ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XI, 4 (1970)

A. JANICKI (Wrocław)

REMARKS ON THE TIME TRANSPORTATION PROBLEM

1. Introduction. Suppose we are given the system (T, M), where $T = (t_{ij})$ is an $(m \times n)$-matrix with real numbers t_{ij}, $M = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)$ and a_i, b_j are positive real numbers such that

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

Any $(m \times n)$-matrix $X = (x_{ij})$ which satisfies the conditions

$$(1)\qquad \begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i, \quad i = 1, 2, \ldots, m, \\ \sum_{i=1}^{m} x_{ij} &= b_j, \quad j = 1, 2, \ldots, n, \end{aligned} \qquad (x_{ij} \geqslant 0)$$

will be called a *solution* of (T, M). The problem is to find an *optimal solution*, i.e. a solution X of (T, M) which minimizes the function

$$(2)\qquad t(X) = \max_{(i,j)\in\Theta} t_{ij}, \qquad \text{where } \Theta = \{(i,j)\colon x_{ij} > 0,\ 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}.$$

Methods of solving time transportation problems are described in [1], [2]. They are not adapted for computer calculations. Therefore a new version of the method for solving TTP, being a modification of that presented in [2], is presented in this paper. This method may be successfully used in computer calculations even for great m and n.

2. Notation and definitions. Let $\Phi = \{(i,j)\colon 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$. Any subset Ω of Φ we call a *set of nodes*. By a *route* $(p, q) - (r, s)$ connecting in Ω nodes $(p, q) \in \Omega$, $(r, s) \in \Omega$ we mean the smallest sequence $\{(i_k, j_k)\}$ $(k = 1, 2, \ldots, l)$ of different nodes from Ω satisfying the conditions

$$(i_1, j_1) = (p, q), \quad (i_l, j_l) = (r, s),$$

for $k = 1, 2, \ldots, l-1$ either $i_k = i_{k+1}$ or $j_k = j_{k+1}$.

If there is in Ω a pair of nodes (p, q), (r, s) for which two different routes $(p, q)-(r, s)$ exist in Ω, then we say that Ω contains a cycle. In that case we mean by a *cycle* the sum of the routes $(p, q)-(r, s)$. A subset B of Φ consisting of $m+n-1$ nodes which contains no cycle is called a *basis*. To each basis B there exist not more than one matrix $X = (x_{ij})$ whose elements satisfy (1) and also the conditions $x_{ij} = 0$ for all $(i, j) \epsilon \Phi - B$. Every matrix X which satisfies these conditions will be called a *basic solution* of (T, M) and denoted by $X(B) = (x_{ij}^B)$. A basis B for which exists a basic solution $X(B)$ of (T, M) will be called a *feasible basis*.

Let

$$\Theta(B) = \{(i, j) \epsilon B: x_{ij}^B > 0\}, \qquad V(B) = \{(i, j) \epsilon B: x_{ij}^B = 0\},$$
$$W(B) = \{(i, j) \epsilon B: t_{ij} > t(X(B))\}$$

for any basic solution $X(B)$.

We shall say that a basic solution of (T, M) is *degenerate* if the set $V(B)$ is not empty. The number of nodes in the set $W(B)$ will be called *degree of degeneration* of $X(B)$ and denoted by dg $X(B)$.

Any node $(k, l) \epsilon B$ which satisfies the condition

$$t_{kl} = \max_{(i,j)\epsilon\Theta(B)} t_{ij} = t(X(B))$$

will be called a *central node* of basis B. The node $(i, j) \epsilon \Phi - B$ which satisfies the conditions

(i) every route $(i, j)-(k, l)$ in the set $B+(i, j)$ contains an even number of nodes,

(ii) (k, l) belongs to the cycle contained in $B+(i, j)$ will be called a *neighbouring node* to a central node (k, l).

