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QUEUEING SYSTEMS WITH FEEDBACK

1. One of the fundamental characteristics of a queueing system is the stochastic process $n(t)$ defined as the number of customers being in the system at moment t. In some cases this process is a Markov process and its investigation is relatively easy. If the process $n(t)$ is not a Markovian one, its "Markovization" is possible and is usually performed by an appriopriate extension of the state of the system or by constructing an appriopriate imbedded Markov chain. The method of state extension which consists in forming a vector process whose one component is the process $n(t)$ gives immediately the interesting characteristics of the process $n(t)$. The method of imbedded Markov chains gives the characteristics of process $n(t)$ in selected moments only, e.g., the system $GI/M/N$ is usually investigated in the arrival moments. Thus, by using this method, the interesting characteristics of the process $n(t)$ in continuous time may be obtained only through additional effort.

Let us notice that the limiting probability distributions of the state of the system immediately before arrival moments of the customers may be used to find the characteristics of waiting times. Thus, the investigation of the state probability distributions of some imbedded Markov chains has some value also in these cases when the state probability distribution in continuous time is known.

For any queueing systems, the mentioned relations between the probability distributions of the state of $n(t)$ and the selected Markov chains are known (Fabens [1], Foster [2], Foster and Perrera [3], Takács [8]). This paper presents a method of finding these relations by the use of extended Markov processes. We illustrate it, applying this method to queueing systems with feedback ([5]-[7]).

2. We consider now two queueing systems. First, let us consider the system $GI/M/\infty$ with feedback of service intensity and queue length. We denote by $G(x)$ the distribution function of interarrival time lengths,

$$\frac{1}{\lambda} = \int_0^\infty x\,dG(x),$$

and by μ_n ($n = 1, 2, \ldots$) the momentary service intensity (in all service channels) under the condition that n units are in the system. The second system considered will be the system $M/G/N$ with feedback of arrival intensity and queue length. Here we denote by $H(x)$ the distribution function of service time,

$$\frac{1}{\mu} = \int_0^\infty x dH(x),$$

and by λ_n ($n = 0, 1, \ldots$) the momentary arrival intensity under the condition that n units are in the system. We assume, in addition, that necessary expected values exists, and that $G(0+) = H(0+) = 0$.

THEOREM 1. *Consider the system $GI/M/\infty$ with feedback of service intensity and queue length. If $X(t)$ denotes the time interval to the next arrival, then the stochastic process $\big(n(t), X(t)\big)$ is a Markov one. The stationary state probabilities of this process*

$$P_n(x) = \Pr\big(n(t) = n,\ X(t) < x\big), \quad n = 0, 1, \ldots, \quad x \geqslant 0,$$

provided they exist, satisfy the following system of differential equations:

$$\text{(1)} \quad \begin{aligned} P_0'(x) - P_0'(0) + \mu_1 P_1(x) &= 0, \\ P_n'(x) - P_n'(0) - \mu_n P_n(x) + G(x) P_{n-1}'(0) + \mu_{n+1} P_{n+1}(x) &= 0, \quad n = 1, 2, \ldots \end{aligned}$$

THEOREM 2. *Let $\{S_r\}$ denote the sequence of arrival moments to the system of Theorem 1 and let $\{n_r\} = \{n(S_r - 0)\}$. The sequence $\{n_r\}$ is a Markov chain and the stationary state probabilities of this chain*

$$p_n^- = \Pr(n_r = n), \quad n = 0, 1, \ldots,$$

provided they exist, are equal to

$$\text{(2)} \quad \begin{aligned} p_{n+1} &= \frac{\lambda}{\mu_{n+1}} p_n^-, \quad n = 0, 1, \ldots, \\ p_0 &= 1 - \sum_{n=0}^{\infty} \frac{\lambda}{\mu_{n+1}} p_n^-, \end{aligned}$$

where $p_n = \Pr\big(n(t) = n\big) = P_n(\infty)$, $n = 0, 1, \ldots$

COROLLARY 1. *For the system $GI/M/N$ (with waiting) we have*

$$\mu_n = \begin{cases} n\mu, & n = 1, 2, \ldots, N, \\ N\mu, & n = N+1, N+2, \ldots, \end{cases}$$

thus

$$p_{n+1} = \begin{cases} \dfrac{\lambda}{(n+1)\mu} p_n^-, & n = 0, 1, \ldots, N-1, \\ \dfrac{\lambda}{N\mu} p_n^-, & n = N, N+1, \ldots \end{cases}$$

