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REMARKS ON THE AMBIGUITY OF DISTANCE DETERMINATION IN FRIEDMANN'S COSMOLOGICAL MODEL

As we have mentioned in the introduction to [3], the distance between two objects in Friedmann's cosmological model is not uniquely determined by the angular and linear diameters.

An ante-meridiem observer, who works before the radius of world reaches its maximum value, applying the method of linear and angular diameters may obtain two numbers, only one of them is the real distance (only under special circumstancies he is able to determine the distance uniquely).

The post-meridiem observer, who works after the moment of the maximum world radius, is sometimes in a similar position though, in general, the method determines then four numbers, only one of them is the real distance from the observer to the source (there is also possible the case when one gets three such numbers). In order to explain it in more detail let us introduce the notion of the observer's antipode. This notion is clear if it is understood geometrically, because the space in which the observer is situated is a spherical one. However, we are interested in a somewhat different notion of an antipode in time. Suppose that the observer received a light signal at a moment at which the world radius was R_1. The antipode of R_1 is, by definition, a world radius R_A such that the light signal emitted from the geometrical antipode of the observer at the moment when the world radius was R_A will be received by the observer at the moment when the world radius will be R_1.

As we have shown with Dulewicz in [1], R_A is an antipode of R_1 if an only if:

1° R_A is an ante-meridiem world radius;

2° R_1 is a post-meridiem world radius (for the definition see [3]);

3° $R_A + R_1 = R_0$,

where R_0 denotes the maximum radius of the spherical world.

Let us note that if R_1 and the linear and angular diameters of an object are given then there exist (in some cases) two moments which precede the moment corresponding to R_A, and two moments which follow it, such that the light from the given object could be emitted at any of these four moments. For some values of linear and angular diameters such two pre-antipodal moments do not exist.

The above assertion is a corollary from an equation relating the radius of the world at the moment of emission, the radius of the world at the moment of reception and the linear diameter as well as the observed angular diameter of the object, and from a theorem on extrema of functions, which will be formulated in the sequel.

The equation in question is obtained in an elementary way from the well-known laws of relativistic optics and of spherical geometry [2].

Let f be a continuous function on the closure $\bar{D}$ of a bounded plane domain D. We shall define the following notions.

Definition 1. A *left point* of $\bar{D}$ is a point whose abscissa is not greater than the abscissa of any point in $\bar{D}$.

Definition 2. A function φ on $\bar{D}$ is *pseudo-increasing* at x_0 from the right side if there exists a sequence $\{x_n\}$ converging to x_0 and such that $x_n > x_0$ and $\varphi(x_n) \geqslant \varphi(x_0)$ for each n $(n = 1, 2, \ldots)$.

Definition 3. A point $(\bar{x}, \bar{y})$ of $\bar{D}$ is a *maximum left point* with respect to the function f if it is a left point of $\bar{D}$ and $f(\bar{x}, \bar{y}) \geqslant f(x, y)$ for all left points (x, y) of $\bar{D}$.

Since the set of all left points of $\bar{D}$ is compact and f is continuous, we have

LEMMA 1. *There exists at least one maximum left point.*

Now we shall prove the following:

LEMMA 2. *If φ is a continuous function on $[a, b]$ and $\varphi(x_1) \leqslant \varphi(x_2)$, and $\varphi(x_3) \geqslant \varphi(x_4)$ for some x_1, x_2, x_3 and x_4 such that $a \leqslant x_1 < x_2 < x_3 < x_4 \leqslant b$, then there exists an x such that $x_1 < x < x_4$, and φ admits a weak local maximum at x.*

Proof. If $\varphi(x_1) = \varphi(x_2)$ or $\varphi(x_3) = \varphi(x_4)$ the assertion is obvious. If $\varphi(x_1) < \varphi(x_2)$ and $\varphi(x_3) > \varphi(x_4)$ then the absolute maximum of φ in $[x_1, x_4]$ cannot be achieved at the endpoints x_1 and x_4. It has to be achieved at some inner point x $(x_1 < x < x_4)$ and the absolute maximum in $[x_1, x_4]$ is at least a local maximum in $[a, b]$.

THEOREM. *If f is a continuous function on the closure $\bar{D}$ of a plane domain D and*

(1) *there do not exist weak local maxima of f in D;*

(2) *for each fixed x_0, belonging to the projection (a, b) of D onto axis x, $f(x_0, y)$ admits an absolute maximum at an inner point of $\bar{D}$;*

(3) *there exists a maximum left point of f in $\bar{D}$, at which f is pseudo-increasing from the right side with respect to the variable x,*

then the function F defined as follows

$$F(x) = \max_{\substack{y \\ (x,y)\in \bar{D}}} f(x,y), \quad \textit{for } x\in(a,b),$$

is increasing in (a,b).

Proof. Note first that F is a continuous function and it does not admit weak local maxima in (a,b). Otherwise there would exist a weak local maximum of f in D, contrary to the assumption (1).

