over the set of all random variables $X = (X_1, \ldots, X_n)$ with independent components satisfying (1). The main aim of the paper is to prove the following

THEOREM. *For* $n = 1, 2, \ldots$

(3) $$\frac{1}{2} - \frac{6}{n(n+4)} \leqslant \varkappa_n \leqslant \frac{1}{2}.$$

The investigation concerning random inequalities was initiated by Hugo Steinhaus in [2]. About other results of this kind see also [1], [3], [4].

To prove the theorem we need some lemmas that may be of some independent interest. To state the first one the definition of p-quantile will be necessary. The number $x^{(p)}$ will be called a *p-quantile* of the random variable X if

$$P(X \leqslant x^{(p)}) \geqslant p \quad \text{and} \quad P(X \geqslant x^{(p)}) \geqslant 1-p.$$

LEMMA 1. *Let X, Y, Z be independent random variables. If $P(X < Y < Z) > a$ and $pP(Y < z^{(q)}) + (1-q)P(Y \geqslant z^{(q)}) \leqslant a$ then $x^{(p)} < z^{(q)}$.*

Proof. Let $pP(Y < z^{(q)}) + (1-q)P(Y \geqslant z^{(q)}) \leqslant a$ and suppose that $x^{(p)} \geqslant z^{(q)}$. Then $P(X < Y < Z, X \geqslant x^{(p)}, Z \leqslant z^{(q)}) = 0$ and

$$\begin{aligned} P(X < Y < Z) &= P(X < Y < Z, X < x^{(p)}, Z \leqslant z^{(q)}) + \\ &\quad + P(X < Y < Z, X < x^{(p)}, Z > z^{(q)}) + \\ &\quad + P(X < Y < Z, X \geqslant x^{(p)}, Z > z^{(q)}) \\ &\leqslant P(X < x^{(p)})P(Y < z^{(q)})P(Z \leqslant z^{(q)}) + \\ &\quad + P(X < x^{(p)})P(Z > z^{(q)}) + P(X \geqslant x^{(p)})P(Y > x^{(p)})P(Z > z^{(q)}) \end{aligned}$$

$$
\begin{aligned}
&= P(X < x^{(p)})P(Y < z^{(q)})P(Z \leqslant z^{(q)}) + \\
&\quad + P(X < x^{(p)})P(Y < z^{(q)})P(Z > z^{(q)}) + \\
&\quad + P(X < x^{(p)})P(Y \geqslant z^{(q)})P(Z > z^{(q)}) + \\
&\quad + P(X \geqslant x^{(p)})P(Y > x^{(p)})P(Z > z^{(q)}) \\
&\leqslant P(X < x^{(p)})P(Y < z^{(q)}) + P(Y \geqslant z^{(q)})P(Z > z^{(q)}) \\
&\leqslant pP(Y < z^{(q)}) + (1-q)P(Y \geqslant z^{(q)}) \leqslant a
\end{aligned}
$$

against the supposition that $P(X < Y < Z) > a$.

COROLLARY 1. *Let* X, Y, Z *be independent random variables. If* $P(X < Y < Z) > \frac{1}{2}$ *then* $x^{(\frac{1}{2})} < z^{(\frac{1}{2})}$.

Proof. It is sufficient to put $p = q = \frac{1}{2}$ in lemma 1.

LEMMA 2. *For each* $n \geqslant 3$ *there exist independent random variables* $X_1, X_2, \ldots, X_n$ *such that*

$$\xi_i \geqslant \frac{1}{2} - \frac{6}{n(n+4)} \qquad (i = 1, 2, \ldots, n)$$

where ξ_i *are defined by* (1).

Proof. Suppose that $n \geqslant 3$. Define the independent random variables $X_1, X_2, \ldots, X_n$ as follows

$$P(X_i = i) = p_i, \quad P(X_i = n+i) = 1 - p_i \quad (i = 1, 2, \ldots, n-1),$$

$$P(X_n = n) = 1,$$

where

$$p_i = \frac{i}{2(i+1)}\left(1 + \frac{4(i+2)}{n(n+4)}\right) \quad \text{for} \quad i \leqslant \frac{n}{2},$$

$$p_i = 1 - p_{n-1} \quad \text{for} \quad i > \frac{n}{2} \quad (i = 1, 2, \ldots, n-1).$$

Put $k_i = \dfrac{4(i+2)}{n(n+4)}$ and suppose that $i \leqslant \dfrac{n}{2} - 1$. We have

$$
\begin{aligned}
p_{i+1}(1-p_i) &= \frac{i+1}{2(i+2)}(1+k_{i+1})\left(1 - \frac{(1+k_i)i}{2(i+1)}\right) \\
&= \frac{1}{4} + \frac{1}{4}\left(k_{i+1} - \frac{ik_i}{i+2}(1+k_{i+1})\right) \\
&< \frac{1}{4} + \frac{1}{4}\left(k_{i+1} - i\frac{ik_i}{i+2}\right) = \frac{1}{4} + \frac{3}{n(n+4)}.
\end{aligned}
$$

Moreover, for $n/2 \leqslant i < n-1$

$$p_{i+1}(1-p_i) = p_{n-i}(1 - p_{n-i-1}) < \frac{1}{4} + \frac{3}{n(n+4)}.$$

The case $\frac{n}{2}-1 < i < \frac{n}{2}$ requires special consideration. It can occur only when n is an odd number. Taking into account that $p_{n-i} = 1-p_i$ we obtain

$$p_{\frac{n+1}{2}}(1-p_{\frac{n-1}{2}}) = (1-p_{\frac{n-1}{2}})^2 = \left(\frac{1}{2}+\frac{2n+3}{n(n+1)(n+4)}\right)^2 < \frac{1}{4}+\frac{3}{n(n+4)}$$

for $n \geqslant 3$.