In particular, for the system $GI/M/1$ we have $\mu_n = \mu$, $n = 1, 2, \ldots$, thus $p_{n+1} = \lambda p_n^- / \mu$, $n = 0, 1, \ldots, p_0 = 1 - \lambda/\mu$ (Foster and Perrera [3]).

COROLLARY 2. *For the system $GI/M/N$ with losses, we have*

$$\mu_n = \begin{cases} n\mu, & n = 1, 2, \ldots, N, \\ \infty, & n = N+1, N+2, \ldots, \end{cases}$$

thus

$$p_{n+1} = \frac{\lambda}{(n+1)\mu} p_n^-, \quad n = 0, 1, \ldots, N-1,$$

$$p_0 = 1 - \sum_{n=0}^{N-1} \frac{\lambda}{(n+1)\mu} p_n^-$$

(Takács [8], p. 182).

Proof of Theorem 1. The behaviour of the process $(n(t), X(t))$ depends after moment t only on the state of this process at moment t; thus it is a Markov process. Assume it to be stationary. An analysis of the state of the system at moments $t+\tau$ and t leads easily to

$$P_0(x) = \Pr(n(t) = 0,\ \tau \leqslant X(t) < x+\tau) + \\ + \mu_1 \tau \Pr(n(t) = 1,\ \tau \leqslant X(t) < x+\tau) + o(\tau),$$

$$P_n(x) = (1 - \mu_n \tau) \Pr(n(t) = n,\ \tau \leqslant X(t) < x+\tau) + \\ + G(x) \Pr(n(t) = n-1,\ X(t) < \tau) + \\ + \mu_{n+1} \tau \Pr(n(t) = n+1,\ \tau \leqslant X(t) < x+\tau) + o(\tau), \quad n = 1, 2, \ldots$$

From this it follows

$$\begin{aligned} P_0(x) &= P_0(x+\tau) - P_0(\tau) + \mu_1 \tau [P_1(x+\tau) - P_1(\tau)] + o(\tau), \\ P_n(x) &= (1 - \mu_n \tau)[P_n(x+\tau) - P_n(\tau)] + G(x) P_{n-1}(\tau) + \\ &\quad + \mu_{n+1} \tau [P_{n+1}(x+\tau) - P_{n+1}(\tau)] + o(\tau), \quad n = 1, 2, \ldots \end{aligned} \tag{3}$$

After derivating and taking the limit for $\tau \to 0$, we have

$$P_0'(x) - P_0'(0) + \mu_1 P_1(x) = 0,$$

$$P_n'(x) - P_n'(0) - \mu_n P_n(x) + G(x) P_{n-1}'(0) + \mu_{n+1} P_{n+1}(x) = 0, \quad n = 1, 2, \ldots$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Let

$$P(x) = \Pr(X(t) < x) = \sum_{n=0}^{\infty} P_n(x).$$

From system (1), after sidewise summation, we obtain

$$P'(x) - P'(0) + G(x) P'(0) = 0$$

and, since $P(0) = 0$, $P(\infty) = 1$, we have

$$P(x) = \lambda \int_0^x \big(1-G(u)\big)\,du. \tag{4}$$

For $x \to \infty$, $P_n(x) \to p_n$ and $P_n'(x) \to 0$. The last relation is obvious because otherwise the probabilities $P_n(x)$ were unlimited. Taking the limit for $x \to \infty$ in system (1), we have

$$-P_0'(0) + \mu_1 p_1 = 0,$$

$$-P_n'(0) - \mu_n p_n + P_{n-1}'(0) + \mu_{n+1} p_{n+1} = 0, \quad n = 1, 2, \ldots,$$