By the *neighbouring set* to (k, l) we mean the set of all neighbouring nodes to the central node (k, l). We shall denote it by $\Psi_{kl}(B)$ or shortly by Ψ_{kl}.

3. Method of solving TTP. We propose the following method of solving a time transportation problem (TTP).

1. Find an initial basic solution $X(B_1)$ by any of the known methods. (Suppose that $t_{k_0 l_0} \neq t(X(B_1))$.)

Now for $h = 1, 2, \ldots$ do the following:

2. Find the central nodes (k, l) of the basic solution $X(B_h)$ and fix as (k_h, l_h) any of them such that if (k_{h-1}, l_{h-1}) is a central node of B_h, then $(k_h, l_h) = (k_{h-1}, l_{h-1})$.

3. Find the set $\Psi_{k_h l_h}$ neighbouring to (k_h, l_h).

4. Choose an arbitrary node (p_h, q_h) belonging to the set

$$\{(p, q): t_{pq} = \min_{(i,j)\in\Psi_{k_h l_h}} t_{ij},\ (p, q)\in\Psi_{k_h l_h}\}.$$

There exist two possibilities (a) and (b).

(a) $$t_{p_h q_h} \geqslant t_{k_h l_h} = t\big(X(B_h)\big).$$

Then stop because the basic solution $X(B_h)$ is an optimal solution. This follows from theorem 2 in section 5.

(b) $$t_{p_h q_h} < t_{k_h l_h}.$$

The set $B_h + (p_h, q_h)$ contains exactly one cycle, say G_h. Divide G_h into two subsets G_h', G_h'' assigning to G_h' nodes (i, j) for which the route $(i, j) - (p_h, q_h)$ in G_h contains an odd number of nodes. Subset G_h'' contains the remaining nodes. Of course, $(p_h, q_h)\in G_h'$, $(k_h, l_h)\in G_h''$.

5. Find

$$\min_{(i,j)\in G_h''} x_{ij}^{B_h} = \bar{x}_h.$$

Determine a set $F_h = \{(r, s)\in G_h'': x_{rs}^{B_h} = \bar{x}_h\}$ and fix as (r_h, s_h) any node from it such that if $(k_h, l_h)\in F_h$, then $(r_h, s_h) = (k_h, l_h)$. Determine a new basis $B_{h+1} = B_h + (p_h, q_h) - (r_h, s_h)$ and a new basic solution $X(B_{h+1})$ defined by the formulae

$$x_{ij}^{B_{h+1}} = \begin{cases} x_{ij}^{B_h} + \bar{x}_h & \text{if } (i, j)\in G_h', \\ x_{ij}^{B_h} - \bar{x}_h & \text{if } (i, j)\in G_h'', \\ x_{ij}^{B_h} & \text{if } (i, j)\notin G_h. \end{cases}$$

Repeat steps 2, 3, 4, and 5.

It will be proved that for any TTP only a finite number of iterations is needed.

4. Commentary. In comparison with the method described in [2] the algorithm presented in this paper includes 3 corrections. Steps 6 and 7 have been omitted. This is very important in computer calculations. It is practically impossible to verify whether a sequence of basic solutions $X(B_1), X(B_2), \ldots, X(B_k)$ contains two identical solutions. Just this difficulty causes that the method described in [2] could not be used in computer calculations. The modifications done in steps 2 and 5 of the algorithm allow to leave out steps 6 and 7 from our algorithm. This will be illustrated by two examples.

Suppose we have following TTP:

$$\begin{array}{ccccc} 8 & 6 & 3 & 1 & 9 \\ 7 & 9 & 7 & 5 & 5 \\ 4 & 5 & 4 & 8 & 9 \\ 3 & 2 & 3 & 9 & 1 \\ 4 & 8 & 4 & 8 & \end{array}$$

(The numbers a_i and b_j are on the right and below the matrix $T = (t_{ij})$, respectively.)

The initial solution is given in Fig. 1.

