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ON COSSERAT CONTINUA

A *Cosserat continuum* is a mechanical medium susceptible to the action of continuously distributed moments of forces. A physical example of such a medium is a model of a magnetized metal with the assumption that it consists of continuously distributed elementary magnets, so that an external magnetic field exercises a moment on each element of the volume. Other examples are given by continuous models of materials with electrically polarized particles, so that the external electric field also exercises moments on each particle. In the theory of shells and rods such continuously distributed moments have been considered for a long time. In this paper we shall discuss, however, three-dimensional Cosserat continua.

The basic equations of equilibrium of such continua, one- two- and three-dimensional, were derived by the brothers, Eugène and François Cosserat at the end of the 19th century and presented finally in their book „*Théorie de corps déformables*”, [2], published in 1909. In the subsequent fifty years their work did not draw much attention, except for a few rather expository articles, e.g., [9], [14]. About the middle of this century a growing number of papers was published dealing with that subject. Some of them are quoted in this paper. A historical account up to 1967 is given by H. Schaefer in [13].

The Cosserats defined a *deformable body* as a continuum to each point of which is attached a frame of three vectors so that the element of the medium has the kinematics of a rigid body. They analyzed the geometry of that continuum and set forth an ambitious plan to derive its mechanics from a single variational principle. The main idea was to postulate that the action integral be invariant with respect to the group of isometries of the Euclidean space. The functional so obtained, called by the authors the “Euclidean action”, depended on six functional arguments so that setting the six variational derivatives equal to zero gave the conditions of equilibrium of the continuum. The first three of the equations, appropriately interpreted, coincided with the classical equa-

tions of elasticity, but with a non-symmetric stress tensor. The second group of the equations contained terms interpreted as another kind of stresses corresponding to the continuously distributed external couples of forces acting per element of the medium. However, in their rather voluminous book, the Cosserats did not discuss the constitutive equations of the continuum nor its dynamics, i.e., equations of motion.

In the subsequent development the theory of Cosserat media has been approached by various methods. Some authors applied the conventional conditions of equilibrium, viz., that the sum of all forces and the sum of all moments be equal to zero, e.g., [1], [12], [16], and obtained basically the same equations. The question whether and why those conditions of equilibrium, taken formally from the statics of a rigid system, apply to a deformable continuum was entertained by few, starting with L. Boltzman (see [8], p. 518 ff, or [18], p. 546, footnote). It was pointed out that those conditions should be actually introduced as an ad hoc axiom in continuum mechanics. In this paper, the mathematical reason and meaning of such assumptions will be discussed from the point of view of the Principle of Virtual Work. Some authors followed the original idea of Cosserats by using variational principles with appropriate invariance conditions (see e.g., [16], [17]). In those principles the lagrangean is assumed to depend on the position vector of the particle of the continuum and its derivatives up to some order. This approach which excludes dissipative forces and non-reversible processes of deformations, has been generalized in various directions. E.g., in [5] there is developed an intricate "multipolar kinematics" motivated by certain analogies with the mechanics of the system of discrete material points. In this paper, a more general, and therefore simpler approach is presented based on a different analysis of rigid systems. This approach also enables us to show that the so generalized Cosserat continua are unique in a certain well defined sense.

Moreover, the approach adopted in this paper leads to clarification of other important aspects of the mechanics of continua, particularly concerning the deformations. As a matter of fact, in the theories of deformable media the very definition of the deformation has an element of arbitrariness. It is not clear why and how some tensors defined as deformations from purely geometrical considerations should relate to the stress tensors defined from mechanical considerations. There is no general agreement among the writers about the way of defining the deformation of the Cosserat media (see e.g., [1], [6], [11], [15]). The situation has been especially critical in the theory of shells which are actually two dimensional Cosserat continua. In the present paper, it is shown that the proper definition of the deformations is determined by such factors

as the dimension of the space, the vanishing of its curvature, and the decomposition of tensors in that space in symmetry parts. Arguments of similar type have been partially considered by Toupin in [16].

The definition of deformations suggested by the generalized principle of virtual work leads in a natural way to a precise definition of an elastic deformation as a reversible process. This idea, which was considered by various authors (see e.g., [8], p. 141), deals with the elastic potential as a scalar-valued function of tensorial arguments, and in conjunction with some basic facts of the theory of invariants gives a more general and thus simpler insight into the nature of the constitutive equations of the deformable continua.

Not many authors consider the equations of motion of Cosserat continua, and if they do, a formalistic approach is given by simply adding to the equations of equilibrium the "inertial forces". This procedure, sanctioned by tradition, seems to contradict some elaborate discussions about the "microstructure" of the kinematics of the continuum. Besides, it is well known that this "D'Alembert principle" is not justified by the general principles of mechanics, unless it is assumed as another ad hoc axiom. In the present paper another approach is given presenting the dynamical characteristics of the continuum, like the density of mass, as quantities of nature analogous with the coefficients of elasticity.

Not all results presented in this paper are claimed to be new. It is rather the point of view and juxtaposition of several known methods and ideas which seem to open new possibilities.

1. Review of statics of rigid systems. Since the concept of the moment of forces or of a couple of forces plays an essential part in the theory of Cosserat continua, we shall start by scrutinizing that concept. It arises in the statics of a rigid system which because, or perhaps in spite of its elementary character is usually taken for granted and escapes critical examination.

The elementary statics of a rigid body is actually an axiomatic theory about "gliding" vectors, based on the assumption that two forces equal in magnitude and acting along the same line in opposite directions are in equilibrium. This assumption leads to the concept of a moment of force, and to the equations of equilibrium requiring that the sum of all forces and the sum of all moments be zero.

The above axiom does not hold for deformable media. Yet, the statics of continuous media is often developed by applying the conditions of equilibrium of a rigid system. For justification, certain ad hoc axioms of "solidification" are sometimes introduced.

In order to avoid such difficulties we shall rather assume another

more general axiom which not only implies the equations of equilibrium of a rigid system, but also can be applied to any other mechanical system. Since the time of Lagrange such an axiom is known under the name of the Principle of Virtual Work, or Virtual Displacements. We shall assume and apply that principle first in its traditional version, postponing a more general formulation to Section 3.

Thus we first define a rigid system by specifying the constraints in the following way. Let a, b, c, \dots be points of a material system, and $\mathbf{x}(a), \mathbf{x}(b), \mathbf{x}(c), \dots$ their position-vectors in a three-dimensional Euclidean space. The system is called *rigid* if it is subject to the constraints

$$(1.1) \quad \delta[(\mathbf{x}(a) - \mathbf{x}(b))(\mathbf{x}(a) - \mathbf{x}(b))] = 0$$

for every pair of points a, b . Here, the symbol δ denotes variation, which is a derivation, i.e., a linear operation satisfying the Leibniz rule for inner products. The constraint (1.1) can be therefore written as

$$(1.2) \quad [\mathbf{x}(a) - \mathbf{x}(b)][\delta\mathbf{x}(a) - \delta\mathbf{x}(b)] = 0.$$

The Principle of Virtual Work is now applied in the following manner. Let $\mathbf{F}(a), \mathbf{F}(b)$ and $\mathbf{F}(c)$ be the external forces acting upon the particles a, b and c , respectively. Let $L(a, b)$ be the Lagrange multiplier corresponding to the constraints (1.2), i.e., some scalar-valued function of the two points a and b . Then, the Principle requires that

$$(1.3) \quad \sum_a \mathbf{F}(a) \delta\mathbf{x}(a) + \sum_a \sum_b L(a, b) [\mathbf{x}(a) - \mathbf{x}(b)][\delta\mathbf{x}(a) - \delta\mathbf{x}(b)] = 0$$

for all $\delta\mathbf{x}(a)$ and $\delta\mathbf{x}(b)$. The summation runs here over the set of all points a, b , of the system. The set of indices may be infinite, and then summation is integration over the set in which existence of a volume measure is stipulated.

The essential feature of the rigid system is that the constraints (1.1) are imposed on every pair of points, and not only on infinitesimally close points. We shall use this remark later on, in Section 2, when discussing deformable continua.