hence

$$P_n'(0) = \mu_{n+1} p_{n+1}, \quad n = 0, 1, \ldots \tag{5}$$

From (4) it follows

$$\begin{aligned} P_n'(0) &= \lim_{\tau\to 0} \frac{1}{\tau} \Pr\big(n(t) = n,\ X(t) < \tau\big) \\ &= \lim_{\tau\to 0} \frac{1}{\tau} \Pr\big(X(t) < \tau\big) \Pr\big(n(t) = n \mid X(t) < \tau\big) \\ &= \lim_{\tau\to 0} \frac{\lambda}{\tau} \int_0^\tau \big(1-G(u)\big)\,du \Pr(n_\tau = n) = \lambda p_n^-. \end{aligned}$$

This and (5) imply (2) which completes the proof of Theorem 2.

THEOREM 3. *Consider the system $M/G/N$ with feedback of arrival intensity and queue length If, for $n(t) > 0$, we denote by $X_1(t), \ldots, X_{\min(n(t),N)}(t)$ the times necessary to finish the services of customers being served at moment t, the stochastic process $[n(t), X_1(t), \ldots, X_{\min(n(t),N)}(t)]$ is a Markov one.*

The stationary state probabilities of this process

$$P_0 = \Pr\big(n(t) = 0\big),$$

$$P_n(x_1, \ldots, x_{\min(n,N)}) = \Pr\big(n(t) = n,\ X_i(t) < x_i,\ i = 1, 2, \ldots, \min(n, N)\big),$$
$$n = 1, 2, \ldots,$$

satisfy, provided they exist, the following system of differential equations:

$$-\lambda_0 P_0 + \frac{\partial}{\partial x_1} P_1(0) = 0, \tag{6}$$

$$\frac{\partial}{\partial x_1} P_1(x_1) - \frac{\partial}{\partial x_1} P_1(0) - \lambda_1 P_1(x_1) + \lambda_0 H(x_1) P_0 + 2 \frac{\partial}{\partial x_2} P_2(x_1, 0) = 0, \tag{7}$$

(8)
$$\sum_{j=1}^{n}\frac{\partial}{\partial x_j}P_n(x_1,\ldots,x_n)-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}P_n(x_1,\ldots,x_{j-1},0,x_{j+1},\ldots,x_n)-$$
$$-\lambda_nP_n(x_1,\ldots,x_n)+\frac{\lambda_{n-1}}{n}\sum_{j=1}^{n}H(x_j)P_{n-1}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n)+$$
$$+(n+1)\frac{\partial}{\partial x_{n+1}}P_{n+1}(x_1,\ldots,x_n,0)=0,\quad n=2,3,\ldots,N-1,$$

(9)
$$\sum_{j=1}^{N}\frac{\partial}{\partial x_j}P_N(x_1,\ldots,x_N)-\sum_{j=1}^{N}\frac{\partial}{\partial x_j}P_N(x_1,\ldots,x_{j-1},0,x_{j+1},\ldots,x_N)-$$
$$-\lambda_NP_N(x_1,\ldots,x_N)+\frac{\lambda_{N-1}}{N}\sum_{j=1}^{N}H(x_j)P_{N-1}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_N)+$$
$$+\sum_{j=1}^{N}\frac{\partial}{\partial x_j}H(x_j)P_{N+1}(x_1,\ldots,x_{j-1},0,x_{j+1},\ldots,x_N)=0,$$

(10)
$$\sum_{j=1}^{N}\frac{\partial}{\partial x_j}P_n(x_1,\ldots,x_N)-\sum_{j=1}^{N}\frac{\partial}{\partial x_j}P_n(x_1,\ldots,x_{j-1},0,x_{j+1},\ldots,x_N)-$$
$$-\lambda_nP_n(x_1,\ldots,x_N)+\lambda_{n-1}P_{n-1}(x_1,\ldots,x_N)+$$
$$+\sum_{j=1}^{N}H(x_j)\frac{\partial}{\partial x_j}P_{n+1}(x_1,\ldots,x_{j-1},0,x_{j+1},\ldots,x_N)=0,$$
$$n=N+1,N+2,\ldots$$

We present Theorem 3 without proof. Equations (6)-(8) are identical with those in Sevastyanov's system, also the proof of (6)-(10) is similar (see Gnedenko and Kovalenko [4], p. 386).