Leaving out the modification done in step 2 we would obtain the sequence of basic solutions given in Figs. 2 and 3. We see that $X(B_1) = X(B_3)$ and $B_1 = B_3$. Using the algorithm presented in this paper and leading from the initial solution $X(B_1)$ we obtain after six iterations the optimal basic solution given in Fig. 4.

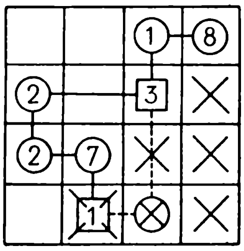

Fig. 1

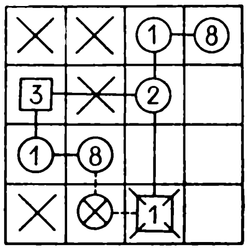

Fig. 2

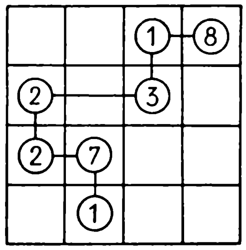

Fig. 3

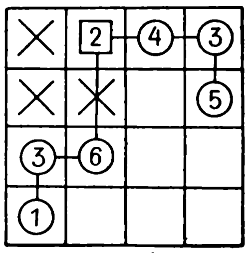

Fig. 4

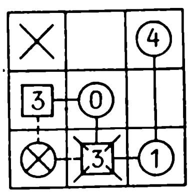

Fig. 5

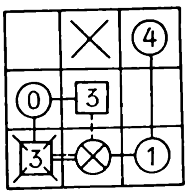

Fig. 6

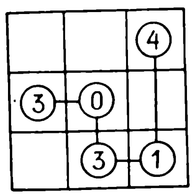

Fig. 7

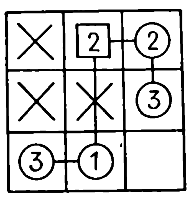

Fig. 8

Suppose we have the TTP

$$\begin{matrix} 5 & 4 & 1 & 4 \\ 6 & 8 & 3 & 3 \\ 2 & 3 & 4 & 4 \\ 3 & 3 & 5 & \end{matrix}$$

with the initial basic solution $X(B_1)$ presented in Fig. 5.

Leaving out the modification done in step 5 we would obtain the sequence of basic solutions given in Figs. 6 and 7. We see that again $X(B_1) = X(B_3)$ and $B_1 = B_3$, while the exact result is that of Fig. 8. (Elements (k_h, l_h), (r_h, s_h), (p_h, q_h) and $(i,j) \in \Psi_{k_h l_h}$ have been denoted in figures by a square, a crossed square, a crossed circle, and crosses, respectively.)

5. Theorems. First, let us state and prove a series of properties of the sequence $X(B_1)$, $X(B_2), \ldots, X(B_h), \ldots$ of basic solutions of (T, M) obtained by using the method given in steps 1-5.

Let $z(X(B)) = \max_{(i,j)\in B} t_{ij}$.

Remark 1. *If* $(i,j) \in W(B)$, *then* $x_{ij}^B = 0$.

Remark 2. $z(X(B)) \geqslant t(X(B))$.

Remark 3. $z(X(B)) = t(X(B))$ *if and only if* $\mathrm{dg}X(B) = 0$.

Proof. $z(X(B)) = t(X(B)) \Leftrightarrow W(B) = \emptyset \Leftrightarrow \mathrm{dg}(X(B)) = 0$.

Corollary 1. $z(X(B_h)) \geqslant z(X(B_{h+1}))$.

Proof. $B_{h+1} = B_h - (r_h, s_h) + (p_h, q_h)$ and $t_{p_h q_h} < t(X(B_h))$. It follows from remark 2 that $t(X(B_h)) \leqslant z(X(B_h))$ and thus $t_{p_h q_h} < z(X(B_h))$.

Corollary 2. $t(X(B_h)) \leqslant z(X(B_1))$ $(h = 1, 2, \ldots)$.

