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A TRANSPORTATION SYSTEM VIEWED AS A QUEUEING SYSTEM

1. The system. Fig. 1 shows a transportation system encountered, e.g., in the building industry. Four elements are distinguished in this system: (a) the loading point consisting of one or more loading devices, (b) the unloading point with a similar structure, (c) trips from the loading point to the unloading point and (d) return trips.

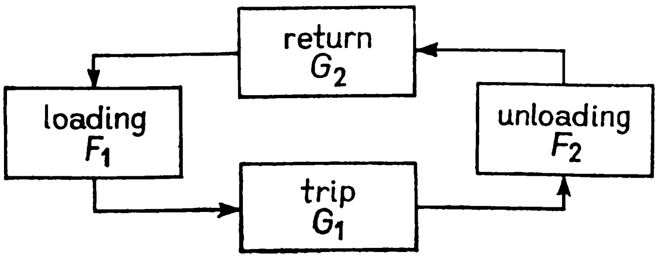

Fig. 1

We assume that the loading, unloading, trip and return times are random variables. The cumulative distribution functions of those random variables are denoted by F_1, F_2, G_1 and G_2, respectively. We assume also that the loading and unloading times are exponentially distributed and that the trip and return times have finite expected values. Moreover, we assume that loading, unloading, trip and return times are independent. This last assumption can usually be made in an automobile transportation but not in a railway transportation where it can not be true.

To complete the description of the system, the vehicle park is described now. Assume that it consists of one tractor and trailer and of $m \geqslant 1$ trailers at the loading point as well as $n \geqslant 1$ trailers at the unloading point. The trailer hauled by the tractor is left at the loading and unloading points, and the tractor takes a new trailer from there. The maneuvering times are added to the appropriate tractor trip times.

The purpose of this paper is to present a probabilistic description of the operation of this system and to propose an efficiency index. This problem has been posed by O. Kapliński at the Winter School on Queueing Theory, Reliability Theory and Similar Problems organized by the Mathematical Institute of the University of Wrocław in Karpacz in January 1971. For practical purposes, generalizations of this system, e. g., by introducing more tractors, would be of great interest.

Similar problems have been considered in queueing theory before. Particular cases of this problem are resolved under the assumption that the number of tractors is equal to the number of trailers. For instance, if the unloading takes place without waiting (multichannel unloading point) or if the unloading times are negligibly short, the unloading, trip and return times can be treated jointly as the tractor working time and the transportation system can be interpreted as a machine repair system (see Barlow and Proschan [2], Kopocińska [4] and Takács [5]). Or, if we assume that the loading, unloading, trip and return times are exponentially distributed, the transportation system leads to a network of waiting lines. Arya and Stachowski [1] have published a solution which was a direct generalization of that considered by Jackson [3].

2. Notation. Let $m(t)$ be the number of loaded trailers at the loading point in the moment t, let $n(t)$ be the number of empty trailers at the unloading point in the moment t and let $X(t) = (m(t), n(t))$. Also, let t_k, $k = 1, 2, \ldots$, denote the departure moments of the tractor from the loading point and let $(m_k, n_k) = (m(t_k+0), n(t_k+0))$, $k = 1, 2, \ldots$

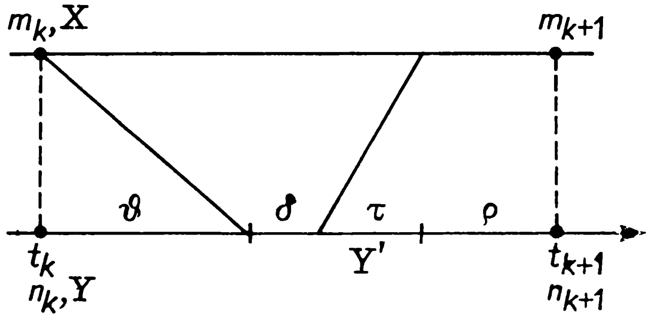

Fig. 2

Fig. 2 shows the operation cycle of the system between consecutive loading moments. On two parallel time axes the departure and arrival moments of the tractor and the corresponding random variables are marked. So t_k and t_{k+1} are tractor departure moments from the loading point, (m_k, n_k) and (m_{k+1}, n_{k+1}) are the corresponding random variables, X is the loading time of the trailer being loaded at moment t_k, Y is, for $n_k < n$, the remaining unloading time (counted from moment t_k), Y'

is the remaining unloading time at the departure moment from the unloading point, ϑ is the trip time, δ is the waiting time for the return trip (if there is no empty trailer at the unloading point), τ is the return trip time, and ϱ is the waiting time for departure from the loading point (if there is no loaded trailer there).