We have proved that $p_{i+1}(1-p_i) < \frac{1}{4}+\frac{3}{n(n+4)}$ for $i = 1, 2, \ldots, n-1$. Then for $i = 1, 2, \ldots, n-3$

$$\xi_i = p_i p_{i+1}+(1-p_{i+1})(1-p_{i+2}) = 1-p_{i+1}(1-p_i)-p_{i+2}(1-p_{i+1}) > \frac{1}{2}-\frac{6}{n(n+4)}.$$

Moreover

$$\begin{aligned}\xi_{n-2} &= P(X_{n-2} < X_{n-1} < X_n) = p_{n-2}p_{n-1} = (1-p_1)(1-p_2)\\ &= \frac{1}{2}-\frac{6}{n(n+4)}+\frac{16}{n^2(n+4)^2} > \frac{1}{2}-\frac{6}{n(n+4)},\\ \xi_{n-1} &= P(X_{n-1} < X_n < X_1) = p_{n-1}(1-p_1) = (1-p_1)^2 > (1-p_1)(1-p_2)\\ &> \frac{1}{2}-\frac{6}{n(n+4)},\\ \xi_n &= P(X_n < X_1 < X_2) = (1-p_1)(1-p_2) > \frac{1}{2}-\frac{6}{n(n+4)}.\end{aligned}$$

Then inequality $\xi_i > \frac{1}{2}-\frac{6}{n(n+4)}$ holds for $i = 1, 2, \ldots, n$ what completes the proof.

Proof of the theorem. Since for $n = 1, 2$ the theorem is obvious we prove it for $n \geqslant 3$.

The left hand inequality in (3) follows immediately from lemma 2. To prove the right side of (3) it is sufficient to notice that inequality $\varkappa_n > \frac{1}{2}$ implies, by corollary 1, the condition

$$x_i^{(\frac{1}{2})} < x_{i+2}^{(\frac{1}{2})} \quad (i = 1, 2, \ldots, n), \tag{4}$$

where indices in (4) are taken modulo n.

For $n = 2k$ this gives

$$x_1^{(\frac{1}{2})} < x_3^{(\frac{1}{2})} < \ldots < x_{2k-1}^{(\frac{1}{2})} < x_1^{(\frac{1}{2})}.$$

For $n = 2k+1$ this gives

$$x_1^{(\frac{1}{2})} < x_3^{(\frac{1}{2})} < \ldots < x_{2k+1}^{(\frac{1}{2})} < x_2^{(\frac{1}{2})} < x_4^{(\frac{1}{2})} < \ldots < x_{2k}^{(\frac{1}{2})} < x_1^{(\frac{1}{2})}.$$

Then in both cases the supposition $\varkappa_n > \frac{1}{2}$ leads to contradiction what ends the proof of the theorem.

From the theorem immediately follows

COROLLARY 2.

$$\lim_{n\to\infty} \varkappa_n = \tfrac{1}{2}. \tag{5}$$

At the end, let us notice that $\sup \min (\xi_1, \xi_2, \ldots, \xi_n)$ will change its value if we remove the supposition of independence of $X_1, X_2, \ldots, X_n$. To prove that define the function $\varrho(x_1, x_2, \ldots, x_n)$ as follows: $\varrho = k$ if exactly k among the following n double inequalities are satisfied

$$x_1 < x_2 < x_3,\ x_2 < x_3 < x_4,\ \ldots,\ x_{n-2} < x_{n-1} < x_n,\ x_{n-1} < x_n < x_1,\ x_n < x_1 < x_2.$$

It is obvious that $\varrho(x_1, x_2, \ldots, x_n) \leqslant n-2$. Furthermore, if μ is the distribution of a random variable $X = (X_1, \ldots, X_n)$ which takes its values in an n-dimensional Euclidean space E_n then

$$n-2 \geqslant \int_{E_n} \varrho\, d\mu \geqslant \xi_1 + \xi_2 + \ldots + \xi_n \geqslant n \min(\xi_1, \xi_2, \ldots, \xi_n)$$

and

$$\min(\xi_1, \xi_2, \ldots, \xi_n) \leqslant \frac{n-2}{n}. \tag{6}$$

From the other side, if X is uniformly distributed on the set of n points with coordinates $1, 2, \ldots, n; 2, 3, \ldots, n, 1; \ldots; n, 1, \ldots, n-1$ then $P(X_i < X_{i+1} < X_{i+2}) = (n-2)/n$ and there exists the random variable X for which

$$\min(\xi_1, \xi_2, \ldots, \xi_n) = \frac{n-2}{n}. \tag{7}$$

Formulas (6) and (7) give

$$\sup \min(\xi_1, \xi_2, \ldots, \xi_n) = \frac{n-2}{n}$$

where the supremum is taken now over the set of all random variables $X = (X_1, X_2, \ldots, X_n)$.

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