THEOREM 4. *Let $\{\sigma_r\}$ be the sequence of moments of service ends in the system defined in Theorem 3 and $\{n_r\}=\{n(\sigma_r-0)\}$. The stationary state probabilities of this chain,*

$$p_n^*=\Pr(n_r=n),\quad n=1,2,\ldots,$$

provided they exist, satisfy

(11)
$$\lambda_np_n=\begin{cases}(n+1)(1-p_0)\mu p_{n+1}^*, & n=0,1,\ldots,N-1,\\ N(1-p_0)\mu p_{n+1}^*, & n=N,N+1,\ldots,\end{cases}$$

where $p_n=\Pr\big(n(t)=n\big)=P_n(\infty,\ldots,\infty)$.

Corollary 3. *In the system $M/G/N$ with limited queue length to $K-N$ places, we have*

$$\lambda_n = \begin{cases} \lambda, & n = 0, 1, \ldots, K-1, \\ 0, & n = K, K+1, \ldots, \end{cases}$$

thus

$$p_n^* = \begin{cases} \dfrac{\lambda p_{n-1}}{n\mu(1-p_0)}, & n = 1, 2, \ldots, N-1, N, \\ \dfrac{\lambda p_{n-1}}{N\mu(1-p_0)}, & n = N+1, \ldots, K. \end{cases}$$

In particular, we have in the system $M/G/N$ with losses

$$p_n^* = \frac{\lambda p_{n-1}}{n\mu(1-p_0)} = \frac{p_n}{1-p_0}, \quad n = 1, 2, \ldots, N.$$

We used here the formula $p_n = \lambda p_{n-1}/n\mu$ (Sevastyanov, see [4], p. 382).

Corollary 4. *In the system $M/G/1$ (without losses) we have*

$$p_n^* = \frac{\lambda p_{n-1}}{\mu(1-p_0)} = p_{n-1}, \quad n = 1, 2, \ldots$$

(Foster [2]).

Proof of Theorem 4. First, we shall prove that

$$P(x) = \Pr\big(X_1(t) < x\big) = p_0 + (1-p_0)\mu \int_0^x \big(1-H(u)\big)\,du. \tag{12}$$

It is easy to verify, by substitution into (6)-(7), that

$$P_n(x_1, \ldots, x_n) = p_0 \frac{\lambda_0 \ldots \lambda_{n-1}}{n!} \prod_{j=1}^{n} \int_0^{x_j} \big(1-H(u)\big)\,du, \tag{13}$$
$$n = 1, 2, \ldots, N.$$

Introducing the functions

$$P_n(x) = \Pr\big(n(t) = n,\ X_1(t) < x\big) = P_n(x, \infty, \ldots, \infty),$$

we may write

$$P_n'(x) - P_n'(0)\big(1-H(x)\big) = 0, \quad n = 1, 2, \ldots, N-1. \tag{14}$$

Substitution of $x_1 = x$, $x_j = \infty$ for $j = 2, 3, \ldots$, into (9) and (10) leads to

$$P_N'(x) - P_N'(0) - (N-1)\frac{\partial}{\partial x_2} P_N(x, 0) - \lambda_N P_N(x) + \tag{15}$$

$$+\frac{\lambda_{N-1}}{N}[H(x)P_{N-1}(\infty)+(N-1)P_{N-1}(x)]+H(x)P'_{N+1}(0)+$$
$$+(N-1)\frac{\partial}{\partial x_2}P_{N+1}(x,0) = 0,$$

$$(16)\quad P_n(x)-P'_n(0)-(N-1)\frac{\partial}{\partial x_2}P_n(x,0)-\lambda_n P_n(x)+\lambda_{n-1}P_{n-1}(x)+$$
$$+H(x)P'_{n+1}(0)+(N-1)\frac{\partial}{\partial x_2}P_{n+1}(x,0) = 0, \quad n = N+1, N+2, \ldots$$