Proof. $t(X(B_h)) \leqslant z(X(B_h)) \leqslant z(X(B_1))$.

Corollary 3. *If* $t_{r_1 s_1} = z(X(B_1))$, *then there exist no basic solution* $X(B_h)$ $(h > 1)$ *such that* $(r_1, s_1) \in B_h$.

Proof. $(r_1, s_1) \notin B_2$, $B_h = B_2 \cdot B_h + (p_{h_1}, q_{h_1}) + (p_{h_2}, q_{h_2}) + \ldots + (p_{h_l}, q_{h_l})$, where $1 \leqslant l \leqslant \min\{h-1, m+n-1\}$, $t_{r_1 s_1} = z(X(B_1)) \geqslant z(X(B_{h_i}))$ and thus $t_{r_1 s_1} > t_{p_{h_i} q_{h_i}}$ or $(r_1, s_1) \neq (p_{h_i}, q_{h_i})$. This means that $(r_1, s_1) \notin B_h$.

Corollary 4. *If* $\mathrm{dg}X(B_1) = 0$ *and if* (k_1, l_1) *is a central node of* B_1 *found in step 2 of the algorithm, then there exist no solution* $X(B_h)$ $(h > 1)$ *for which* $x_{k_1 l_1}^{B_h} = 0$.

Proof. Let $i > 1$ be the smallest number such that $(k_1, l_1) \in B_i$ and $x_{k_1 l_1}^{B_i} = 0$. $t_{k_1 l_1} = z(X(B_1)) \geqslant z(X(B_h)) \geqslant z(X(B_i)) \geqslant t_{k_1 l_1}$ and thus from corollary 3 it follows that $(k_1, l_1) \in B_h$ and, for $1 \leqslant h < i$, $x_{k_1 l_1}^{B_h} > 0$. In particular, $x_{k_1 l_1}^{B_{i-1}} > 0$, and thus (k_1, l_1) is a central node of B_{i-1}. But $x_{k_1 l_1}^{B_i} = x_{k_1 l_1}^{B_{i-1}} - \bar{x}_{i-1} = 0$. This means that $x_{k_1 l_1}^{B_{i-1}} = x_{r_{i-1} s_{i-1}}^{B_{i-1}}$ and because of the modification introduced in step 5 of the algorithm $(k_1, l_1) \notin B_i$ must hold because $(r_{i-1}, s_{i-1}) = (k_1, l_1)$. This contradiction completes the proof.

THEOREM 1. *The sequence $X(B_1), X(B_2), \ldots, X(B_h), \ldots$ of basic solutions of the TTP (T, M) constructed by the method given in steps $1-5$ contains no two identical solutions $X(B_i), X(B_j)$ for $i \neq j$.*

Proof. Suppose that in the sequence $X(B_1), X(B_2), \ldots, X(B_e)$, $(e > 1)$ of basic solutions of (T, M) obtained by using the discussed method there is $X(B_1) = X(B_e)$, $B_e = B_1$. From corollary 1 it follows that $z(X(B_1)) = z(X(B_2)) = \ldots = z(X(B_e))$.

Let $Z(B_h) = \{(i,j) \in B_h : t_{ij} = z(X(B_h))\}$. Of course, always $Z(B_h) \neq \emptyset$. Take then $Z(B_h), Z(B_{h+1})$ $(1 \leqslant h \leqslant e-1)$. We have $B_{h+1} = B_h - (r_h, s_h) + (p_h, q_h)$, $t_{p_h q_h} < t(X(B_h)) \leqslant z(X(B_h)) = z(X(B_{h+1}))$, $(p_h, q_h) \notin Z(B_{h+1})$, and thus $Z(B_{h+1}) \subset Z(B_h)$.

