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A GEOMETRICAL APPROACH TO THE SPECIAL STABLE DISTRIBUTIONS (1)

The convolution formulas for the Cauchy distribution and for the distribution of the reciprocal of the square of a standard normal random variable are here derived from the geometry of circularly symmetric distributions on the plane. Other known derivations seem quite different, and in a sense less elementary, though some are by no means less interesting. A few other illustrations of the geometrical method are also included.

Let a, a', b, and b' be points on a circle with $a \neq a'$ and $b \neq b'$, and suppose that the chords (a, a') and (b, b') intersect in a point c not exterior to the circle and not the same as a or b. Then the angle (a, c, b) is half the sum of the arcs (a, b) and (a', b'). This fact seems old and widely known. The special case in which $a' = b' = c$ is very familiar indeed (Euclid III 26 and 27), and the general case is easily inferred from the special one with the aid of the construction line (a, b').

If a and b are two points in the Euclidean plane, the *direction* from a to b is the vector from a to b normalized to unit length, and the *unoriented direction* is the unordered couple consisting of the direction and its negative, so that the unoriented direction from a to b is also the unoriented direction from b to a. The notion of a uniformly distributed random direction needs no explicit definition here, and a random unoriented direction will be called uniformly distributed if it has the same distribution as the unoriented direction associated with a uniformly distributed random direction. In this terminology, the geometric fact of the preceeding paragraph has the following probabilistic interpretation.

LEMMA 1. *If a random point P is uniformly distributed on the periphery of a circle in the plane and c is not exterior to that circle, then the unoriented direction from c to P is uniformly distributed.*

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For the present form of the lemma and of the argument leading to it, I thank E. J. G. Pitman and Paul Lévy, respectively.

Let Z be a random vector distributed with circular symmetry in the Euclidean vector plane. Assume, to avoid unimportant complications, that $\Pr(Z = 0) = 0$. Let X and Y be a pair of Cartesian coordinates of Z, and let R and A be the corresponding polar coordinates, so that: $X = R\cos A$; $Y = R\sin A$; R and A are independent; $\Pr(R > 0) = 1$; and A is uniformly distributed on the reals modulo 2π.

For this note, interest centers on the example in which X and Y are independent and normally distributed with mean 0 and variance 1, which will be called the *standard example*. The more general situation is treated mainly to emphasize the structure of the arguments.

Let "$\sim$" mean "is distributed like".

THEOREM 1. *For all real f and g, $fX+gY \sim (f^2+g^2)^{1/2}X$.*

Proof. Aside from the trivial possibility that $f = g = 0$, $fX+gY = (f^2+g^2)^{1/2}R\cos(A-a)$, where $a = \arctan f/g$. But $A-a \sim A$ and is independent of R. Since $X = R\cos A$, this completes the proof.

Specialized to the standard example, Theorem 1 is the convolution formula for the normal distribution. The proof, however familiar, helps to set the stage.

LEMMA 2. *If σ and τ are nonnegative and not both 0,*

$$\frac{XY}{(\sigma^2 X^2+\tau^2 Y^2)^{1/2}} \sim \frac{X}{\sigma+\tau}.$$

Proof. Expressed in polar coordinates, the lemma says that R times a certain function of A is distributed like $R\cos A$ or, equivalently, $R\sin A$. Since R and A are independent, it will be adequate to show that the function of A, namely

$$h(A) = \frac{(\sigma+\tau)\sin A\cos A}{(\sigma^2\cos^2 A+\tau^2\sin^2 A)^{1/2}},$$

is distributed like $\sin A$. But

$$h(A/2) = \frac{(\sigma+\tau)\sin A}{\{2(\sigma^2+\tau^2)+2(\sigma^2-\tau^2)\cos A\}^{1/2}} = \frac{\sin A}{\{1+2\varrho\cos A+\varrho^2\}^{1/2}},$$

where, $\varrho = (\sigma-\tau)/(\sigma+\tau)$. Since $(\sin 2A, \cos 2A) \sim (\sin A, \cos A)$, what needs to be shown is that $h(A/2) \sim \sin A$ for each ϱ in $[-1, 1]$.