From the assumptions it follows that the random variables X, Y, Y', ϑ and τ have the following distributions:

$$\begin{aligned} P(X < x) &= F_1(x) = 1-\exp(-\lambda_1 x), \quad x \geqslant 0,\ \lambda_1 > 0,\\ P(Y < x) &= F_2(x) = 1-\exp(-\lambda_2 x), \quad x \geqslant 0,\ \lambda_2 > 0,\\ P(Y' < x) &= F_2(x), \quad n_k < n,\\ P(\vartheta < x) &= G_1(x),\\ P(\tau < x) &= G_2(x). \end{aligned}$$

The random variables δ and ϱ can be expressed as

$$\tag{1} \begin{aligned} \delta &= \begin{cases} 0, & n_k > 0,\\ \max(0,\ Y-\vartheta), & n_k = 0, \end{cases}\\ \varrho &= \begin{cases} 0, & m_k > 0,\\ \max(0,\ X-\vartheta-\delta-\tau), & m_k = 0. \end{cases} \end{aligned}$$

Introduce the notation

$$\tag{2} X^* = \delta_{0,m_k} X \quad \text{and} \quad Y^* = \delta_{0,n_k} Y,$$

where

$$\delta_{i,j} = \begin{cases} 0, & i \neq j,\\ 1, & i = j. \end{cases}$$

Now, the random variables δ and ϱ can be written as follows:

$$\tag{3} \begin{aligned} \delta &= \max(0,\ Y^*-\vartheta),\\ \varrho &= \max(0,\ X^*-\vartheta-\delta-\tau). \end{aligned}$$

3. Imbedded Markov chain. The state of the system is described by the two-dimensional stochastic process $X(t) = (m(t), n(t))$, where $m(t)$ denotes the number of loaded trailers at the loading point in the moment t and $n(t)$ is the number of empty trailers at the unloading point in the moment t. Under our assumptions the process $X(t)$ need not be Markovian. We investigate it knowing that the stochastic chain (m_k, n_k) is an imbedded Markov chain. In fact, it is easily seen that if the state of the system in the moment t_k+0 is known, then the state of the system in the moment $t_{k+1}+0$ depends only upon ϑ, Y, Y', τ and X. All these random variables do not depend upon the state of the chain up to the moment t_k; for the random variables Y, Y' and X, this fact follows from the exponentiality of their distributions.

The transition matrix P for the random variables $X(t_k+0) = (m_k, n_k)$ and $X(t_{k+1}+0) = (m_{k+1}, n_{k+1})$ can be expressed as the product of the matrices of consecutive transitions between the 7 random variables defined as the state of the system in appropriately chosen moments. These random variables and the transition matrix notation are given in Table 1.

TABLE 1

State of the system	Transition matrix
$X(t_k+0)$	
	P_1
$X(t_k+\vartheta-0)$	
	P_2
$X(t_k+\vartheta+\delta-0)$	
	P_3
$X(t_k+\vartheta+\delta+0)$	
	P_4
$X(t_k+\vartheta+\delta+\tau-0)$	
	P_5
$X(t_k+\vartheta+\delta+\tau+\varrho-0)$	
	P_6
$X(t_k+\vartheta+\delta+\tau+\varrho+0)$	

In the numerical analysis, the states of the process $X(t)$ are to be linearly ordered in any fixed manner. However, during the calculation of the transition probabilities $P_1, P_2, \ldots, P_6$, the two-dimensional form of the process is more convenient. The transition matrices are expressed by means of Poisson probabilities and cumulative distribution function:

$$p(k, \lambda z) = \frac{(\lambda z)^k}{k!} e^{-\lambda z},$$

$$P(k, \lambda z) = \sum_{i=k+1}^{\infty} p(i, \lambda z), \qquad k = 0, 1, \ldots, \lambda > 0.$$