Suppose that $(i,j) \in Z(B_h)$ and $(i,j) \notin Z(B_{h+1})$. Corollary 3 allows us to state that $(i,j) \notin B_e$. On the other hand, we have

$$B_h = B_1 \cdot B_h + (p_{h_1}, q_{h_1}) + \ldots + (p_{h_l}, q_{h_l})$$

$$(1 \leqslant h_i \leqslant h,\ 1 \leqslant l \leqslant \min\{h, m+n-1\})$$

$$t_{p_{h_i} q_{h_i}} < z(X(B_{h_i})) = t_{ij}$$

and thus $(i,j) \in B_1 \cdot B_h$. This means that $(i,j) \in B_1$. However, $B_1 = B_e$. This allows us to assert that $Z(B_h) \subset Z(B_{h+1})$. Finally,

$$Z(B_1) = Z(B_2) = \ldots = Z(B_e) = Z. \tag{3}$$

Now the proof splits in two parts.

(a) Suppose that $\operatorname{dg} X(B_1) = d > 0$. This allows us to assert that $x_{ij}^{B_1} = 0$ for $(i,j) \in Z$. If $t(X(B_h)) = z(X(B_h))$ for some $1 < h < e$, then $\operatorname{dg} X(B_h) = 0$ (remark 3), the central node (k_h, l_h) of B_h belongs to Z and from corollary 4 it follows that there exists no solution $X(B_i)$ $(i > h)$ for which $x_{k_h l_h}^{B_i} = 0$. This means that $x_{k_h l_h}^{B_e} \neq 0$. But we know that $X(B_1) = X(B_e)$ and thus $x_{k_h l_h}^{B_e} = 0$. This contradiction shows that $t(X(B_h)) < z(X(B_h))$ or, otherwise, for all $(i,j) \in Z$ and $h = 1, 2, \ldots, e$ we have $Z \subset W(B_h) \subset B_h$, $x_{ij}^{B_h} = 0$. Thus, we see that

$$X^1(B_1), X^1(B_2), \ldots, X^1(B_e) \qquad (X^1(B_i) = X(B_i) \text{ for } i = 1, 2, \ldots, e)$$

is a sequence of basic solutions to TTP (T^1, M), where $t_{ij}^1 = t_{ij}$ for $(i,j) \notin Z$, $t_{ij}^1 = c$ for $(i,j) \in Z$, c is an arbitrary number such that $c < t(X(B_1))$ and $T^1 = (t_{ij}^1)$, and it could have been obtained by using steps $1-5$, because $(k_h, l_h) \notin Z$, $(r_h, s_h) \notin Z$, $(p_h, q_h) \notin Z$ for $h = 1, 2, \ldots, e$. Of course, $t(X(B_1)) = t(X^1(B_1))$ and $z(X(B_1)) > z(X^1(B_1))$.

If we repeat our argumentation, then after p $(p \leqslant d)$ steps we come to the conclusion that there exist the $(m \times n)$-matrix T^p and the sequence

$X^p(B_1), X^p(B_2), \ldots, X^p(B_e)$ of basic solutions of (T^p, M) which can be obtained by using the discussed method such that $t(X^p(B_1)) = z(X^p(B_1))$. This means that $\mathrm{dg}X^p(B_1)) = 0$.

All that allows us to assume that $\mathrm{dg}X(B_1) = 0$ in the sequence $X(B_1), X(B_2), \ldots, X(B_e)$ of basic solutions of (T, M).