The equations of equilibrium can be now deduced from (1.3) in the following way. Write

$$(1.4) \quad L(a, b) + L(b, a) = P(a, b).$$

Then, condition (1.3) can be written as

$$(1.5) \quad \sum_a \left\{ \mathbf{F}(a) + \sum_b P(a, b) [\mathbf{x}(a) - \mathbf{x}(b)] \right\} \delta\mathbf{x}(a) = 0$$

for every $\delta\mathbf{x}(a)$. This gives the equations of equilibrium

$$(1.6) \quad \mathbf{F}(a) + \sum_b P(a, b) [\mathbf{x}(a) - \mathbf{x}(b)] = 0, \quad P(a, b) - P(b, a) = 0.$$

These are not the traditional equations of the elementary statics of rigid body, i.e.,

$$(1.7) \quad \sum_a \mathbf{F}(a) = 0 \quad \text{and} \quad \sum_a \mathbf{F}(a) \times \mathbf{x}(a) = 0,$$

where \times denotes the cross product. These traditional equations, however, follow directly from (1.6). Indeed, the first of equations (1.7) follows from the fact that the scalar $P(a, b)$ is symmetric in a, b , while the vector $\mathbf{x}(a) - \mathbf{x}(b)$ is antisymmetric and thus their products cancel in the double summation over a, b , of (1.6). Similarly, the second equation (1.7) follows from the symmetry of $P(a, b)$ and the antisymmetry of the cross products $\mathbf{x}(a) \times \mathbf{x}(b)$ with respect to a, b , so that their products also cancel in the double summation over all a, b .

However, the equations in the form (1.6) which are usually not considered in the literature, give more detailed information about the equilibrium than do the traditional equations (1.7). In fact, the above analysis shows that the conventional equations of equilibrium (1.7) are "global" in that sense that they impose conditions upon the whole material system and not on each individual point. In contrast, the equations of equilibrium in the form (1.6) are "local" since they require that the force $\mathbf{F}(a)$ acting at the point a must be balanced by the sum of the reactions of the remaining points. These equations also contain the law of action and reaction because the reaction of the point b upon a defined as

$$(1.8) \quad \mathbf{P}(a, b) = P(a, b)[\mathbf{x}(a) - \mathbf{x}(b)]$$

is antisymmetric in a, b , and directed along the line joining the points a and b .

It should be emphasized that the global equations (1.7) are only necessary, but not sufficient conditions for equilibrium as defined by the Principle of Virtual Work with which the local equations (1.6) are equivalent. And finally, notice that the local equations (1.6) do not require any concept of a moment of force.

We shall now use another argument which leads to the concept of moments and to another form of local equations of equilibrium equivalent to equations (1.6). This argument, which is of a geometric character, is based on a theorem so well known and so elementary that it is seldom stated and proved. We shall call it *Poisson's theorem*:

The constraints (1.1) defining the rigid system are equivalent to the following two conditions:

There exists a vector $\delta\omega$ such that, for every pair of points a, b ,

$$(1.9) \quad \delta\mathbf{x}(a) - \delta\mathbf{x}(b) - \delta\omega \times [\mathbf{x}(a) - \mathbf{x}(b)] = 0.$$

(1.10) *The vector $\delta\omega$ is common for the whole rigid system, i.e., it does not depend on the points a, b .*

Proof. Condition (1.9) is sufficient for (1.1) for any vector $\delta\omega$. This follows by direct substitution of (1.9) into (1.2) which is equivalent to (1.1).

In order to prove the necessity, notice first that condition (1.2) implies that the vector $\delta\mathbf{x}(a) - \delta\mathbf{x}(b)$ is orthogonal to the vector $\mathbf{x}(a) - \mathbf{x}(b)$; thus, there always exists a vector $\delta\omega(a, b)$, depending a priori on the points a, b , such that (1.9) holds, i.e.,

$$(i) \quad \delta\mathbf{x}(a) - \delta\mathbf{x}(b) = \delta\omega(a, b) \times [\mathbf{x}(a) - \mathbf{x}(b)].$$

In order to show that $\delta\omega$ in fact does not depend on the points a, b , apply equality (i) to another pair of points b, c , and add both sides of these equalities to get

$$(ii) \quad \delta\mathbf{x}(a) - \delta\mathbf{x}(c) = \delta\omega(a, b) \times [\mathbf{x}(a) - \mathbf{x}(b)] + \delta\omega(b, c) \times [\mathbf{x}(b) - \mathbf{x}(c)].$$

On the other hand, apply (i) to the pair (a, b) , and write it as follows:

$$(iii) \quad \delta\mathbf{x}(a) - \delta\mathbf{x}(c) = \delta\omega(a, c) \times [\mathbf{x}(a) - \mathbf{x}(b)] + \delta\omega(a, c) \times [\mathbf{x}(b) - \mathbf{x}(c)].$$

Multiply (ii) and (iii) scalarily by $\mathbf{x}(b) - \mathbf{x}(c)$ and subtract to conclude that for every three points a, b and c there must be

$$[\delta\omega(a, b) - \delta\omega(a, c)] \{[\mathbf{x}(a) - \mathbf{x}(b)] \times [\mathbf{x}(b) - \mathbf{x}(c)]\} = 0.$$

This implies that $\delta\omega(a, b) = \delta\omega(a, c)$, thus $\delta\omega$ does not depend on the second points b and c of the pairs (a, b) and (a, c) . Similarly, by taking the dot product of (ii) and of (iii) with $\mathbf{x}(a) - \mathbf{x}(b)$, we can show that $\delta\omega$ does not depend on the first argument a, c , of the pairs (a, c) and (b, c) ; therefore, $\delta\omega$ is the same throughout the entire rigid system, q.e.d.

For further reference, we shall remark that the vector $\delta\omega$ in (1.9) can be uniquely determined by the formula

$$(1.11) \quad \delta\omega = \sum_{0, a, b, c} \frac{1}{2\Delta} e^{abc} \{[\delta\mathbf{x}(a) - \delta\mathbf{x}(0)][\mathbf{x}(b) - \mathbf{x}(0)]\} [\mathbf{x}(c) - \mathbf{x}(0)]$$

in which the sum is extended over four points $0, a, b$ and c located at the vertices of a tetrahedron with non-vanishing volume Δ , and where e^{abc} is equal to $+1$ or -1 , depending on the orientation of the points $0, a, b$ and c .

Also notice that the proof of Poisson's theorem can be carried over to spaces of higher dimensions provided that the vector $\delta\omega$ be replaced by an appropriate antisymmetric tensor.

Let us now apply the Poisson's theorem to the Principle of Virtual Displacements to get another form of the equations of equilibrium for

a rigid system. Let $\mathbf{Q}(a, b)$ be the Lagrange multiplier corresponding to the constraints (1.9). The Principle of Virtual Work requires that

$$\sum_a \mathbf{F}(a) \delta \mathbf{x}(a) + \sum_a \sum_b \mathbf{Q}(a, b) [\delta \mathbf{x}(a) - \delta \mathbf{x}(b) - \delta \boldsymbol{\omega} \times (\mathbf{x}(a) - \mathbf{x}(b))] = 0$$

for all $\delta \mathbf{x}(a)$ and $\delta \boldsymbol{\omega}$. Notice now that $\mathbf{Q}(a, b) = -\mathbf{Q}(b, a)$ because the left-hand side of constraint (1.9) is antisymmetric in a, b . Write, therefore

$$(1.12) \quad \mathbf{Q}(a, b) - \mathbf{Q}(b, a) = \mathbf{R}(a, b) = -\mathbf{R}(b, a),$$

and then the Principle of Virtual Work can be written as

$$\sum_a [\mathbf{F}(a) + \sum_b \mathbf{R}(a, b)] \delta \mathbf{x}(a) + \delta \boldsymbol{\omega} \sum_a \sum_b \mathbf{R}(a, b) \times [\mathbf{x}(a) - \mathbf{x}(b)] = 0$$

for all $\delta \mathbf{x}(a)$ and $\delta \boldsymbol{\omega}$. Hence, the other form of local equations of equilibrium follows:

$$(1.13) \quad \mathbf{F}(a) + \sum_b \mathbf{R}(a, b) = 0, \quad \sum_a \sum_b \mathbf{R}(a, b) \times [\mathbf{x}(a) - \mathbf{x}(b)] = 0, \\ \mathbf{R}(a, b) + \mathbf{R}(b, a) = 0.$$

The second of these equations involves, explicitly, the moment of the reactive force $\mathbf{R}(a, b)$. Again, equations (1.7) of global equilibrium follow directly from the local equations (1.13), but are not equivalent to them.