Let $P_r(i, j; i+k, j+l)$, $i, i+k = 0, 1, \ldots, m$; $j, j+l = 0, 1, \ldots, n$; $r = 1, 2, \ldots, 6$, be the probabilities of transition from the state (i, j) to the state $(i+k, j+l)$ in the transition matrix P_r. They are found as follows:

1. Transition from the state of the process $X(t)$ in the moment t_k+0 to that in the moment $t_k+\vartheta-0$. In the considered time interval only loadings or unloadings can be finished, therefore, the probabilities of

transition from the state (i, j) to the state $(i+k, j+l)$ are

$$P_1(i, j; i+k, j+l) = \begin{cases} \int_0^\infty P(i, j; i+k, j+l; z)\, dG_1(z), & k \geqslant 0,\ l \geqslant 0, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$P(i, j; i+k, j+l; z) = \begin{cases} p(k, \lambda_1 z)p(l, \lambda_2 z), & i+k < m,\ j+l < n, \\ P(m-i, \lambda_1 z)p(l, \lambda_2 z), & i+k = m,\ j+l < n, \\ p(k, \lambda_1 z)P(n-j, \lambda_2 z), & i+k < m,\ j+l = n, \\ P(m-i, \lambda_1 z)P(n-j, \lambda_2 z), & i+k = m,\ j+l = n. \end{cases}$$

2. Transition from the state of the process $X(t)$ in the moment $t_k + \vartheta - 0$ to that in the moment $t_k + \vartheta + \delta - 0$. If at the beginning of this time interval there holds $j \geqslant 1$, then $\delta = 0$; thus the state of the system does not change. If $j = 0$, then δ is the random variable with exponential distribution $F_2(x)$ with parameter λ_2. Thus

$$P_2(i, j; i+k, j+l) = \begin{cases} 1, & j \geqslant 1,\ k = l = 0, \\ \int_0^\infty P(i, 0; i+k, 0; z)\, dF_2(z), & j = 0,\ k \geqslant 0,\ l = 1, \\ 0 & \text{otherwise}. \end{cases}$$

3. Transition from the state of the process $X(t)$ in the moment $t_k + \vartheta + \delta - 0$ to that in the moment $t_k + \vartheta + \delta + 0$. A departure from the unloading point takes place in this time interval; thus, $(i, j) \to (i, j-1)$, $j \geqslant 1$. Therefore,

$$P_3(i, j; i+k, j+l) = \begin{cases} 1, & k = 0,\ j \geqslant 1,\ l = -1, \\ 0 & \text{otherwise}. \end{cases}$$

4. Transition from the state of the process $X(t)$ in the moment $t_k + \vartheta + \delta + 0$ to that in the moment $t_k + \vartheta + \delta + \tau - 0$. During that time only loadings and unloadings are finished. Therefore,

$$P_4(i, j; i+k, j+l) = \begin{cases} \int_0^\infty P(i, j; i+k, j+l; z)\, dG_2(z), & k \geqslant 0,\ l \geqslant 0, \\ 0 & \text{otherwise}. \end{cases}$$

5. Transition from the state of the process $X(t)$ in the moment $t_k + \vartheta + \delta + \tau - 0$ to that in the moment $t_k + \vartheta + \delta + \tau + \varrho - 0$. If at the beginning of this time interval there holds $i \geqslant 1$, then $\varrho = 0$; thus the

state of the system does not change. If $i = 0$, then ϱ is the random variable with exponential distribution $F_1(x)$ with parameter λ_1. Thus

$$P_5(i,j;i+k,j+l) = \begin{cases} 1, & i \geqslant 1,\ k = l = 0, \\ \int\limits_0^\infty P(0,j;0,j+l;z)\,dF_1(z), & i = 0,\ k = 1,\ l \geqslant 0, \\ 0 & \text{otherwise}. \end{cases}$$