(b) Suppose now that $X(B_1)$ is a degenerate basic solution of (T, M), $(B_1 - \Theta(B_1) \neq \emptyset)$ and that $\mathrm{dg}X(B_1) = 0$. We know (see (3)) that $z(X(B_1)) = \ldots = z(X(B_e))$ and $Z(B_1) = \ldots = Z(B_e) = Z$. We assumed $\mathrm{dg}X(B_1) = 0$ and thus the central node (k_1, l_1) of B_1 belongs to Z. From corollary 4 it follows that for any $1 \leqslant h \leqslant e$ the condition $x_{k_1l_1}^{B_h} = 0$ is not true. The modification done in step 2 of the algorithm allowed us to assert that (k_1, l_1) is a central node of all B_h $(1 \leqslant h \leqslant e)$. Now, $x_{k_1l_1}^{B_{h+1}} = x_{k_1l_1}^{B_h} - \bar{x}_h$, $\bar{x}_h \geqslant 0$, and thus $x_{k_1l_1}^{B_{h+1}} \leqslant x_{k_1l_1}^{B_h}$. But $x_{k_1l_1}^{B_1} = x_{k_1l_1}^{B_e}$, and thus $x_{k_1l_1}^{B_1} = x_{k_1l_1}^{B_2} = \ldots = x_{k_1l_1}^{B_e}$ or

$$(4) \qquad \bar{x}_h = 0 \quad \text{for } h = 1, 2, \ldots, e.$$

It is clear now that $x_{ij}^{B_{h'}} = x_{ij}^{B_{h''}}$ for $1 \leqslant h', h'' \leqslant e$ and $\Theta(B_1) = \Theta(B_2) = \ldots = \Theta(B_e) = \Theta$.

Let us divide the subset U of all nodes $(i, j) \in B_1 - \Theta$ for which there exists the route $(i, j) - (k_1, l_1)$ in the set $\Theta + (i, j)$ into two subsets

$$U_1 = \left\{(i, j) \in U \colon (i, j) - (k_1, l_1) \begin{array}{c}\text{contains an even}\\ \text{number of nodes}\end{array}\right\},$$

$$U_2 = \left\{(i, j) \in U \colon (i, j) - (k_1, l_1) \begin{array}{c}\text{contains an odd}\\ \text{number of nodes}\end{array}\right\}.$$

Of course, $U_1 + U_2 = U \neq \emptyset$ because $B_1 - \Theta(B_1) \neq \emptyset$.

Let $U_1 \neq \emptyset$ and $(i, j) \in U_1$. (k_1, l_1) is a central node of all $B_h (1 \leqslant h \leqslant e)$ and thus $(r_h, s_h) = (i, j)$ can never be true. This means that $(i, j) \in B_h$ $(1 \leqslant h \leqslant e)$.

Let $U_2 \neq \emptyset$ and $(i, j) \in U_2$. Suppose that $(r_h, s_h) = (i, j)$ for $1 \leqslant h \leqslant e$. Then $(i, j) \notin B_{h+1}$. The route $(i, j) - (k_1, l_1)$ in the set $B_{h'} + (i, j)$ $(h < h' \leqslant e)$ contains an odd number of nodes. But $B_1 = B_e$ and thus there exists a number $h' (h < h' \leqslant e)$ such that $(i, j) \in \Psi_{k_1l_1}$. This is, however, impossible because the node (i, j) for which the route $(i, j) - (k_1, l_1)$ contains an even number of nodes would belong to $\Psi_{k_{h'}l_{h'}}$. Finally, $U \subset B_h$ for $h = 1, 2, \ldots, e$.

Let us now introduce the TTP (T, M^1), where $M^1 = (a_1^1, \ldots, a_m^1, \ldots, b_n^1)$ is defined by the following formulae:

$$\text{if } (i, j) \in U, \text{ then } a_i^1 = a_i + 1 \text{ and } b_j^1 = b_j + 1,$$
$$\text{if } (i, j) \notin U, \text{ then } a_i^1 = a_i \text{ and } b_j^1 = b_j.$$

B_1 is a feasible basis for (T, M^1) and a basic solution $X^1(B_1)$ can be defined as follows:

$$x_{ij}^1 = \begin{cases} x_{ij}^{B_1}+1 & \text{for } (i,j)\in U, \\ x_{ij}^{B_1} & \text{for } (i,j)\notin U. \end{cases}$$