Comparison with equations (1.6) of local equilibrium shows that the reaction $\mathbf{R}(a, b)$ in (1.12) is not given in such an explicit form as the reaction $P(a, b) [\mathbf{x}(a) - \mathbf{x}(b)]$ in (1.6). However, it should be emphasized that these two local equations, (1.6) and (1.12), are equivalent since, by Poisson's theorem, the constraints (1.1) and (1.9) are equivalent. This is true for a rigid system. For a deformable continuum, as we shall discuss in Section 2, the constraints analogous to (1.1) and (1.9) are not equivalent. Their equivalence must be stipulated by a separate axiom. This is, in fact, the essence of Boltzmann's axiom mentioned in the introduction.

2. Deformable continuum. Guided by analogies, and differences, with the rigid system, we shall now discuss the equilibrium of a deformable continuum. By such a continuum we understand a three-dimensional manifold embedded in the three-dimensional Euclidean space. More specifically, let the particle a of the continuum be labelled by local coordinates a^k ($k = 1, 2, 3$) and let its position-vector in the Euclidean space be $\mathbf{x}(a)$. In contrast with the rigid system, where only a volume measure was needed, we now assume that the material manifold is equipped with a metric tensor defined by

$$(2.1) \quad g_{kl}(a) = \partial_k \mathbf{x} \partial_l \mathbf{x} \quad (\partial_k = \partial / \partial a^k).$$

Thus, we not only have the volume measure $da = \sqrt{g} da^1 da^2 da^3$ in which

$$(2.2) \quad g = \det(g_{kl}),$$

but also covariant differentiation ∇_k , the Christoffel symbols Γ_{kl}^m , and the entire tensor analysis.

In analogy with the constraints (1.1) in Section 1, for a rigid system, we now define a deformable continuum by the constraints

$$(2.3) \quad \delta(dx dx) = 0.$$

This can be transformed to a more convenient form as follows. Since $dx = \nabla_k x da^k$ and since (2.3) must hold for all particles a , the constraint (2.3) can be written as

$$\delta(\nabla_k x \nabla_l x) = \nabla_k \delta x \nabla_l x + \nabla_k x \nabla_l \delta x = 0.$$

Let δx^l be the component of δx in the basis $\nabla_l x$, i.e.,

$$(2.4) \quad \delta x(a) = \delta x^l \nabla_l x,$$

so that $\nabla_k \delta x = \nabla_k \delta x^l \nabla_l x$. Then, the constraints (2.3) are equivalent to

$$(2.5) \quad \nabla_k \delta x_l + \nabla_l \delta x_k = 0, \quad \mathbf{1}$$

and thus the virtual displacement δx_k is a Killing vector in the tangent space of the deformable continuum. The constraints (2.5) actually define the mathematical model of the deformable continuum.

The equations of equilibrium of that continuum are derived by the Principle of Virtual Work in the following way.

Assume that the particles a of the deformable continuum occupy a region \mathcal{A} with the boundary $\partial\mathcal{A}$. Let $F = F^k \nabla_k x$ be the density of forces acting per volume element $da = da^1 da^2 da^3 \sqrt{g}$ and let $f = f^k \nabla_k x$ be the force acting per surface element $d\sigma(a)$ of the boundary having a unit normal vector ν_k directed inside the region \mathcal{A} . The functions $F^k(a)$ are sufficiently regular inside the region \mathcal{A} and on its boundary $\partial\mathcal{A}$, while the components $f^k(a)$ are defined and regular on the boundary $\partial\mathcal{A}$ only. Suppose, for simplicity, that the constraints (2.5) hold inside the region \mathcal{A} , while on its boundary the virtual displacements δx are „free”, i.e., not subject to any constraints. Denoting by $L^{kl}(a)$ the Lagrange multiplier corresponding to the constraint (2.5), the Principle of Virtual Work requires that, for all δx ,

$$(2.6) \quad \int_{\mathcal{A}} da [F \delta x + L^{kl} (\nabla_k \delta x_l + \nabla_l \delta x_k)] + \int_{\partial\mathcal{A}} d\sigma(a) f \delta x = 0.$$

To deduce from this the equations of equilibrium, write

$$(2.7) \quad L^{kl} + L^{lk} = -P^{lk} = -P^{kl}.$$

Integrate (2.6) by parts to conclude that for all δx_l there must be

$$\int_{\mathcal{A}} da (F^k + \nabla_l P^{lk}) \delta x_k + \int_{\partial \mathcal{A}} d\sigma(a) (f^k + \nu_l P^{lk}) \delta x_k = 0.$$

Thus, we obtain the classical equations of equilibrium in the theory of elasticity

$$(2.8) \quad \begin{aligned} F^k + \nabla_l P^{lk} &= 0, & P^{kl} - P^{lk} &= 0 & \text{in } \mathcal{A} \cup \partial \mathcal{A}, \\ f^k + \nu_l P^{lk} &= 0 & & \text{on the boundary } \partial \mathcal{A}. \end{aligned}$$

The tensors P^{lk} , as Lagrange multipliers, are interpreted as the internal stresses of the continuum in reaction to the corresponding constraints (2.5). They have been taken here with the sign minus to conform with the traditional convention fixing the directions of the stresses. The symmetry of the stress P^{lk} follows from the symmetry of the constraint, and not from any consideration of moments. In fact, there have been no moments introduced at all.

Equations (2.8) are the continuous analog to equations (1.6) of local equilibrium of rigid system which, strangely enough, are less known. There are, however, also differences. In the local equations (1.6) of rigid system, the reaction $P(a, b)$ was symmetric with respect to the two points a, b , and expressed the principle of action and reaction. Here, the symmetry of the tensor P^{kl} is of different nature and it refers to the geometrical symmetry of the constraints (1.5).

Now, we shall consider the continuous counterpart of the constraint (1.9), based on Poisson's theorem. In the deformable continuum only the first part, viz. (1.9), of that theorem can be carried over. In fact, the constraints (2.5) imply that $\nabla_k \delta x_l$ is an antisymmetric tensor $\delta \omega_{kl}$. Let $\delta \omega^m$ be its dual, i.e., let

$$(2.9) \quad \nabla_k \delta x_l - \varepsilon_{klm} \delta \omega^m = 0,$$

where ε_{klm} is the antisymmetric discriminant tensor of the manifold. Equation (2.9) is the continuous analog to equation (1.9) of Poisson's theorem. However, the second part, requiring that the vector

$$(2.10) \quad \delta \omega = \delta \omega^k \nabla_k x$$

be independent of the point a of the deformable continuum, need not be satisfied. On the contrary, the vector $\delta \omega^m(a)$ can be any function of the point $a = (a^1, a^2, a^3)$. Any condition imposed upon the vector $\delta \omega$ defines actually a mathematical model of the deformable continuum. Such a condition corresponds to what some authors refer to as "microstructure" (see, e.g., [11] and [17]).

Guided by the analogy with Poisson's theorem, let us assume that, in addition to the constraints (2.9), we have also the constraints

$$(2.11) \quad \nabla_k \delta \omega^l = 0.$$

We will show that the deformable continuum defined by constraints (2.9) and (2.11) is the Cosserat continuum.

