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ON A NETWORK OF $M/M/n$ FACILITIES WITH THE INTENSITY OF NEW ARRIVALS DEPENDING ON THE TOTAL NUMBER OF WAITING UNITS

1. Introduction. Let us consider a network system which consists of N service facilities or stations such that a unit after being served in one of them may either leave the system or immediately apply for servicing in another station.

Jackson [3] studied a special case of a system where all stations are arranged in one line. New arrivals always enter the first station and then pass all the line so that the only output is after servicing in the last station. In another system considered by Jackson [2] arriving units may enter immediately any of N stations with possible output after the service being completed as well as the transfer between any pair of stations. In both cases a Poisson input with constant arrival rate λ was assumed and there was no restriction on the capacity of waiting rooms at the stations of the system. Under such assumptions the steady-state probability distributions were obtained for the possible states of the systems.

In this paper authors generalize the results of Jackson [2] allowing for the dependence of the arrival rate on the total number of waiting units. Results presented here were obtained by the first author but only for a special case of a specified system with three stations (see Example 1, p. 170). The second author, being a referee of the former manuscript sent in by the first author, obtained the general solution, formulated Example 2 and prepared the present version of the paper.

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2. Specification of the network system and steady-state probability distribution of its states. We consider a network system which fulfils the following assumptions:

1° The system consists of N stations.

2° There are k_i service channels at the i-th station ($i = 1, 2, \ldots, N$), each of them having the exponential service-time distribution with the service rate μ_i.

3° There is no limitation on the length of queues at the stations of the system.

4° Units entering the system form a Poisson process with the arrival rate $\lambda f(l)$ depending on the total number l of the units actually present in the system (λ is a positive constant and $f(l)$ is a bounded function having non-negative values and defined for non-negative integer arguments l).

5° Arriving unit with probability λ_i/λ applies for servicing in i-th station ($i = 1, 2, \ldots, N$; $\lambda_1+\lambda_2+\ldots+\lambda_N = \lambda$).

6° A unit, having the service completed in the i-th station, with probability Q_{ij} immediately applies for servicing in the j-th station ($i, j = 1, 2, \ldots, N$) and with probability

$$Q_i^* = 1-\sum_{j=1}^{N} Q_{ij} \tag{1}$$

it leaves the system.

7° All random decisions concerning the first application of entering units as well as their consecutive transfers or outputs are independent of each other and of the actual state of the system.

The work of the defined system can be described by a multidimensional stochastic process

$$\nu(t) = [\nu_1(t), \nu_2(t), \ldots, \nu_N(t)], \tag{2}$$

where $\nu_i(t)$, $i = 1, 2, \ldots, N$, stands for the number of units in the i-th station (in service channels and in the queue) at the moment t. From the above-mentioned assumptions it follows that (2) is a Markov process. Let

$$P(n_1, n_2, \ldots, n_N; t) = \Pr\{\nu_1(t) = n_1, \nu_2(t) = n_2, \ldots, \nu_N(t) = n_N\} \tag{3}$$

be the probability of the state $(n_1, n_2, \ldots, n_N)$ at the moment t and let

$$P(n_1, n_2, \ldots, n_N) = \lim_{t\to\infty} P(n_1, n_2, \ldots, n_N; t) \tag{4}$$

be the steady-state probability, assuming it exists, of the same state. We prove the following

THEOREM. *If there exists a steady-state probability distribution for the process $\nu(t)$, it has to be of the form*

$$P(n_1, n_2, \ldots, n_N) = \frac{\prod_{l=0}^{n-1} f(l) \prod_{i=1}^{N} \Gamma_i^{n_i}}{\prod_{i=1}^{N} \beta(k_i, n_i)\mu_i^{n_i}} P(0, 0, \ldots, 0), \tag{5}$$

where $n = n_1 + n_2 + \ldots + n_N$,

$$\beta(k_i, n_i) = \begin{cases} n_i! & \text{for } n_i \leqslant k_i, \\ (k_i)!\, k_i^{n_i - k_i} & \text{for } n_i > k_i, \end{cases} \tag{6}$$

and Γ_i are the solutions of the system of equations

$$\Gamma_i = \lambda_i + \sum_{j=1}^{N} Q_{ij}\Gamma_j, \quad i = 1, 2, \ldots, N. \tag{7}$$