The $(m\times n)$-matrix X such that $x_{ij} = x_{ij}^1$ for $(i,j)\in U+\Theta$ and $x_{ij} = 0$ for $(i,j)\notin U+\Theta$ is of course a solution to (T, M^1). We know that $U+\Theta\subset B_h$ and thus B_h is a feasible basis of (T, M^1) and $X^1(B_1)$ is a basic solution of (T, M^1) for $h = 1, 2, \ldots, e$. Therefore the sequence $X^1(B_1), X^1(B_2), \ldots, X^1(B_e)$ is a sequence of basic solutions of (T, M^1) obtained by using steps $1-5$. Also, $t(X^1(B_1)) = t(X(B_1)) = z(X(B_1)) = z(X^1(B_1))$ and thus $\mathrm{dg}\,X^1(B_1) = 0$. Moreover, $\Theta^1(B_1) = \Theta(B_1)+U$ and $U\neq\emptyset$.

Bearing this in mind we can assert that if we repeat our argumentation given in part (b) of the proof, then after r steps we will come to the conclusion that there exists a TTP (T, M^r) such that the sequence $X^r(B_1), X^r(B_2), \ldots, X^r(B_e)$ is a sequence of basic solutions of (T, M^r) obtained by using the discussed method and for which $\Theta^r(B_1) = B_1$. This means that the basic solution $X^r(B_1)$ of (T, M^r) is not degenerate. All which was said in parts (a) and (b) of our proof allows us to assume that

(i) in the sequence $X(B_1), X(B_2), \ldots, X(B_e)$ of basic solutions of (T, M) holds $X(B_1) = X(B_e)$, $B_1 = B_e$ $(e > 1)$,

(ii) $\mathrm{dg}\,X(B_1) = 0$ and $X(B_1)$ is an undegenerate basic solution of (T, M).

We know (see (4)) that in this case $x_{r_h s_h} = \bar{x}_h = 0$ for $h = 1, 2, \ldots, e$.

On the other hand, $X(B_1)$ is a non-degenerate basic solution of (T, M) and thus $x_{ij}^{B_1} > 0$ for $(i,j)\in B$ and $x_{r_1 s_1}^{B_1} = \bar{x}_1 > 0$.

This contradiction completes the proof.

THEOREM 2. *If (k, l) is a central node of a basis B, $\Psi_{kl}(B)$ is a set neighbouring to (k, l) and if $\Psi_{kl}(B)\subset\Pi$, then there exists no solution $X = (x_{ij})$ such that $x_{ij} = 0$ for all $(i,j)\in\Pi+(k,l)$.*

The proof of this theorem can be found in [2].

THEOREM 3. *If TTP is solved by the method given in steps $1-5$, then the number of iterations leading from $X(B_1)$ to the optimal solution is finite.*

Proof. The number of all basic solutions of (T, M) is finite. The sequence $X(B_1), X(B_2), \ldots, X(B_h), \ldots$ of basic solutions of (T, M) obtained by using the method given in steps $1-5$ consists of different elements (see theorem 1) and, therefore, is a finite sequence. It means that there exists a number $h\geqslant 1$ such that the basic solution $X(B_h)$ of (T, M) satisfies the condition formulated in step 4, (a) of our algorithm. Theorem 2 shows that $X(B_h)$ is an optimal solution of (T, M). The proof is completed.

References

[1] W. Grabowski, *Problem of transportation in minimum time*, Bull. Acad. Polon. Sci. 12 (1964), pp. 107-108.

[2] W. Szwarc, *The time transportation problem*, Zastosow. Matem. 8 (1966), pp. 231-242.

DEPT. OF NUMERICAL METHODS
UNIVERSITY OF WROCŁAW

Received 26. 6. 1970.