In fact, let, as before, F be the external force per volume element, corresponding to the displacement δx , and let $M = M^k \nabla_k x$ be the external generalized force corresponding to the displacement $\delta \omega$ inside the region \mathcal{A} . Let the corresponding factors on the boundary $\partial \mathcal{A}$ be f and m , respectively. Again, supposing for simplicity that the virtual displacements δx and $\delta \omega$ on the boundary are not subject to any constraints, let us denote the Lagrange multipliers corresponding to the constraints (2.9) and (2.11) by $-T^{kl}$ and $-M_{,l}^k$, respectively. Then, the Principle of Virtual Work requires that

$$(2.12) \quad \int_{\mathcal{A}} da [F^k \delta x_k + M_l \delta \omega^l - T^{kl} (\nabla_k \delta x_l - \varepsilon_{klp} \delta \omega^p) - M_{,l}^k \nabla_k \delta \omega^l] + \\ + \int d\sigma(a) (f^k \delta x_k + m_l \delta \omega^l) = 0$$

for all δx_k and $\delta \omega^l$.

Again, integrating by parts, we obtain the following equations of equilibrium:

$$(2.13) \quad \begin{aligned} F^k + \nabla_l T^{kl} &= 0, & M_k + \varepsilon_{kpq} T^{pq} + \nabla_l M_{,k}^l &= 0 & \text{inside } \mathcal{A}, \\ f^k + \nu_l T^{kl} &= 0, & m_k + \nu_l M_{,k}^l &= 0 & \text{on the boundary } \partial \mathcal{A}. \end{aligned}$$

These are exactly the equations of equilibrium of the Cosserat continua if we interpret, as before, the Lagrange multipliers T^{kl} and $M_{,k}^l$ as the stresses corresponding to the constraints. The tensor T^{kl} , which is no more symmetric, is the force-stress, while $M_{,k}^l$ is the couple-stress. If the external moments M_k and the couple-stresses $M_{,k}^l$ are zero, then equations (2.13) become the classical ones, (2.8).

However, in contrast to the classical equations (2.8), the Cosserat equations (2.13) are not a continuous analog to equations (1.13) of local equilibrium of a rigid system. This is obvious if only for the reason that for the rigid system both local equations (1.6) and (1.13) are equivalent, and this is not true for equations (2.8) and (2.13). Moreover, in the equations of equilibrium of rigid systems only one type of reaction occurs, while in the Cosserat equations two type of stresses, viz. T^{kl} and $M_{,k}^l$ appear.

The fact that the Cosserat continuum is defined by the constraints (2.9) and (2.11) yields one more equation which is not mentioned by other authors. To see that, write equation (2.11) in the form

$$(2.14) \quad \delta \omega^m = \frac{1}{2} \varepsilon^{klm} \nabla_k \delta x_l,$$

which, by the way, is analogous to the formula (1.11) for rigid system. Since the space is Euclidean, we have $\nabla_{km} = \nabla_{mk}$; hence (2.14) implies that in the constraint (2.11) there is

$$(2.15) \quad \nabla_k \delta \omega^k = 0.$$

Thus, the Lagrange multiplier $M_{.l}^k$ corresponding to that constraint must satisfy the equation

$$(2.16) \quad M_{.k}^k = 0,$$

i.e., the couple-stress tensor must be traceless.

Let us have another look at the constraints (2.5) and (2.11) defining the Cosserat continuum. By (2.14) the vector $\delta \omega^k$ can be eliminated and the constraints can be expressed in terms of the virtual displacements δx_l only:

$$(2.17) \quad \nabla_k \delta x_l + \nabla_l \delta x_k = 0, \quad \frac{1}{2} \varepsilon^{klp} \nabla_{kp} \delta x_l = 0.$$

This shows that the constraints defining the Cosserat continuum contain derivatives of the displacements δx_l of first and second order, while the constraints defining the classical medium, i.e. (2.5), contain those derivatives in the first order only. It is, therefore, natural to classify the deformable media by the order of the derivatives of the virtual displacements occurring in the constraints.

Some authors (e.g., [17], p. 86-7, or [5], p. 117 ff.) considered also a classification based on the order of some derivatives or of some tensors, but their arguments are based on different ideas. We shall discuss these questions in Section 4. Before that, however, we shall formulate the Principle of Virtual Work in a more precise fashion.

3. The Principle of Virtual Work. The Principle of Virtual Work, or of Virtual Displacements, formulated by Lagrange almost two centuries ago ([10], chap. I, II) is from methodological point of view an axiom defining the mathematical model of a physical system and its equilibrium. It is usually expressed in traditional terminology using words like "virtual" as opposed to "actual", without explaining either one, or "constraints" and corresponding "Lagrange multipliers", etc. Such language, however suggestive in the narrative, is not essential in describing the mathematical formalism involved.

In order to arrive at more precise formulation, let us analyze the examples of applications of the Principle of Virtual Work given in Sections 1 and 2, and filter out the mathematical content of it. The "virtual displacements" are variations of some functions of the position-vectors, and are assumed infinitesimal in order to make the "constraints" linear with respect to these displacements. The "constraints" themselves are

written in form of equations with zero on the right-hand side of the equations by the "Lagrange multipliers". And only those products are needed, but not the equations, to be added to the "external work", i.e., the products of the "external forces" by the corresponding displacement. In this way, we build up the total "Virtual Work" which is essentially a scalar-valued function resulting from the application of a linear differential operator to the virtual displacement. The Virtual Work is then integrated over the entire region of the parameters defining the material continuum, and this basic "trick" of integration by parts serves the only purpose to produce the adjoint operator, which turns out to be multiplicative with respect to the virtual displacements. Thus, equating the total virtual work to zero is equivalent to requiring that the adjoint operator be zero.

This discussion motivates the following formal approach. A material continuum is defined as a differentiable manifold embedded in a three-dimensional Euclidean space, so that each point a of the manifold has a position $\mathbf{x}(a)$, which is vector in the Euclidean space. The coordinates a^k ($k = 1, 2, 3$) of a particle a in a local coordinate system are called the *material* or *Lagrange's coordinates*. They are the continuous analog to the letters a labeling the individual particles of a discrete system considered in Section 1. As in Section 2, we assume that the manifold has a metric tensor and all the tensor analysis associated with it. The virtual displacements $v_k(a)$, $k = 1, 2, 3$, are vectors in the tangent space of the material manifold. They have been denoted in Section 2 by the symbols $\delta x_k(a)$. It should be emphasized that the virtual displacement $v_k(a)$ is any vector in the tangent space and need not be related to the position $\mathbf{x}(a)$ of the particle which is a vector in a different space, viz., the embedding Euclidean space. As a matter of fact, the position-vectors $\mathbf{x}(a)$ are only needed in the mathematical model of a deformable continuum for the geometry of the model, viz., the metric tensor. Sometimes, though, the components $x^k(a)$ are used for identification of the particle a instead of its material coordinates. The use of the x^k , called then the field or Euler's variables, however convenient for some purposes, is not appropriate in our approach because it brings in non-linear appearance into the linear operators.

The physical properties of the material continuum are defined by a linear operator Λ^k which to each virtual displacement $v_k(a)$ attaches a scalar-valued function $\varphi(a) = (\Lambda^k v_k)(a)$ defined on the manifold. Using modern terminology we say that the operator Λ^k maps the tangent bundle of the manifold into the algebra of scalar-valued functions defined on the manifold, all appropriate regularity conditions granted. Next, we assume that those scalar-valued functions ψ and ω belong to a Hilbert space with a given inner product $\langle \psi, \omega \rangle$.