Obviously,

$$P(x) = \Pr\left(X_1(t)<x\right) = \sum_{n=1}^{\infty} P_n(x);$$

thus summation of (14), (15) and (16) gives

$$P'(x)-P'(0)\left(1-H(x)\right)+$$
$$+\left[(N-1)\frac{\partial}{\partial x_2}P_N(x,0)-\frac{\lambda_{N-1}}{N}\left(H(x)P_{N-1}(\infty)-(N-1)P_{N-1}(x)\right)-\right.$$
$$\left.-P'_N(0)H(x)\right] = 0.$$

From (13) it follows that the expression in the square brackets equals zero, hence we obtain the differential equation $P'(x) = P'(0)\left(1-H(x)\right)$, which is solved under the conditions $P(0) = 0$, $P(0+) = p_0$, $P(\infty) = 1$. We have

$$P(x) = p_0+(1-p_0)\mu\int_0^x \left(1-H(u)\right)du,$$

which completes the proof of (12).

Taking in (6)-(10) the limit for $x_j \to \infty$, $j = 1, 2, \ldots$, and using the fact that

$$\frac{\partial}{\partial x_j}P_n(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots,) \to P'_n(0)$$

and

$$\frac{\partial}{\partial x_j}P_n(x_1, \ldots, x_n) \to 0$$

for every j, we obtain the following system of equations:

$$-\lambda_0 p_0+P'_1(0) = 0,$$
$$-nP'_n(0)-\lambda_n p_n+\lambda_{n-1}p_{n-1}+(n+1)P'_{n+1}(0) = 0, \quad n = 1, 2, \ldots, N-1,$$
$$-NP'_n(0)-\lambda_n p_n+\lambda_{n-1}p_{n-1}+NP'_{n+1}(0) = 0, \quad n = N, N+1, \ldots$$

Hence

$$\lambda_n p_n = \begin{cases} (n+1)P'_{n+1}(0), & n = 0, 1, \ldots, N-1, \\ NP'_{n+1}(0), & n = N, N+1, \ldots \end{cases} \tag{17}$$

From (12) it follows that

$$\begin{aligned} P'_n(0) &= \lim_{\tau \to 0} \frac{1}{\tau} \Pr\big(n(t) = n,\ X_1(t) < \infty, \ldots, X_{j-1}(t) < \infty, \\ &\qquad\qquad X_j(t) < \tau,\ X_{j+1}(t) < \infty, \ldots, X_n(t) < \infty\big) \\ &= \lim_{\tau \to 0} \frac{1}{\tau} \Pr\big(0 < X_1(t) < \tau\big) \Pr\big(n(t) = n \mid X_1(t) < \tau,\ X_2(t) < \infty, \ldots\big) \\ &= (1-p_0)\mu p_n^*. \end{aligned}$$

This completes the proof of Theorem 4.

3. In the system defined by Theorem 3 we have an immediate feedback of arrival intensity and queue length. Assuming that the arrival intensity at moment t depends upon the state of the system at moment $t-\tau$, one has the case of feedback with constant delay τ. Assume now that in the system with immediate feedback of arrival intensity and queue length every customer before entering the system must undergo a quarantine, the time of which is a random variable with the distribution $K(x)$. Such a system (without the quarantine) is called a *system with delayed feedback of arrival intensity and queue length* [7]. As before, let $n(t)$ denote the number of customers being in the system at moment t and let $m(t)$ denote the number of customers being in quarantine at moment t. The system $M/M/1$ has been considered in [7] under the assumption that the time of being in quarantine is exponentially distributed, $K(x) = 1-\exp(-\nu x)$. The stochastic process $\big(m(t), n(t)\big)$ is then a Markov process, and the stationary state probabilities of this process,