6. Transition from the state of the process $X(t)$ in the moment $t_k+\vartheta+\delta+\tau+\varrho-0$ to that in the moment $t_k+\vartheta+\delta+\tau+\varrho+0$. During this time a departure from the loading point takes place; thus, $(i,j) \to (i-1,j)$, $i \geqslant 1$. Therefore,

$$P_6(i,j;i+k,j+l) = \begin{cases} 1, & i \geqslant 1,\ k = -1,\ l = 0, \\ 0 & \text{otherwise}. \end{cases}$$

4. Efficiency index. Indices of efficiency of the system can be based on the limit probability distribution of the state of the chain (m_k, n_k) and on the behaviour of the process $X(t)$ during one operation cycle. These indices can serve, then, to formulation of system optimization problems.

A simple efficiency index can be based on the tractor waiting times as follows:

$$p = \frac{\mathrm{E}(\delta+\varrho)}{\mathrm{E}(\vartheta+\tau+\delta+\varrho)}.$$

The values of $\mathrm{E}\vartheta$ and $\mathrm{E}\tau$ are determined from the probability distributions made in the assumptions about the system. It remains, thus, to determine the expected value of the random variable $\delta+\varrho$. From (3) it follows

$$\begin{aligned} (4) \qquad \delta+\varrho &= \max(0,\ Y^*-\vartheta)+\max\big(0,\ X^*-\vartheta-\max(0,\ Y^*-\vartheta)-\tau\big) \\ &= \max\big(\max(0,\ Y^*-\vartheta),\ X^*-\vartheta-\tau\big) \\ &= \max\big(\max(\vartheta,\ Y^*),\ X^*-\tau\big)-\vartheta. \end{aligned}$$

If the steady-state probabilities of the chain (m_k, n_k) are known — and they can be found from the transition matrix of the chain — then, using (2), we find

$$\begin{aligned} (5) \quad & \mathrm{E}\max\big(\max(\vartheta,\ Y^*),\ X^*-\tau\big) \\ &= \mathrm{P}(m_k = 0,\ n_k = 0)\mathrm{E}\max\big(\max(\vartheta,\ Y),\ X-\tau\big)+ \\ &\quad +\mathrm{P}(m_k > 0,\ n_k = 0)\mathrm{E}\max(\vartheta,\ Y)+ \\ &\quad +\mathrm{P}(m_k = 0,\ n_k > 0)\mathrm{E}\max(\vartheta,\ X-\tau)+\mathrm{P}(m_k > 0,\ n_k > 0)\mathrm{E}\vartheta. \end{aligned}$$

This allows to calculate $E(\delta+\varrho)$.

We express now the expected values of the right-hand side of (5) explicitly. The calculation is based on the Laplace and Laplace-Stieltjes transforms of G_1 and G_2.

Let

$$G_i^*(s) = \int_0^\infty e^{-sx} G_i(x)\,dx \quad \text{and} \quad g_i^*(s) = \int_0^\infty e^{-sx}\,dG_i(x) = sG_i^*(s),$$

where $i = 1, 2$.

It is easily verified that

$$\text{(6)} \quad \mathrm{E}\max(\vartheta, Y) = \int_0^\infty \big[1-\mathrm{P}\big(\max(\vartheta, Y) < z\big)\big]\,dz$$

$$= \int_0^\infty [1-G_1(z)+G_1(z)\exp(-\lambda_2 z)]\,dz = \mathrm{E}\vartheta + G_1^*(\lambda_2),$$

$$\text{(7)} \quad \mathrm{E}\max(\vartheta, X-\tau) = \int_0^\infty [1-G_1(z)\mathrm{P}(X-\tau<z)]\,dz$$

$$= \int_0^\infty \Big[1-G_1(z)\int_0^\infty \big(1-\exp(-\lambda_1 z-\lambda_1 t)\big)\,dG_2(t)\Big]\,dz$$

$$= \int_0^\infty \big[1-G_1(z)\big(1-\exp(-\lambda_1 z)g_2^*(\lambda_1)\big)\big]\,dz = \mathrm{E}\vartheta + G_1^*(\lambda_1)g_2^*(\lambda_1),$$