In particular, if Λ^k is a linear differential operator acting on the vectors v_k defined in a region \mathcal{A} in the manifold and on its boundary $\partial\mathcal{A}$, then it is given by the formula

$$(3.1) \quad (\Lambda^k v_k)(a) = \begin{cases} F^k(a)v_k(a) + F^{lk}(a)\nabla_l v_k(a) + F^{mlk}(a)\nabla_{ml} v_k(a) + \dots & \text{for } a \in \mathcal{A}, \\ f^k(a)v_k(a) + f^{lk}(a)\nabla_l v_k(a) + \dots & \text{for } a \in \partial\mathcal{A}, \end{cases}$$

in which $F^k, F^{lk}, F^{mlk}, \dots, f^k, f^{lk}, \dots$ are tensor-valued functions given in the region \mathcal{A} and its boundary $\partial\mathcal{A}$, respectively, and $\nabla_l, \nabla_{ml}, \dots$ are the covariant derivatives which occur up to some given order N called the *order* of the differential operator. Then, we assume that the inner product is given by the formula

$$(3.2) \quad \langle \Lambda^k v_k, \psi \rangle = \int_{\mathcal{A}} da \Lambda^k v_k \psi + \int_{\partial\mathcal{A}} d\sigma(a) \Lambda^k v_k \psi.$$

The adjoint operator Λ_*^k mapping the scalar-valued functions ψ into the cotangent bundle of the manifold is defined by the condition

$$(3.3) \quad \langle \Lambda^k v_k, \psi \rangle = \langle v_k, \Lambda_*^k \psi \rangle.$$

In particular, if Λ^k is a linear differential operator of second order of the form given in (3.1), then its adjoint operator Λ_*^k acts by the formula

$$(3.4) \quad (\Lambda_*^k \psi)(a) = \begin{cases} F_*^k(a)\psi(a) + F_*^{lk}(a)\nabla_l \psi(a) + F_*^{mlk}(a)\nabla_{ml} \psi(a) & \text{for } a \in \mathcal{A}, \\ f_*^k(a)\psi(a) + f_*^{lk}(a)\nabla_l \psi(a) & \text{for } a \in \partial\mathcal{A}, \end{cases}$$

where

$$(3.5) \quad \begin{aligned} F_*^k &= F^k - \nabla_l F^{lk} + \nabla_{lm} F^{mlk}, \\ F_*^{lk} &= F^{lk} - \nabla_m (F^{mlk} + F^{lmk}), \\ F_*^{mlk} &= F^{mlk} && \text{inside the region } \mathcal{A}, \\ f_*^k &= f^k - \nu_l F^{lk} - \nabla_l (f^{lk} - \nu_m F^{mlk}) + \nu_l \nabla_m F^{mlk}, \\ f_*^{lk} &= -f^{lk} + \nu_m (F^{mlk} + F^{lmk}) && \text{on the boundary } \partial\mathcal{A}. \end{aligned}$$

Here, ν_l is the unit vector normal to the boundary $\partial\mathcal{A}$. The coefficients F^k and f^k are interpreted as the components of external generalized forces acting per volume element da of the region \mathcal{A} , or per area element $d\sigma(a)$ of the boundary $\partial\mathcal{A}$, respectively. The tensors F^{lk}, F^{mlk} and f^{lk} are interpreted as internal stresses in the region and on the boundary, respectively.

The integration by parts, so essential in the use of the Principle of Virtual Work, is equivalent to the requirement that in formulae (3.4) the adjoint operator Λ_*^k acts on the function $\psi = \text{const}$. Then, that formula (3.4) reduces to

$$(3.6) \quad \Lambda_*^k \psi = \begin{cases} F_*^k \psi & \text{inside } \mathcal{A}, \\ f_*^k \psi & \text{on the boundary } \partial\mathcal{A}. \end{cases}$$

We shall elevate this observation to the rank of an axiom defining the equilibrium of the material system:

To say that *the deformable continuum represented by the operator \mathcal{A}^k is in equilibrium* means that

$$(3.7) \quad \psi = \text{const} \quad \text{and} \quad \mathcal{A}_*^k = 0.$$

The first condition, however, strange it might look, opens some possibilities for a generalization of this formalism to quantum mechanics, which should be subject to a separate study.

Let us illustrate the formalism by the classical model of a deformable continuum as considered in the theory of elasticity, and discussed previously in Section 2 (cf. formulae (2.5)-(2.8)). The operator \mathcal{A}^k is here, according to (2.6) and (2.7), given by the formula

$$(3.8) \quad \mathcal{A}^k v_k = \begin{cases} F^k u_k - P^{kl} (\nabla_l u_k + \nabla_k u_l) & \text{inside } \mathcal{A}, \\ f^k u_k & \text{on the boundary } \partial\mathcal{A}. \end{cases}$$

The adjoint operator \mathcal{A}_*^k given by (3.6), when substituted into the condition of equilibrium (3.7), yields exactly equations (2.8) previously derived.

The above-mentioned definition allows us to classify the mathematical models of deformable continua by the type of the linear operator \mathcal{A}^k defining them. Thus, if \mathcal{A}^k is a differential operator, as given by (3.1), the material continua can be first classified by the order of that operator.

As mentioned at the end of Section 2, some authors call deformable continuum "of grade N " if it is defined by a Lagrangean which depends on the position-vector $x(a)$ and its derivatives up the order N . In the approach here, we do not introduce any Lagrangean whatsoever. As a matter of fact, the Lagrangean is not used itself in any theory; it is only its variation which is needed for the derivation of the corresponding basic equations, and in those equations only the derivatives of the Lagrangean appear. Furthermore, the only use of the variation of the Lagrangean is to produce a linear operator acting upon the variations of the functional arguments of that Lagrangean, and the infinitesimal character of the variations is introduced solely for the purpose to make that operator linear. Therefore, if we assume that the operator is linear from the onset, then there is no more need of considering any Lagrangean or to treat as infinitesimal the virtual displacements. That is the reason why the virtual displacements have been here defined as vectors in the tangent space of the material manifold. Also, the use of a Lagrangean restricts the linear operator by presuming that its coefficients are partial derivatives of that Lagrangean. The formalism used here is more general in this respect that it enables us to consider, among other things, dissipative forces and non-reversible processes. Moreover, it suggests generali-

zations by introducing deformable continua defined by other linear operators, not necessarily differential. The physical idea underlying the use of a differential operator consists in the assumption that the interaction between the particles of the continuum is local, i.e., one part of the continuum acts upon the other only on the surface dividing those parts. A non-local interaction can be described by a Fredholm-type operator, for example:

$$(3.9) \quad (\mathcal{A}^l v_l)(a) = \int_{\mathcal{A}} db K^l(a, b) v_l(b) + \int_{\partial \mathcal{A}} d\sigma(b) k^l(a, b) v_l(b).$$

Here, the kernels K^l and k^l are functions of two points a, b inside the region \mathcal{A} and on the boundary $\partial \mathcal{A}$, respectively. The differential operator, given in (3.1), can be considered as a particular case of (3.9) if we use for the kernels the delta-distributions. The operator \mathcal{A}_*^l adjoint to (3.9) and mapping the scalar-valued functions ψ into the virtual displacements v_l is

$$(4.10) \quad (\mathcal{A}_*^l \psi)(a) = \int_{\mathcal{A}} db K^l(b, a) \psi(b) + \int_{\partial \mathcal{A}} d\sigma(b) k^l(b, a) \psi(b).$$

More generally, one can consider an integro-differential operator, e.g.,

$$(\mathcal{A}^l v_l)(a) = \int_b db [K^l(a, b) v_l(b) + K^{ml}(a, b) \nabla_m v_l(b)] + \int_{\partial \mathcal{A}} d\sigma(b) k^l(b) v_l(b).$$

We shall not elaborate that subject but only indicate these formal possibilities. In what follows, we shall confine ourselves to considering deformable continua defined by linear differential operators of the type given by (3.1).

4. Classification of deformable continua. The classification of deformable continua according to the order of their differential operators can be refined by distinguishing within each order certain types. We shall show that in addition to the order, the form of the operator is restricted by three circumstances: by the decomposition of the tensors into parts of various symmetries, by the dimension of the continuum and by the curvature of the embedding space. Here, we shall discuss three-dimensional continua embedded in a three-dimensional Euclidean space and confine ourselves to considering linear differential operators of first and second order only. Higher order operators can be handled in a completely analogous way.