$$p_{m,n} = \Pr\big(m(t) = m,\ n(t) = n\big), \quad m, n = 0, 1, \ldots,$$

provided they exist, satisfy the following system of linear equations:

$$\begin{aligned} -\lambda_0 p_{0,0} + \mu p_{0,1} &= 0, \\ -(\lambda_0 + m\nu) p_{m,0} + \mu p_{m,1} + \lambda_0 p_{m-1,0} &= 0, \\ -(\lambda_n + \mu) p_{0,n} + \mu p_{0,n+1} + \nu p_{1,n-1} &= 0, \\ -(\lambda_n + m\nu + \mu) p_{m,n} + \mu p_{m,n+1} + (m+1)\nu p_{m+1,n-1} + \lambda_n p_{m-1,n} &= 0, \\ m, n &= 1, 2, \ldots \end{aligned} \tag{18}$$

THEOREM 5. *Consider the system $M/M/1$ with delayed feedback of arrival intensity and queue length, where the delay time is distributed as*

$K(x)$. If, for $m(t) > 0$, we denote by $X_1(t), \ldots, X_{m(t)}(t)$ the times necessary to end the quarantines the customers being there at moment t, the stochastic process $[m(t), n(t), X_1(t), \ldots, X_{m(t)}(t)]$ is a Markov one. The stationary state probabilities of this process,

$$P_{0,n} = \Pr\big(m(t) = 0,\ n(t) = n\big),$$

$$P_{m,n}(x_1, \ldots, x_m) = \Pr\big(m(t) = m, n(t) = n,\ X_1(t) < x_1, \ldots, X_m(t) < x_m\big),$$
$$m = 1, 2, \ldots, \quad n = 0, 1, \ldots,$$

satisfy, provided they exist, the system of equations

$$-(\lambda_n + \mu)P_{0,n} + \frac{\partial}{\partial x_1} P_{1,n-1}(0) + \mu P_{0,n+1} = 0,$$

$$\frac{\partial}{\partial x_1} P_{1,n}(x_1) - \frac{\partial}{\partial x_1} P_{1,n}(0) - \big(\lambda_n + (1-\delta_{0,n})\mu\big)P_{1,n}(x_1) +$$
$$+ \lambda_n P_{0,n} K(x_1) + 2\frac{\partial}{\partial x_2} P_{2,n-1}(x_1, 0) + \mu P_{1,n+1}(x_1) = 0,$$

$$(19)\quad \sum_{j=1}^{m} \frac{\partial}{\partial x_j} P_{m,n}(x_1, \ldots, x_m) - \sum_{j=1}^{m} \frac{\partial}{\partial x_j} P_{m,n}(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n) -$$
$$-\big(\lambda_n + (1-\delta_{0,n})\mu\big)P_{m,n}(x_1, \ldots, x_m) +$$
$$+ \frac{\lambda_n}{m} \sum_{j=1}^{m} P_{m-1,n}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m) K(x_j) +$$
$$+ (m+1)\frac{\partial}{\partial x_{m+1}} P_{m+1,n-1}(x_1, \ldots, x_m, 0) + \mu P_{m,n+1}(x_1, \ldots, x_m) = 0.$$

The proof of this theorem is similar to those of Theorems 1 and 3.

THEOREM 6. *Consider the system $M/M/1$ with delayed feedback of arrival intensity and queue length under the assumption that the quarantine times are exponentially distributed with parameter ν. If $\{S_r\}$ are the arrival moments of customers at service (the moments of ending the quarantine) and if $\{(m_r, n_r)\} = \{(m(S_r-0), n(S_r-0))\}$, and $p^-_{m,n} = \Pr(m_r = m, n_r = n)$ we have*

$$(20)\qquad p^-_{m,n} = \frac{p_{m,n}}{1-p_{0.}}, \quad m = 1, 2, \ldots, \quad n = 0, 1, \ldots,$$

where $p_{m,n}$ are given by system (18), and $p_{0.} = \sum_{n=0}^{\infty} p_{0,n}$.