The probability $P(0, 0, \ldots, 0)$ of the system to be empty can be evaluated from the condition

$$\sum_{n_1, n_2, \ldots, n_N = 0}^{\infty} P(n_1, n_2, \ldots, n_N) = 1.$$

Proof. From a straightforward consideration of the ways in which the system can reach the state $(n_1, n_2, \ldots, n_N)$ it turns out that for an arbitrary moment $t \geqslant 0$ and $\tau > 0$ we have

$$\begin{aligned} (8) \quad & P(n_1, n_2, \ldots, n_N; t+\tau) \\ &= \Big\{1 - f(n)\lambda\tau - \sum_{i=1}^{N} \mu_i a(k_i, n_i)(1 - Q_{ii})\tau\Big\} P(n_1, n_2, \ldots, n_N; t) + \\ &+ \sum_{i=1}^{N} \mu_i a(k_i, n_{i+1}) Q_i^* P(n_1, n_2, \ldots, n_i+1, \ldots, n_N; t)\tau + \\ &+ f(n-1) \sum_{i=1}^{N} \lambda_i \gamma(n_i) P(n_1, n_2, \ldots, n_i - 1, \ldots, n_N; t)\tau + \\ &+ \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \mu_j a(k_j, n_j+1) Q_{ji} \gamma(n_i) P(n_1, n_2, \ldots, n_j+1, \ldots, n_i - 1, \ldots, n_N; t)\tau \\ &+ o(\tau), \end{aligned}$$

where $a(k_i, n_i) = \min(k_i, n_i)$ and $\gamma(n_i) = a(1, n_i)$.

Following the usual procedure of transferring $P(n_1, n_2, \ldots, n_N; t)$ from the right to the left-hand side of equation (8), dividing both sides by τ and passing to the limit as τ approaches zero, one obtains a system of differential equations (which will not be given here). Assuming the existence of the steady-state probabilities (4) one comes to the conclusion that they have to obey the system of algebraic linear equations

$$
\begin{aligned}
(9)\qquad & \Big\{f(n)\lambda+\sum_{i=1}^{N}\mu_i a(k_i, n_i)(1-Q_{ii})\Big\}P(n_1, n_2, \ldots, n_N)\\
&=\sum_{i=1}^{N}\mu_i a(k_i, n_i+1)Q_i^* P(n_1, n_2, \ldots, n_i+1, \ldots, n_N)+\\
&+f(n-1)\sum_{i=1}^{N}\lambda_i\gamma(n_i)P(n_1, n_2, \ldots, n_i-1, \ldots, n_N)+\\
&+\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\mu_j a(k_j, n_j+1)Q_{ji}\gamma(n_i)P(n_1, n_2, \ldots, n_j+1, \ldots, n_i-1, \ldots, n_N).
\end{aligned}
$$

We shall shaw that functions (5) fulfil system (9) of equations.
By substitution of (5) into (9) and some reductions we get

$$
\begin{aligned}
&\Big\{f(n)\lambda+\sum_{i=1}^{N}\mu_i a(k_i, n_i)(1-Q_{ii})\Big\}\frac{\prod_{l=0}^{n-1}f(l)\prod_{r=1}^{N}\Gamma_r^{n_r}}{\prod_{r=1}^{N}\beta(k_r, n_r)\mu_r^{n_r}}\\
&=\sum_{i=1}^{N}\mu_i a(k_i, n_i+1)Q_i^*\frac{\prod_{l=0}^{n}f(l)\prod_{r=1}^{N}\Gamma_r^{n_r}\Gamma_i}{\prod_{r=1}^{N}\beta(k_r, n_r)\mu_r^{n_r}a(k_i, n_i+1)\mu_i}+\\
&+f(n-1)\sum_{i=1}^{N}\lambda_i\gamma(n_i)\frac{\prod_{l=0}^{n-2}f(l)\prod_{r=1}^{N}\Gamma_r^{n_r}/\Gamma_i}{\prod_{r=1}^{N}\beta(k_r, n_r)\mu_r^{n_r}/a(k_i, n_i+1)\mu_i}+\\
&+\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\mu_j a(k_j, n_j+1)Q_{ji}\gamma(n_i)\frac{\prod_{l=0}^{n-1}f(l)\prod_{r=1}^{N}\Gamma_r^{n_r}\dfrac{\Gamma_j}{\Gamma_i}}{\prod_{r=1}^{N}\beta(k_r, n_r)\mu_r^{n_r}\dfrac{a(k_j, n_j+1)\mu_j}{a(k_i, n_i)\mu_i}}.
\end{aligned}
$$