Thus, we start with a differential operator of first order defined by

$$(4.1) \quad (\mathcal{A}^k v_k)(a) = \begin{cases} F^k(a) v_k(a) + F^{lk}(a) \nabla_l v_k(a) & \text{for } a \in \mathcal{A}, \\ f^k(a) v_k(a) & \text{for } a \in \partial \mathcal{A}. \end{cases}$$

Decompose the tensor F^{lk} in symmetric, $F^{(lk)}$, and antisymmetric part $F^{[lk]}$, viz.

$$(4.2) \quad F^{(lk)} = \frac{1}{2}(F^{lk} + F^{kl}), \quad F^{[lk]} = \frac{1}{2}(F^{lk} - F^{kl}).$$

Similarly, for the tensor $\nabla_l v_k$, write

$$(4.3) \quad \nabla_{(l} v_{k)} = \frac{1}{2}(\nabla_l v_k + \nabla_k v_l), \quad \nabla_{[l} v_{k]} = \frac{1}{2}(\nabla_l v_k - \nabla_k v_l).$$

The parts of different symmetries are orthogonal in the sense that $F^{(lk)} \nabla_{[l} v_{k]} = 0$ and $F^{[lk]} \nabla_{(l} v_{k)} = 0$, so that $F^{lk} \nabla_l v_k = F^{(lk)} \nabla_{(l} v_{k)} + F^{[lk]} \nabla_{[l} v_{k]}$.

In the three-dimensional space the tensors dual to those antisymmetric parts are vectors, viz.

$$(4.4) \quad M_p = \frac{1}{2} \varepsilon_{p lk} F^{lk}, \quad \omega^p = \frac{1}{2} \varepsilon^{p lk} \nabla_l v_k,$$

where $\varepsilon_{p lk}$ and $\varepsilon^{p lk}$ are the antisymmetric discriminant tensors of the manifold. Thus, the differential operator (4.1) can be written as

$$(4.5) \quad (\Lambda^k v_k)(a) = \begin{cases} F^{lk} v_k + \frac{1}{2} \varepsilon^{p lk} M_p \nabla_l v_k + F^{(lk)} \nabla_l v_k & \text{for } a \in \mathcal{A}, \\ f^k v_k & \text{for } a \in \partial \mathcal{A}. \end{cases}$$

Hence, the conditions of equilibrium given by (3.7) and (3.6) are

$$\begin{aligned} F_*^k(a) &= F^{lk} - \nabla_l (F^{(lk)} + \frac{1}{2} \varepsilon^{p lk} M_p) = 0 & \text{for } a \in \mathcal{A}, \\ f_*^k(a) &= f^k - v_l (F^{(lk)} + \frac{1}{2} \varepsilon^{p lk} M_p) = 0 & \text{for } a \in \partial \mathcal{A}. \end{aligned}$$

Write

$$(4.6) \quad F^{(lk)} + \frac{1}{2} \varepsilon^{p lk} M_p = -T^{lk} \quad \text{for } a \in \mathcal{A} \cup \partial \mathcal{A}.$$

Then, the equations of equilibrium become

$$(4.7) \quad \begin{aligned} F^{lk} + \nabla_l T^{lk} &= 0 & \text{for } a \in \mathcal{A}, \\ f^k + v_l T^{lk} &= 0 & \text{for } a \in \partial \mathcal{A}, \quad M_p + \varepsilon_{p lk} T^{lk} &= 0 & \text{for } a \in \mathcal{A} \cup \partial \mathcal{A}. \end{aligned}$$

Let us compare these equations with equations (2.8) and (2.13), by interpreting the tensor T^{kl} as a stress-tensor, and the vector M_p as an external moment acting per volume element. The deformable continuum, defined by (4.1), is thus more general than the classical one defined by equations (2.8) in which the stress tensor P^{lk} is symmetric. But, the continuum defined by (4.1) is less general than the Cosserat continuum satisfying equations (2.13) where also couple-stresses $M_{,k}^l$ occur. This completes the classification of the continua defined by a differential operator of first order.

We pass now to the examination of deformable continua defined by a differential operator of second order, viz.

$$(4.8) \quad (\Lambda^k v_k)(a) = \begin{cases} F^{lk} v_k + F^{lk} \nabla_l v_k + F^{mlk} \nabla_{ml} v_k & \text{inside } \mathcal{A}, \\ f^k v_k + f^{lk} \nabla_l v_k & \text{on the boundary } \partial \mathcal{A}. \end{cases}$$

Again, decompose the second-order tensors F^{lk} and $\nabla_l v_k$ in the symmetric and antisymmetric parts given by (4.3). As for the third-order tensors F^{mlk} and $\nabla_{ml} v_k$, they can be decomposed into three parts, viz., the completely symmetric $F^{(mlk)}$, $\nabla_{(ml} v_k)$, the completely antisymmetric $F^{[mlk]}$, $\nabla_{[ml} v_k]$, and the "mixed" part denoted here by $F^{\{mlk\}}$, $\nabla_{\{ml} v_k\}$, respectively. Those parts are given by the formulae

$$(4.9) \quad \begin{cases} F^{(mlk)} = \frac{1}{6}(F^{mlk} + F^{mkl} + F^{kml} + F^{klm} + F^{lkm} + F^{lmk}), \\ F^{[mlk]} = \frac{1}{6}[F^{mlk} - F^{mkl} + F^{kml} - F^{klm} + F^{lkm} - F^{lmk}], \\ F^{\{mlk\}} = \frac{1}{3}\{2F^{mlk} - F^{kml} - F^{lkm}\}, \end{cases}$$

and, analogously, for $\nabla_{ml} v_k$.

This decomposition corresponds to the three non-equivalent irreducible unitary representations of the symmetric group S_3 of permutations of three indices m, l and k , or to the three forms of the Young's tableaux⁽¹⁾ (cf. [19], p. 647-8). The parts with different symmetries are mutually orthogonal in the sense that, e.g.,

$$(4.10) \quad F^{(mlk)} \nabla_{[ml} v_k] = 0, \quad F^{\{mlk\}} \nabla_{(ml} v_k) = 0 \quad \text{etc.}$$

Thus,

$$(4.11) \quad F^{mlk} \nabla_{ml} v_k = F^{(mlk)} \nabla_{(ml} v_k) + F^{[mlk]} \nabla_{[ml} v_k] + F^{\{mlk\}} \nabla_{\{ml} v_k\}.$$

Because the space is Euclidean, the antisymmetric part $\nabla_{[ml} v_k]$ in (4.11) is zero since the mixed second covariant derivatives are equal, $\nabla_{ml} = \nabla_{lm}$. For the same reason, the last term in (4.11) can be transformed as follows:

$$F^{\{mlk\}} \nabla_{\{ml} v_k\} = \begin{cases} \frac{1}{3}\{F^{mlk} - F^{klm} + F^{lmk} - F^{lkm}\} \nabla_{ml} v_k, \\ \frac{1}{3} \varepsilon^{mkp} \varepsilon_{pqr} (F^{qlr} + F^{lqr}) \nabla_{ml} v_k. \end{cases}$$

Write

$$(4.12) \quad \frac{1}{3} \varepsilon_{pqr} (F^{qlr} + F^{lqr}) = \frac{1}{2} M^l_{.p}.$$

This tensor is traceless, i.e.,

$$(4.13) \quad M^l_{.l} = 0$$

because the sum of the F 's (4.12) is symmetric in q and l .

⁽¹⁾ Toupin in [16], p. 390-392, quotes a decomposition of a tensor of third order in four parts of different symmetries. However, even after correcting all the misprints in his formulae (4.11) the parts quoted there are not mutually orthogonal, viz., the parts labelled there P and \bar{P} are not orthogonal.