Proof. For $x_j \to \infty$, $j = 1, 2, \ldots$, we have

$$\frac{\partial}{\partial x_j} P_{m,n}(x_1, \ldots, x_m) \to 0,$$

thus, taking the limit in system (19) leads to [1],

$$(21)\qquad -mP'_{m,n}(0)-(\lambda_n+\mu)P_{m,n}+\lambda_n P_{m-1,n}+ \\ +(m+1)P'_{m+1,n}(0)+\mu P_{m,n+1}=0, \qquad m,n=1,2,\ldots,$$

where

$$P'_{m,n}(0)=\lim_{\substack{x_1\to\infty\\ \cdots\\ x_m\to\infty}}\frac{\partial}{\partial x_j}P_{m,n}(x_1,\ldots,x_{j-1},0,x_{j+1},\ldots,x_m).$$

From the assumption $K(x)=1-e^{-\nu x}$ it follows that

$$P_{m,n}(x_1,\ldots,x_m)=p_{m,n}\prod_{j=1}^{m}(1-e^{-\nu x_j}),$$

since such a substitution transforms system (19) into system (18). Hence

$$\begin{aligned}P'_{m,n}(0)&=\lim_{\tau\to 0}\frac{1}{\tau}P_n(\tau,\infty,\ldots,\infty)\\ &=\lim_{\tau\to 0}\Pr\big(0<X_1(t)<\tau\big)\Pr\big(m(t)=m,\,n(t)=n\,|\,0<X_1(t)<\tau\big)\\ &=(1-p_{0.})\nu\Pr(m_r=m,\,n_r=n)=(1-p_{0.})\nu p^{-}_{m,n},\end{aligned}$$

which completes the proof of Theorem 6.

THEOREM 7. *Consider the system $M/G/1$ with delayed feedback of arrival intensity and queue length, where the quarantine times are exponentially distributed with parameter ν. If we denote for $n(t)>0$ by $X(t)$ the time necessary to end the service of a customer being in service at moment t, the stochastic process $\big(m(t),n(t),X(t)\big)$ is a Markov process. The stationary state probabilities of this process,*

$$P_{m,0}=\Pr\big(m(t)=m,\,n(t)=0\big),$$
$$P_{m,n}(x)=\Pr\big(m(t)=m,\,n(t)=n,\,X(t)<x\big),\quad m=0,1,\ldots,\ n=1,2,\ldots,$$

satisfy, provided they exist, the system of differential equations

$$\begin{aligned}(22)\qquad & -(m\nu+\lambda_0)P_{m,0}+\lambda_0 P_{m-1,0}+P'_{m,1}=0,\\ & P'_{m,1}(x)-P'_{m,1}(0)-(m\nu+\lambda_1)P_{m,1}(x)+\lambda_1 P_{m-1,1}(x)+\\ &\qquad +(m+1)\nu P_{m+1,0}(x)H(x)+P'_{m,2}(0)H(x)=0,\\ & P'_{m,n}(x)-P'_{m,n}(0)-(m\nu+\lambda_n)P_{m,n}(x)+\lambda_n P_{m-1,n}(x)+\\ &\qquad +(m+1)\nu P_{m+1,n-1}(x)+P'_{m,n+1}(0)H(x)=0,\\ &\qquad\qquad m=0,1,\ldots,\ n=2,3,\ldots,\end{aligned}$$

where $P_{-1,0}=P_{-1,n}(x)=0$, $n=1,2,\ldots$

[1] We limit ourselves to presenting the equations for $m,n=1,2,\ldots$ and we do not present the boundary equations for $m=0$ or $n=0$.

We omit the proof of Theorem 7.