We shall show now that F neither admits weak local minima in (a,b). Suppose, on the contrary, that there exists a weak local minimum of F in (a,b), say at point x_4. Hence there exists x_3 in (a,b) such that $x_3 < x_4$ and $F(x_3) \geqslant F(x_4)$.

Let (x_1, y_1) be a maximum left point of f at which the function f, considered as a function of a single variable x, is pseudo-increasing. Then there exists a point x_2 such that $x_2 < x_3$ and $f(x_1, y_1) \leqslant f(x_2, y_1)$, and $f(x_2, y_1) \leqslant F(x_2)$.

From the definition of F and the maximum left point we have also $f(x_1, y_1) = F(x_1)$, so that finally $F(x_1) \leqslant F(x_2)$.

Applying lemma 2 to the points x_1, x_2, x_3 and x_4, defined above, we conclude that F admits a weak local maximum in the open interval (x_1, x_4). Thus f admits a weak local maximum in D, which is a contradiction to the previous statement.

Since F in (a,b) admits neither weak local minima nor weak local maxima it has to be monotone. The function F cannot be decreasing since $F(x_1) \leqslant F(x_2)$. Thus it has to be increasing which ends the proof of the theorem.

Remark. In practice we may apply a simple criterion for the verification as to whether a function is a pseudo-increasing one by means of the estimation of its derivative or the difference quotient. In order to verify that f does not admit weak local minima it suffices to prove that both partial derivatives of f do not equal to zero simultaneously. On the other hand, the assumption that f is differentiable and both its partial derivatives do not vanish simultaneously neither simplifies the formulation of the theorem nor makes it easier to prove it. The assumption of this kind implies the non-existence of a maximum of F, i. e. the $\max\max f(x,y)$, but it does not imply the non-existence of a minimum of F (there are examples of f and such points (x,y) in D at which f admits $\min\max f$ while the partial derivatives are not equal to zero simultaneously).

If the linear diameter of an object is fixed, then the angular diameter of the object depends on the distance between the object and the observer and on the moment of observation or, what is equivalent, it depends on the moment of emission of the signal. We shall not write this dependence in its explicit form since the analytic representation of the function would be different in many subcases which ought to be considered.

If we want to use the above theorem for the investigation of this dependence we must first verify the assumptions (1), (2) and (3), which is rather complicated and a purely calculatory problem [2]. However, let us note the following situation, which is rather typical for many problems concerning ambiguity of the distance. Function f, considered as a function of only one variable x (the moment of emission), admits two local maxima and therefore its inverse f^{-1} is in general a four-valued function. The domain D may be docomposed into two domains D_1 and D_2 such that $f|D_i$, $i = 1, 2$, admits only one (local) maximum; let us denote them by

$$F_1(x) = \max_{\substack{y \\ (x,y)\in D_1}} f(x, y) \quad \text{and} \quad F_2(x) = \max_{\substack{y \\ (x,y)\in D_2}} f(x, y).$$

The projections of D_1 and D_2 into axis x coincide. Let (a, b) denote this common projection and let c be the middle point of (a, b).

Function F_2 is symmetric with respect to c, i. e.

$$F_2(c-v) = F_2(c+v) \quad \text{for } |v| < (b-a)/2$$

Let us decompose the closure $\bar{D}_2$ of the domain D_2 into $\bar{D}_2'$ and $\bar{D}_2''$, $\bar{D}_2'$ and $\bar{D}_2''$ being the domains lying over $[a, c]$ and $[c, b]$, respectively. Function f satisfies the assumptions (1), (2) and (3) in each of the domains $\bar{D}_2'$ and $\bar{D}_2''$ and, moreover, $F_2(a) = F_1(b)$.

According to the theorem both F_1 and F_2 are increasing functions. Hence the values of F_2 on $(a, c]$ are greater than the values of F_1 on $[c, b)$. Since F_2 is symmetric with respect to c, the values of F_2 are greater than the values of F_1 on the whole interval (a, b). Thus, for angular diameters greater than $F_1(x)$ and less than $F_2(x)$, function f admits each value between $F_1(x)$ and $F_2(x)$ exactly at two points belonging to the domain D_2. For the angular diameters less than $F_1(x)$ the function f admits each value less than $F_1(x)$ exactly at two points of D_1 and exactly at two points of D_2.

References

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UWAGI O WIELOZNACZNOŚCI WYZNACZANIA ODLEGŁOŚCI W KOSMOLOGICZNYM MODELU FRIEDMANNA

STRESZCZENIE

Jak wspomniano we wstępie do poprzedniej pracy [3], odległość dwu obiektów w kosmologicznym modelu Friedmanna nie jest jednoznacznie wyznaczalna metodą pomiaru średnicy kątowej i liniowej danego obiektu. W niniejszej pracy pokazano, że — w zależności od momentu obserwacji — można otrzymać dwie lub nawet cztery różne wartości, z których tylko jedna jest rzeczywistą odległością obiektu od obserwatora.