A. JANICKI (Wrocław)

UWAGI O ZAGADNIENIU TRANSPORTOWYM Z KRYTERIUM CZASU

STRESZCZENIE

Zagadnienie transportowe z kryterium czasu można sformułować w następujący sposób: Dany jest układ (T, M), gdzie $T = (t_{ij})$ jest macierzą typu $m \times n$ o elementach rzeczywistych, a $M = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m)$ jest układem $m+n$ liczb dodatnich a_i, b_j, przy czym

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

Problem polega na znalezieniu minimum funkcji

$$t(X) = \max_{(i,j)\in\Theta} t_{ij}, \quad \text{gdzie } \Theta = \{(i,j): x_{ij} > 0,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$$

określonej na zbiorze macierzy $X = (x_{ij})$ typu $m \times n$ spełniających dla $i = 1, 2, \ldots, m$ i $j = 1, 2, \ldots, n$ następujące warunki:

$$x_{ij} > 0, \quad \sum_{j=1}^{n} x_{ij} = a_i, \quad \sum_{i=1}^{m} x_{ij} = b_j.$$

Przedstawiona w niniejszej pracy metoda rozwiązania zagadnienia transportowego z kryterium czasu jest metodą iteracyjną. Przy pomocy dowolnej ze znanych metod konstruuje się rozwiązanie początkowe $X(B_1)$, a następnie wyznacza się skończony ciąg $X(B_1), X(B_2), \ldots, X(B_k)$ rozwiązań bazowych, stosując do tego celu wzory zawarte w punktach 2-5 algorytmu opisanego w rozdziale 3. Po wykonaniu skończonej ilości iteracji otrzymujemy szukane rozwiązanie optymalne $X(B_k)$. Odpowiednie twierdzenia można znaleźć w rozdziale 5.

Omawiana tutaj metoda jest zmodyfikowaną i znacznie uproszczoną wersją metody opublikowanej w [2]. Modyfikacje wprowadzono z myślą o adaptacji metody dla obliczeń wykonywanych przy pomocy maszyny cyfrowej. Uzyskaną w ten sposób metodę można z powodzeniem stosować w obliczeniach wykonywanych na maszynach cyfrowych nawet dla dużych m i n.

А. ЯНИЦКИ (Вроцлав)

ЗАМЕЧАНИЯ О ТРАНСПОРТНОЙ ЗАДАЧИ С КРИТЕРИЕМ ВРЕМЕНИ

РЕЗЮМЕ

Транспортная задача с критерием времени формулируется следующим образом: задана система (T, M), где $T = (t_{ij})$ матрица порядка $m \times n$, элементы которой действительны, а $M = (a_1, a_2, \ldots, a_m, b_1, b_1, \ldots, b_n)$ — система $m+n$ положительных чисел a_i, b_j, причем

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

Следует найти минимум функции

$$t(x) = \max_{(i,j)\in\Theta} t_{ij}, \text{ где } \Theta = \{(i,j): x_{ij} > 0,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$$

определенной в совокупности всех матриц $X = (x_{ij})$ порядка $m \times n$ удовлетворяющих для $i = 1, 2, \ldots, m$ и $j = 1, 2, \ldots, n$ следующим условиям

$$x_{ij} \geqslant 0, \quad \sum_{j=1}^{n} x_{ij} = a_i, \quad \sum_{i=1}^{m} x_{ij} = b_j.$$

В статье приведен итерационный метод решения транспортной задачи с критерием времени.

С начала любым известным методом находится начальное решение $X(B_1)$, а потом конечную последовательность $X(B_1), X(B_2), \ldots, X(B_k)$ базисных решений, применяя к этой цели формулы приведенные в пунктах 2-5 алгоритма описанного в главе 3.

Базисное решение $X(B_k)$ является искомым оптимальным решением. Соответствующие теоремы можно найти в главе 5.

Метод, приведенный в статье, является модифицированной, и значительно упрощенной версией метода опубликованного в [2]. Модификации приведены с целью приспособить метод к вычислениям на электронных вычислительных машинах.