Geometrically, if A is thought of as a random point P on the unit circle, then $h(A/2)$ is the sine of the angular coordinate B of that point as viewed from the point $c = (-\varrho, 0)$. If the angular coordinate associated with the direction from c to P is B, that associated with the opposite direction is $B+\pi$, the sine of which is the negative of that of B. Therefore, $|\sin B|$ depends only on the unoriented direction from c to P. But according to Lemma 1, this unoriented direction is distributed just as it would be if B were uniformly distributed, that is, $|\sin B| \sim |\sin A|$. This, together with the remark that the distribution of $\sin A$ and $\sin B$ are both symmetric, completes the proof of the present lemma.

COROLLARY 1. *If M and N are independent and normal with mean* 0, *then* $L = MN/(M^2+N^2)^{1/2}$ *is normal with mean* 0 *and* s.d. $(L) = \{[\text{s.d.}(M)]^{-1}+[\text{s.d.}(N)]^{-1}\}^{-1}$.

This corollary has been proved (cf. [5]), by recognizing its obvious equivalence to the standard-example application of the next theorem, which itself is obviously equivalent to the lemma.

THEOREM 2. *For nonnegative p and q,*

$$\frac{p}{X^2}+\frac{q}{Y^2} \sim \frac{(p^{1/2}+q^{1/2})^2}{X^2}.$$

Applied to the standard example, the theorem asserts that the distribution of the reciprocal of the square of a standard normal variable is stable of order 1/2. Doetsch [1] attributes the fact to Cesaro and gives two interesting, but not very probabilistic, demonstrations. Lévy [3] independently discovered the fact in the course of a beautiful investigation of the Wiener process. For some further information, pursue "Stable distributions of order 1/2" in the Index of Feller [2].

The standard Cauchy distribution is the distribution of $C = Y/X = \tan A$.

THEOREM 3. *For all real f and g not both* 0,

$$(fC+g)^{-1} \sim \frac{fC+g}{(f^2+g^2)}.$$

Proof:

$$(fC+g)^{-1} = \frac{X}{fY+gX} = \frac{f}{(f^2+g^2)}\frac{(-gY+fX)}{(fY+gX)}+\frac{g}{f^2+g^2}$$
$$\sim \frac{f}{(f^2+g^2)}\frac{Y}{X}+\frac{g}{f^2+g^2},$$

as in the proof of Theorem 1.

Theorem 3 has been proved ([4], p. 1270) by direct calculation with the Cauchy density.

THEOREM 4. *If $Z' = (X', Y')$ is independent of Z and it too is distributed with circular symmetry, then, for all real f and g,*

$$f\frac{X'}{X}+g\frac{Y'}{Y} \sim (|f|+|g|)\frac{X'}{X}. \tag{1}$$

Proof. The left side of (1) is U/V, where

$$U = \frac{fYX' + gXY'}{(g^2X^2 + f^2Y^2)^{1/2}}$$

and

$$V = \frac{XY}{(g^2X^2 + f^2Y^2)^{1/2}}.$$

Given X and Y, U is distributed like X', as Theorem 1 shows. Therefore U is independent of V and distributed like X'. According to Lemma 2, $V \sim X/(|f| + |g|)$. This proves the theorem.

COROLLARY 2. *If C and C' are independent and have the standard Cauchy distribution, then for all f and g, $fC + gC' \sim (|f| + |g|)C$.*

This popular elementary fact has been demonstrated in several other ways. See for example Feller ([2], p. 50). The proof given here is easily extended to the multivariate Cauchy distribution, that is, the distribution of a normally distributed vector divided by an independent, standard normally distributed number.

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