Summing up all the decompositions into all the symmetries, we write the operator (4.8) in the form

$$(4.14) \quad (\Lambda^k v_k)(a) = \begin{cases} F^k v_k + \frac{1}{2} \varepsilon^{pik} M_p \nabla_i v_k + F^{(lk)} \nabla_l v_k + F^{(mlk)} \nabla_{ml} v_k + \frac{1}{2} M_p^q \varepsilon^{pik} \nabla_{iq} v_k & \text{for } a \in \mathcal{A}, \\ f^k v_k + \frac{1}{2} \varepsilon^{pik} m_p \nabla_i v_k + f^{(lk)} \nabla_l v_k & \text{for } a \in \partial \mathcal{A}. \end{cases}$$

Here, M_p is given by (4.4), and m_p is defined analogously on the boundary $\partial \mathcal{A}$, as the antisymmetric part of f^{lk} . Hence, the conditions of equilibrium can be obtained by condition (3.7). Using formula (3.6), we obtain the following equations of equilibrium:

$$(4.15) \quad \begin{aligned} F^k - \nabla_l [F^{(lk)} + \frac{1}{2} \varepsilon^{pik} M_p - \nabla_m F^{(mlk)} - \frac{1}{2} \varepsilon^{pik} \nabla_q M_p^q] &= 0 \quad \text{inside } \mathcal{A}, \\ f^k - \nabla_l [f^{(lk)} + \frac{1}{2} \varepsilon^{pik} m_p] + \nabla_l [F^{(mlk)} + \frac{1}{2} v_q \varepsilon^{pik} M_p^q] + \\ &+ v_l [F^{(mlk)} + \frac{1}{2} \nabla_q \varepsilon^{pik} M_p^q] = 0 \quad \text{on the boundary } \partial \mathcal{A}. \end{aligned}$$

We shall show that these equations are a generalization of equations (2.13) and (2.16) for Cosserat continua. To this end, denote the expression in the bracket in (4.15) as follows:

$$(4.16) \quad F^{(lk)} + \frac{1}{2} \varepsilon^{pik} M_p - \nabla_m F^{(mlk)} - \frac{1}{2} \varepsilon^{pik} \nabla_q M_p^q = -T^{lk} \quad \text{for } \mathcal{A} \cup \partial \mathcal{A}.$$

Then, the equations of equilibrium including (4.13) are

$$(4.17) \quad \begin{aligned} F^k + \nabla_l T^{lk} &= 0, \quad M_k + \varepsilon_{kpq} T^{pq} + \nabla_l M_{.k}^l = 0, \quad M_{.k}^k = 0 \quad \text{for } a \in \mathcal{A}, \\ f^k + v_l T^{lk} &= 0, \quad m_k + v_l M_{.k}^l - \nabla_l (f^{(lk)} - v_q F^{(qlk)}) = 0 \quad \text{for } a \in \partial \mathcal{A}. \end{aligned}$$

Comparison with equations (2.13) and (2.16) shows that the deformable continuum defined here is a generalization of the Cosserat continuum considered in Section 2 in the following respects.

Formulae (4.16) and (4.12) show that the stresses T^{lk} and $M_{.k}^l$ contain a tensor F^{mlk} of third order, and this finer structure was not assumed in equations (2.13). Such higher order tensors have been considered by some authors, e.g., [16] and [5]. Here, it has been shown how the Euclidean character of the embedding space and the symmetry considerations restrict the use of those tensors to only some of their symmetry parts.

This concludes the classification of the continua defined by a differential operator of second order. We could proceed in a similar fashion with examining the deformable continua called sometimes "multipolar" [5], corresponding to operators of higher order. Since no reasonably practical example of such continua has been exhibited so far, we shall not go into the technical details of applying the general method illustrated on operators of first and second order.

The same method can be applied to the cases where the deformable continuum or the embedding space are of dimensions other than three. Thus, e.g., two- and one-dimensional continuum embedded in a three-dimensional Euclidean space give the mathematical models of shells and rods; a continuum embedded in four-dimensional Minkowski or Riemannian space give the mathematical models of relativistic continuum mechanics, electromagnetic fields and gravitation. Those topics will be discussed in separate papers, see also [3] and [4].

5. Deformations. Constitutive equations. Dynamics. The decomposition of the tensors into their symmetry parts leads to a natural definition of the deformations of a continuum. We first define the "actual" displacement of a deformable continuum as a vector $u_k(a)$ in the tangent space. If the continuum is described by an operator Λ^k , then the scalar-valued function $(\Lambda^k u_k)(a)$ is interpreted as the density of energy. To fix the idea, consider a linear differential operator of second order. By decomposing the occurring tensors in their symmetry parts, we can write the density of energy corresponding to the displacement u_k in the form given by (4.14) and (4.4), for $a \in \mathcal{A}$,

$$(5.1) \quad (\Lambda^k u_k)(a) = F^k u_k + M_p \omega^p + F^{(lk)} \nabla_{(l} u_{k)} + F^{(mlk)} \nabla_{(ml} u_{k)} + M_p^a \cdot \frac{1}{2} \varepsilon^{pik} \nabla_{ik} u_k.$$

The part of the energy density containing tensors of first order, i.e., the vectors u_k and ω^p , is interpreted as the external work, and the corresponding coefficients F^k and M_p — as external generalized forces. The part of the energy density containing tensors of order two and higher, if any, is interpreted as the internal work. The general idea behind this interpretation is that the external forces are considered as "given" and the internal work is considered as the reaction of the medium to those external factors. It is worth recalling that the reason why M_p and ω^p in the "external" work are vectors is the fact that the dimension of the continuum is three, as can be seen from formulae (4.4).

The tensorial coefficients $F^{(lk)}$, $F^{(mlk)}$ and M_p^a in the internal work are called *stresses*. The corresponding tensors built up from the derivatives of the displacement vector u_k are defined as components of the deformation or *strain*. Let us denote these components as follows:

$$(5.2) \quad \begin{aligned} e_{(lk)} &= \frac{1}{2} (\nabla_l u_k + \nabla_k u_l), & e_{(mlk)} &= \frac{1}{3} (\nabla_{ml} u_k + \nabla_{lk} u_m + \nabla_{mk} u_l), \\ \omega_k^l &= \frac{1}{2} \varepsilon^{pql} \nabla_{kp} u_q = \nabla_k \omega^l. \end{aligned}$$

Here the parantheses () around the indices indicate, as before, the complete symmetry between the enclosed indices. Notice that the expression for $e_{(mlk)}$ contains only three terms and not six as $F^{(mlk)}$ in for-

mula (4.9), because the space is Euclidean (flat) and $\nabla_{ml} = \nabla_{lm}$. For the same reason the trace of the tensor ω_k^l and, consequently, the divergence of the vector ω^k vanish:

$$(5.3) \quad \omega_k^k = \nabla_k \omega^k = 0.$$

In this way, for any displacement vector u_k the components of the deformations are well-defined tensors. This definition can be easily extended to the cases of operators of higher order.

However, not every tensor can be a component of a deformation because the existence of a displacement vector u_k generating the deformation implies some identities between the derivatives of that tensor. Those identities, known as "compatibility conditions", are based on the equalities of mixed derivatives, i.e., on the fact that the curvature tensor of the space is zero.

Thus, by using (5.1) and (5.2), the internal work is

$$(5.4) \quad \varphi = F^{(lk)} e_{lk} + F^{(mlk)} e_{mlk} + M_{.l}^k \omega_k^l.$$

With this definition, however, an additional assumption is made that the coefficients $F^{(lk)}$, $F^{(mlk)}$ and $M_{.l}^k$ are functions of the deformations, i.e., functions on the tangent bundle to the material continuum. In contrast, in formula (4.1) those coefficients in the operator defining the continuum were considered as functions of the particle, i.e., functions on the manifold itself. The physical interpretation behind the former assumption is as follows. The external factors F^k and M_k induce the displacement u_k and the rotation ω^k , which in turn produce the strain, defined by the components of deformation (5.2). The material continuum reacts to that strain with the stresses.

The functional relationship between the stresses and deformations, called *constitutive equations*, can be of various form. We shall consider here the case of linear elasticity only. A continuum is called *elastic* if there exists a scalar-valued function Φ of the deformations such that

$$(5.5) \quad F^{(lk)} = \frac{\partial \Phi}{\partial e_{lk}}, \quad F^{(mlk)} = \frac{\partial \Phi}{\partial e_{mlk}}, \quad M_{.l}^k = \frac{\partial \Phi}{\partial \omega_k^l}.$$

The function Φ is called the *elastic potential*. If the potential Φ is a homogeneous quadratic polynomial of the deformations, then the constitutive equations (5.5) are linear with respect to the deformations. Such an elastic continuum is called *linear*.