Further transformation leads to

$$
\begin{aligned}
f(n)\lambda+\sum_{i=1}^{N}\mu_i a(k_i, n_i)(1-Q_{ii}) &= f(n)\sum_{i=1}^{N}Q_i^*\Gamma_i+\sum_{i=1}^{N}\lambda_i\mu_i a(k_i, n_i)\frac{1}{\Gamma_i}+\\
&+\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\mu_i a(k_i, n_i)Q_{ji}\frac{\Gamma_j}{\Gamma_i}
\end{aligned}
$$

which is obviously true since from the definition of Q_i^* and Γ_i it follows that

$$\sum_{i=1}^{N} Q_i^* \Gamma_i = \sum_{i=1}^{N}\Big(1-\sum_{j=1}^{N} Q_{ij}\Big)\Gamma_i = \sum_{i=1}^{N}\Big[\Gamma_i - \sum_{j=1}^{N} Q_{ij}\Gamma_i\Big]$$

$$= \sum_{j=1}^{N}\Big[\Gamma_j - \sum_{i=1}^{N} Q_{ij}\Gamma_i\Big] = \sum_{j=1}^{N} \lambda_j = \lambda$$

and

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \mu_i a(k_i, n_i) Q_{ji} \frac{\Gamma_j}{\Gamma_i} = \sum_{i=1}^{N} \mu_i a(k_i, n_i) \frac{1}{\Gamma_i} \sum_{\substack{j=1 \\ j\neq i}}^{N} Q_{ji}\Gamma_i$$

$$= \sum_{i=1}^{N} \mu_i a(k_i, n_i) \frac{1}{\Gamma_i} (\Gamma_i - \lambda_i - Q_{ii}\Gamma_i)$$

$$= \sum_{i=1}^{N} \mu_i a(k_i, n_i)(1-Q_{ii}) - \sum_{i=1}^{N} \lambda_i \mu_i a(k_i, n_i) \frac{1}{\Gamma_i}.$$

To complete the proof of the theorem it is now necessary to show that system (9) has a unique solution. For this purpose let us consider an imbedded Markov chain defined at the moments being the integer multiples of a given number h. The argumentation similar to that given by R.R.P. Jackson in [3] leads to the conclusion that there exists a steady-state distribution of the defined chain and it does not depend on h. From there it follows that the steady-state distribution of the process is uniquely determined.

3. Particular cases.

I. The assumption

$$f(l) = 1 \quad \text{for } l = 0, 1, 2, \ldots$$

leads to a network system already considered by J. R. Jackson in [2].

II. Another specification of the function $f(l)$,

$$f(l) = \begin{cases} 1 & \text{for } 0 \leqslant l < K, \\ 0 & \text{for } L \geqslant K, \end{cases}$$

leads to the case of a limited capacity of the system. The intensity of new arrivals is constant as long as the total number of units present in the system does not reach the number K and there is a complete balking when this level is reached. This assumption results in a finite number of possible states of the system. This is valid of course under a more general assumption

$$f(l) = \begin{cases} \text{arbitrary} & \text{for } 0 \leqslant l < K, \\ 0 & \text{for } \quad l \geqslant K, \end{cases}$$

which allows also a partial balking until the system is completely filled up.

Example 1. Let us consider now a network system consisting of three stations S_1, S_2, S_3. Units enter the system from outside according to a Poisson process with a constant arrival rate λ, and all new arrivals apply first for servicing at the first station ($\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 = 0$).

All station are equiped with single channels ($k_1 = k_2 = k_3 = 1$) and at each of them the first-in-first-out queueing discipline is being obeyed. After being served at the station S_1 unit may leave the system of apply for servicing at S_2 or S_3 ($Q_{11} = 0$). Those who have being served at S_2 may leave the system or enter S_3 ($Q_{21} = Q_{22} = 0$) while those who have being served at S_3 must leave the system ($Q_{31} = Q_{32} = Q_{33} = 0$, $Q_{33}^* = 1$). A complete balking is assumed when the total number of units in the system reaches the level K.

The above assumptions may be visualized by the following practical situation. A mathematical model is thought of for a mechanical workshop with three service counters S_1, S_2, S_3. All entering units have to be served first at the counter S_1 where the technical diagnosis and simple repairment is performed. The diagnosis may demand the specialized repairment which has to be performed either at S_2 or at S_3. It is also possible that the specialized repairment on both counters S_2 and S_3 is necessary but always first on S_2 and then on S_3.

Applying formulae (5)-(7) we get

$$P(n_1, n_2, n_3) = \begin{cases} \varrho_1^{n_1}\varrho_2^{n_2}\varrho_3^{n_3}P(0,0,0) & \text{if } n_1+n_2+n_3 \leqslant k, \\ 0 & \text{if } n_1+n_2+n_3 > k, \end{cases}$$

where $\varrho_i = \Gamma_i/\mu_i$ $(i = 1, 2, 3)$, $\Gamma_1 = \lambda$, $\Gamma_2 = \lambda Q_{12}$, $\Gamma_3 = \lambda(Q_{13}+Q_{12}Q_{23})$.