THEOREM 8. *Let $\{\sigma_r\}$ be the sequence of the moments of finishing the services of customers in the system defined in Theorem 7 and let $\{(m_r', n_r')\} = \{(m(\sigma_r - 0), n(\sigma_r - 0))\}$. The sequence $\{(m_r', n_r')\}$ is a Markov chain and the stationary probabilities of this chain,*

$$P_{m,n}^* = \Pr(m_r' = m, n_r' = n), \quad m, n = 0, 1, \ldots,$$

provided they exist, satisfy the system of equations

$$(23) \qquad -(1-P_{.0})\mu P_{m,n}^* - (m\nu + \lambda_n)P_{m,n} + \lambda_n P_{m-1,n} + \\ +(m+1)\nu P_{m+1,n-1} + (1-P_{.0})\mu P_{m,n+1}^* = 0,$$

where $P_{m,n} = \Pr\big(m(t) = m,\ n(t) = n,\ X(t) < \infty\big) = P_{m,n}(\infty)$, *and* $P_{.0} = \sum_{m=0}^{\infty} P_{m,0}$.

Proof. Let

$$P(x) = \Pr\big(0 < X(t) < x\big) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} P_{m,n}(x).$$

Summarize of the equations of (22) gives

$$P'(x) = \big(1-H(x)\big)\Big[\sum_{m=0}^{\infty}\sum_{n=2}^{\infty} P_{m,n}'(0) + \nu\sum_{m=1}^{\infty} mP_{m,0}\Big] = 0,$$

and, since $P(0) = 0$, $P(\infty) = 1 - P_{.0}$, we have

$$P(x) = (1-P_{.0})\mu\int_0^x \big(1-H(u)\big)\,du.$$

For $x \to \infty$, $P_{m,n}'(x) \to 0$, thus, taking the limit in system (22), we obtain

$$P_{m,n}'(0) - (m\nu + \lambda_n)P_{m,n} + \lambda_n P_{m-1,n} + (m+1)\nu P_{m+1,n-1} + P_{m,n+1}'(0) = 0,$$

where

$$\begin{aligned} P_{m,n}'(0) &= \lim_{\tau\to 0}\frac{1}{\tau}P_{m,n}(\tau) \\ &= \lim_{\tau\to 0}\frac{1}{\tau}\Pr\big(0 < X(t) < \tau\big)\Pr\big(m(t) = m,\ n(t) = n \,|\, 0 < X(t) < \tau\big) \\ &= (1-P_{.0})\mu\Pr(m_r' = m,\ n_r' = n) = (1-P_{.0})\mu P_{m,n}^*. \end{aligned}$$

This completes the proof of Theorem 8.

COROLLARY 5. *In the system $M/M/1$ with delayed feedback of arrival intensity and queue length, where the quarantine times are exponentially distributed with parameter ν,*

$$(24) \qquad p_{m,n} = (1-p_{.0})P^*_{m,n}, \qquad m = 0, 1, \ldots, n = 1, 2, \ldots,$$

where $p_{m,n}$ are given in the system of equations (18) *and where* $p_{.0} = \sum_{m=0}^{\infty} p_{m,0}$.

Actually, under the assumption of Corollary 5 the probabilities $P_{m,n}$ are equal to probabilities $p_{m,n}$ given in system (18), also systems (18) and (23) are equivalent. Equations (24) follow immediately from this.

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SYSTEMY OBSŁUGI MASOWEJ ZE SPRZĘŻENIEM ZWROTNYM

STRESZCZENIE

Praca poświęcona jest relacjom między stacjonarnymi rozkładami prawdopodobieństwa liczby jednostek w systemie, rozpatrywanej w czasie ciągłym, i stacjonarnymi rozkładami prawdopodobieństwa odpowiednio wybranych włożonych łańcuchów stochastycznych. Rozpatrzono w niej trzy systemy obsługi masowej: 1° system $GI/M/\infty$ ze sprzężeniem intensywności obsługi z długością kolejki, w którym włożony łańcuch (Markowa) zdefiniowano jako stan systemu bezpośrednio przed momentami zgłoszeń jednostek do systemu; 2° system $M/G/N$ ze sprzężeniem intensywności zgłoszeń z długością kolejki, w którym włożony łańcuch zdefiniowano jako stan systemu bezpośrednio przed momentami zakończenia obsługi; 3° system $M/M/1$ z opóźnionym sprzężeniem intensywności zgłoszeń z długością kolejki, w którym wzięto pod uwagę oba tu zdefiniowane włożone łańcuchy stochastyczne.