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INFINITE G_δ-GAMES WITH IMPERFECT INFORMATION(1)

1. Introduction. Games with an infinite number of moves and perfect information have been studied by several writers [1], [2], [3], [4], [6], but just which games of this type have a value remains open. Indeed Mycielski and Steinhaus [3] have proposed the axiom that every such game has a value. In this paper we study games with an infinite number of moves and imperfect information of a simple kind. Our result, analogous to that of Wolfe [6] for perfect information, is that a win-lose game has a value if one player's winning set is a G_δ.

2. Statement of the Theorem. Let I, J be two nonempty finite sets, let $Z = I \times J$, let Ω be the space $Z^{\aleph_0}$ of all infinite sequences $\omega = (z_1, z_2, \ldots)$, $z_n \in Z$, and let φ be a bounded Baire function on Ω. The function φ defines a zero-sum two person game, denoted by $G(\varphi)$, played as follows:

Initially, player A chooses $i_1 \in I$ and, simultaneously, player B chooses $j_1 \in J$. Then the result $z_1 = (i_1, j_1)$ is announced to both players. Then, simultaneously, A chooses $i_2 \in I$ and B chooses $j_2 \in J$. Then $z_2 = (i_2, j_2)$ is announced to both players, etc. The result of the infinite sequence of simultaneous choices is a point $\omega = (z_1, z_2, \ldots) \in \Omega$, and B pays A the amount $\varphi(\omega)$.

Denote by S the set of all *positions*, i.e. finite sequences $s = (z_1, \ldots, z_n)$, $n = 0, 1, 2, \ldots$ A strategy $\alpha(\beta)$ for $A(B)$ associates with each position s a corresponding probability distribution on $I(J)$: when the current position is s, $A(B)$ will make his next choice according to $\alpha(s)$ $(\beta(s))$. A pair (α, β) defines a probability distribution $P_{\alpha\beta}$ on Ω, and an *expected income*

$$v(\alpha, \beta, \varphi) = \int \varphi(\omega)\, dP_{\alpha\beta}(\omega)$$

for A in $G(\varphi)$ when he uses α and B uses β.

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The lower and upper values of $G(\varphi)$ are

$$L(\varphi) = \sup_{\alpha}\inf_{\beta} v(\alpha, \beta, \varphi),$$

$$U(\varphi) = \inf_{\beta}\sup_{\alpha} v(\alpha, \beta, \varphi).$$

Always $L(\varphi) \leqslant U(\varphi)$; if they are equal, their common value is called the *value* of $G(\varphi)$. We shall prove the

THEOREM. *Let T be any subset of S, and define φ on Ω by $\varphi(z_1, z_2, \ldots) = 1$ if $(z_1, \ldots, z_n) \in T$ for infinitely many n, $\varphi(z_1, z_2, \ldots) = 0$ otherwise. Then $G(\varphi)$ has a value, i.e. $L(\varphi) = U(\varphi)$.*

As noted by Wolfe [6], the ω-set $\{\varphi = 1\}$ is a G_δ, and every G_δ in Ω is the set $\{\varphi = 1\}$ for some T, so that our result may be stated as: if φ is the indicator of a G_δ, then $G(\varphi)$ has a value.

3. Proof. For any position s, denote by $u(s)$ the upper value of the game $G(\varphi)$ starting from position s:

$$u(s) = U(\varphi_s),$$

where φ_s is defined by $\varphi_s(\omega) = \varphi(s\omega)$. Associate with each s the game G_s^* which starts at s and which continues until a later position $t \in T$ is reached. If this happens, play stops and A receives $u(t)$. If it never happens, play continues indefinitely and A receives 0. Formally, G_s^* is $G(\psi)$, with $\psi(z_1, z_2, \ldots) = u(s, z_1, \ldots, z_k)$ if $s, z_1, \ldots, z_i \notin T$ for $1 \leqslant i < k$ and $s, z_1, \ldots, z_k \in T$, $\Psi(z_1, z_2, \ldots) = 0$ if $s, z_1, \ldots, z_k \notin T$ for any $k \geqslant 1$. It is easily checked that ψ is lower semi-continuous, so that, from a general minimax theorem of Sion [5], G_s^* has a value and B has an optimal strategy. Denote the value of G_s^* by $w(s)$. We shall show that

$$w(s) \geqslant u(s),$$

i.e. for any $\varepsilon > 0$, we shall describe a strategy β for B, starting from s, such that, for every α, the probability that T is hit infinitely often does not exceed $w(s) + \varepsilon$. Let B, starting from s, play optimally in G_s^*, until T is hit after s, say at t. Then B plays to keep the probability that T is hit infinitely often to $u(t) + \varepsilon$. Thus he restricts the probability that T is hit infinitely often to $w(s) + \varepsilon$. Actually $w = u$, but we shall not need this.

We now, for any $\varepsilon > 0$, describe a strategy α for A which hits T infinitely often with probability $u(e) - \varepsilon$, where e denotes the empty sequence, i.e. the starting position. Since $u(e) = U(\varphi)$, this will complete the proof. Put $\varepsilon_n = \varepsilon/2^n$. Let A start by playing an ε_1 optimal strategy in G_e^*. If T is hit after e, say at t_1, A then plays an ε_2-optimal strategy in $G_{t_1}^*$. If T is hit after t_1, say at t_2, A then plays an ε_3 optimal strategy in $G_{t_2}^*$, etc. Let B use any strategy β, and denote the resulting play by $z_1, z_2, \ldots$

Define $x_0 = w(e)$ and, for $k \geqslant 1$, $x_k = u(t_k)$ if $z_1, z_2, \ldots$ hits T for the kth time at t_k, $x_k = 0$ if $z_1, z_2, \ldots$ hits T less than k times. Then

$$E(x_k | x_0, \ldots, x_{k-1}) \geqslant x_{k-1} - \varepsilon_k. \tag{1}$$

This is clear if $x_{k-1} = 0$. If $x_{k-1} > 0$, T was hit for the $k-$1st time, say at t, and A then played in G_t^* to get at least $w(t) - \varepsilon_k \geqslant x_{k-1} - \varepsilon_k$. Since his payoff in G_t^* is then x_k, (1) follows. From (1) we conclude $E(x_k) \geqslant$ $\geqslant E(x_{k-1}) - \varepsilon_k$, so that

$$E(x_k) \geqslant x_0 - (\varepsilon_1 + \ldots + \varepsilon_k) > w(e) - \varepsilon. \tag{2}$$

Since $0 \leqslant x_k \leqslant 1$ and $x_k = 0$ unless T is hit k times, (2) implies

$$P\{T \text{ is hit } k \text{ times}\} > w(e) - \varepsilon.$$

Letting $k \to \infty$ yields

$$P\{T \text{ is hit infinitely often}\} \geqslant w(e) - \varepsilon,$$

completing the proof.

We conjecture that every $G(\varphi)$ (with φ a Baire function) has a value, but even the general case of φ for which, for every constant c, $\{\varphi > c\}$ is a G_δ, remains unsolved.

References

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