III. If we assume the function $f(l)$ to be of the form

$$f(l) = \begin{cases} N-l & \text{for } 0 \leqslant l < K, \\ 0 & \text{for } \quad l \geqslant K, \end{cases}$$

we obtain a model of the system with the limited number of input sources. There are K identical independent input sources each of them equiped with a single unit which is either inside the system or ready to enter it with the intensity λ. If at a given moment there are l units inside the system the remaining $K-l$ units provide the input intensity $(N-l)\lambda$.

A similar model may be applied to describe a closed system in which a constant number of K units continuously circles among N stations.

To find the steady-state probability distribution of possible states of a closed system we may apply the theorem proved here to the modified system consisting of $N-1$ stations. The remaining station, say station S_N with k_N channels, has to be excluded from the system and treated now as a source of entering units.

For the modified system we have:

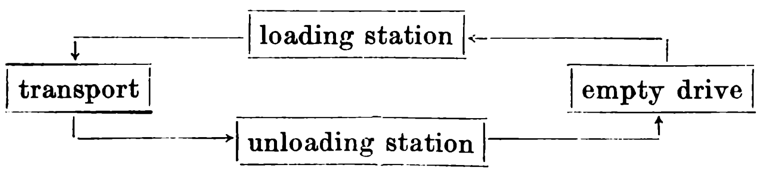
$$\lambda = (1-Q_{NN})\mu_N, \quad \lambda_i = Q_{N_i}\mu_N \ (i = 1, 2, \ldots, N-1),$$

and

$$f(l) = \begin{cases} a(k_N, k-l) & \text{for } 0 \leqslant l < k, \\ 0 & \text{for } l \geqslant k, \end{cases}$$

where l denotes the number of units at $N-1$ station $S_1, S_2, \ldots, S_{N-1}$

Example 2. Let us consider a circular transportation model as represented on the following scheme:



We shall assume the existence of K transporters and exponential distribution of service times on all the four service stations:

S_1 – the loading station with k_1 channels each of them having the service rate μ_1;

S_2 – transport with K channels (thus no queue ever possible at this station) and service rate μ_2;

S_3 – unloading station with k_2 channels and service rate μ_3;

S_4 – empty drive with K channels (no queue) and service rate μ_4.

All transporters circle in the system so that the transition probabilities Q_{ij} are $Q_{12} = Q_{23} = Q_{34} = Q_{41} = 1$ and $Q_{ij} = 0$ for all the remaining cases.

The modified system consists here of only three stations S_1, S_2, S_3. This enables us to calculate the steady-state distribution of the possible states of the system.

References

[1] T. B. Crabil, *Sufficient conditions for positive recurrence and recurrence of specially structured Markov chains*, Opns. Res. 16 (1968), p. 858-868.

[2] J. R. Jackson, *Network of waiting line*, ibidem 4 (1957), p. 518-522.

[3] R. R. P. Jackson, *Queueing processes with phase type service*, J. Roy. Stat. Soc., Ser. B, 4 (1956), p. 129-132.

[4] T. L. Saaty, *Elements of queueing theory with applications*, McGraw-Hill, New York 1961.

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O SIECI URZĄDZEŃ OBSŁUGI MASOWEJ TYPU $M/M/n$ Z INTENSYWNOŚCIĄ NOWYCH WEJŚĆ ZALEŻNĄ OD LICZBY JEDNOSTEK W SYSTEMIE

STRESZCZENIE

Autorzy rozpatrują ogólny system sieciowy $M/M/n$, analogiczny do systemu opisanego w [2], w którym 1° strumień przebyć jest strumieniem Poissona z parametrem $\lambda f(l)$ zależnym od liczby jednostek l w systemie w danej chwili (λ — stała dodatnia, $f(l)$ — ograniczona funkcja o nieujemnych wartościach, określona dla l całkowitych nieujemnych) oraz 2° przebywające jednostki są obsługiwane najpierw w i-tym dziale z prawdopodobieństwem λ_i/λ ($i = 1, 2, \ldots, N$; $\lambda_1+\lambda_2+\ldots+\lambda_N = \lambda$).

Autorzy podają wzory na rozkład prawdopodobieństwa stanu systemu liczby jednostek w poszczególnych działach w warunkach stacjonarnych oraz ilustrują otrzymane wyniki dwoma przykładami.