The assumption that the arguments of Φ are tensors and that its value is a scalar restricts the elastic potential. In fact, since the value of Φ is invariant, it must depend on the invariants of the tensorial argument only. By a Hilbert theorem (see e.g., [7], p. 235), a complete system of invariants of a finite system of tensors consists of a finite set of basic

invariants. And, by a theorem of Cramlet ([7], p. 188), every basic invariant can be obtained by multiplication, total alternation, and subsequent total contraction of those tensors.

For a linear elastic continuum there are seven invariants, which are quadratic in the components of deformations, viz.

$$\begin{aligned} I_1 &= (e_m^m)^2 = g^{lk} e_m^m e_{lk}, & I_2 &= e^{lk} e_{lk}, \\ I_3 &= e_{km}^k e_i^{lm}, & I_4 &= e^{mlk} e_{mlk}, & I_5 &= \omega_{.l}^k \omega_k^l, \\ I_6 &= \omega_{.l}^k \omega_{.k}^l, & L_7 &= e_{(lk)} \omega^{lk} = e_{(lp)} g^{kp} \omega_k^l. \end{aligned}$$

The invariant of ω_k^l , analogous to I_1 , is zero by (5.3). Also, notice that the invariants I_5 and I_6 are different because the tensor ω_k^l has no symmetry, viz., $\omega_{.l}^k = g^{pk} g_{lp} \omega_p^q \neq \omega_{.k}^l$. Thus, the quadratic elastic potential is

$$(5.7) \quad \Phi = \lambda I_1 + 2\mu I_2 + \alpha_3 I_3 + \alpha_4 I_4 + \alpha_5 I_5 + \alpha_6 I_6 + \alpha_7 I_7,$$

where the coefficients λ, μ and $\alpha_3, \dots, \alpha_6$ are scalar-valued functions which may depend on the point a of the continuum. The constitutive equations are by (5.5) as follows:

$$(5.8) \quad F^{(lk)} = \lambda g^{lk} e_m^m + 2\mu e^{(lk)} + \alpha_7 g^{kp} \omega_p^l, \quad F^{(mlk)} = \alpha_3 g^{ml} e_p^{pk} + \alpha_4 e^{mlk},$$

$$M_{.l}^k = \alpha_5 \omega_{.l}^k + \alpha_6 \omega_l^k + \alpha_7 e_{(lp)} g^{pk}.$$

This is a generalization of the Hooke's law in classical elasticity, where only two coefficients of elasticity, viz. λ and μ , occur. If the coefficients are constants, then the continuum is called *homogeneous*. We shall show that in this case there are only five essential coefficients of elasticity. In fact, substitute the stresses given by (5.8) into the equations of equilibrium given by (4.15), while at the same time expressing the deformations through the displacement vector u_k only, as given by formulae (5.2). Then, after appropriate calculations, taking into account (5.3), we obtain the differential equations of equilibrium inside the region \mathcal{A} ,

$$(5.9) \quad X^k - (\lambda + \mu) \nabla_l^k u^l - \mu \nabla_l^l u^k + \nabla_m^m [(\lambda_1 + \mu_1) \nabla_l^k u^l + \mu_1 \nabla_l^l u^k - \tfrac{1}{2} \alpha_7 \omega^k] = 0,$$

where the following notations have been used:

$$(5.10) \quad -X^k = F^k - \tfrac{1}{2} e^{mlk} \nabla_l M_m,$$

$$(5.11) \quad \lambda_1 = \tfrac{1}{3}(\alpha_3 + \alpha_4) - \tfrac{1}{2}\alpha_5, \quad \mu_1 = \tfrac{1}{3}(\alpha_3 + \alpha_4) + \tfrac{1}{2}\alpha_5,$$

and $\nabla^k = g^{kp} \nabla_p$, so that ∇_m^m is the Laplace operator.

The conditions on the boundary $\partial\mathcal{A}$ can be obtained in a similar manner. It is worth noticing that in (5.10) only the curl of the external moments M_k occurs with the external force F^k . Equations (5.9) are a generalization of the classical Lamé equations, in which only two elastic coefficients λ and μ occur, and the three others λ_1 , μ_1 and α_7 are zero.

We conclude this section by discussing the equations of motion of a deformable continuum. Usually, one writes down those equations by simply adding to the equations of equilibrium the “inertial terms”, i.e. the product of the density by the acceleration of the particle. This procedure, called *D'Alembert's principle*, can be, however, obtained by extending the definition of the deformation. We shall here only indicate the idea.

Thus, in addition to the coordinates a^k ($k = 1, 2, 3$) identifying the particle a , consider the time t in an interval $[t_0, t_1]$. The virtual displacement is now considered as a vector-valued function $v_k(a, t)$ defined on the Cartesian product $\mathcal{A} \times [t_0, t_1]$. The linear differential operator (3.1), defining the deformable continuum, contains now in addition to the previous terms also terms with the time derivatives $\partial_t = \partial/\partial t$. To only illustrate the idea, assume that in the operator \mathcal{A} there is only one additional such term, viz.

$$(5.12) \quad P^k(a, t) \partial_t v_k(a, t) \quad \text{for } a \in \mathcal{A} \text{ and } t_0 < t < t_1,$$

with some value on the boundary not to be specified here. Then, the adjoint operator to (5.12) is $-\partial_t P_k(a, t)$, and this additional term has to be added to the equations of equilibrium to obtain the equations of motion. This corresponds to the idea that a motion is an equilibrium in the four-dimensional space including time.

The coefficients $P_k(a, t)$ are interpreted as momenta of motion. If the “actual” displacement is $u_k(a, t)$, we can consider $\partial_t u_k$ as a deformation analogous to $\nabla_l u_k$, and assume an analogy to Hooke's law, i.e. that

$$(5.13) \quad P_k(a, t) = \varrho(a, t) \partial_t u_k(a, t).$$

Here, the coefficient $\varrho(a, t)$ is interpreted as the density of matter. And then we obtain the “inertial term” $-\partial_t(\varrho u_k)$ which has to be added to the equations of equilibrium in order to get equations of motion as the D'Alembert's principle requires.

This point of view seems to be in spirit of classical Newtonian mechanics where the mass (or its density) is understood as a “resistance” of the medium to the change of the momentum. Here, Newton's second law is analogous to Hooke's law.

Moreover, we can extend this point of view by introducing into the operator \mathcal{A} other “momenta”, like $P^{lk}(a, t) \partial_t \nabla_l v_k(a, t)$, $Q^k(a, t) \partial_t v_k(a, t)$ etc. For example, the antisymmetric part of the first of them is of the form $G_m(a, t) \partial_t \omega^m$ and is interpreted as the density of the angular mo-

menta. The corresponding "Hooke's law" is $G_m(a, t) = \sigma(a, t) \partial_t \omega_m$, and the coefficient of "elasticity" is the density of the local „moments of inertia" of a deformable continuum.

There is no reason why those terms should not be introduced into the equations of motion of the Cosserat continua. A more systematic discussion of those topics should be subject to a separate study.

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O OŚRODKACH COSSERATÓW**STRESZCZENIE**

Rozpatrując mechanikę Cosseratów z punktu widzenia Zasady Prac Wirtualnych, autor bada, jak więzy nałożone na przemieszczenia wirtualne prowadzą do ogólnych matematycznych modeli ośrodków zorientowanych i do ich klasyfikacji. Wyjaśnia zarazem, jaką rolę w mechanice ośrodków zorientowanych gra rozkład tensorów na części o różnych symetriach oraz płaskość przestrzeni. Przyjęta metoda pozwala także na ogólne wyprowadzenie warunków elastyczności przy pomocy teorii niezmienników oraz na uogólnienie zasady D'Alemberta.