$$\text{(8)} \quad \mathrm{E}\max\big(\max(\vartheta, Y), X-\tau\big)$$

$$= \int_0^\infty \big[1-\mathrm{P}\big(\max(\vartheta, Y) < z\big)\mathrm{P}(X-\tau<z)\big]\,dz$$

$$= \int_0^\infty \Big[1-G_1(z)\big(1-\exp(-\lambda_2 z)\big)\int_0^\infty \big(1-\exp(-\lambda_1 z-\lambda_1 t)\big)\,dG_2(t)\Big]\,dz$$

$$= \mathrm{E}\vartheta + G_1^*(\lambda_2) + G_1^*(\lambda_1)g_2^*(\lambda_1) - G_1^*(\lambda_1+\lambda_2)g_2^*(\lambda_1).$$

5. The case $m = 1$ and $n = 1$. In this case, directly after the departure of the tractor from the loading point, the loading of the next trailer begins, the chain (m_k, n_k) is thus determined only on two states $(0, 0)$ and $(0, 1)$; therefore, only the chain $\{n_k\}$ is of interest. The state probabilities of it can be written explicitly, also in the case of any arbitrary loading time distribution. It can be easily verified that

$$\text{(9)} \quad \mathrm{P}(n_k = 0) = \mathrm{P}(Y' > \tau+\varrho) = \mathrm{P}\big(Y' > \tau+\max(0, X-\vartheta-\delta-\tau)\big)$$

$$= \mathrm{P}\Big(Y' > \max\big(\tau, X-\vartheta-\max(0, Y^*-\vartheta)\big)\Big)$$

$$= \mathrm{P}(n_k = 0)\mathrm{P}\big(Y' > \max(\tau, X-\max(\vartheta, Y))\big) +$$

$$+\big(1-\mathrm{P}(n_k = 0)\big)\mathrm{P}\big(Y' > \max(\tau, X-\vartheta)\big).$$

Assuming stationarity of the chain $\{n_k\}$, we have

$$\mathrm{P}(n_{k+1} = 0) = \mathrm{P}(n_k = 0) = \pi.$$

Hence, from (9) it follows

$$\pi = \frac{\mathrm{P}\big(Y' > \max(\tau, X-\vartheta)\big)}{1-\mathrm{P}\big(Y' > \max(\tau, X-\max(\vartheta, Y))\big)+\mathrm{P}\big(Y' > \max(\tau, X-\vartheta)\big)}. \tag{10}$$

The calculation of (10), and also of the efficiency index p, using (4) and (5), is usually cumbersome, though the used operations are simple: composition of distributions, expected value and comparison of random variables. Easy computation procedures can be given if the random variables are discrete ones and take on the values an, $n = 0, 1, \ldots, a > 0$. The exponential distribution is then replaced by the geometric distribution.

If in the considered case we assume, similarly as before in this paper, an exponential unloading time, then after suitable transformations we obtain

$$\begin{aligned} &\mathrm{P}\big(Y' > \max(\tau, X-\max(\vartheta, Y))\big) \\ &\qquad = \lambda_2\{[1-g_1^*(\lambda_2)+\lambda_2 G_1^*(\lambda_2)]G_2^*(\lambda_2)- \\ &\qquad\qquad -[g_1^*(\lambda_1)-g_1^*(\lambda_1+\lambda_2)+G_1^*(\lambda_1+\lambda_2)]G_2^*(\lambda_1+\lambda_2)\}, \\ &\mathrm{P}\big(Y' > \max(\tau, X-\vartheta)\big) = \lambda_2 G_2^*(\lambda_2)-\lambda_2 g_1^*(\lambda_1)G_2^*(\lambda_1+\lambda_2). \end{aligned} \tag{11}$$

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O PEWNYM SYSTEMIE TRANSPORTU JAKO SYSTEMIE OBSŁUGI MASOWEJ

STRESZCZENIE

System transportu, przeanalizowany w pracy, składa się z czterech elementów — punktu załadowczego, złożonego z jednej lub kilku linii załadunku, jazdy, wyładunku i jazdy powrotnej. W systemie pracuje jeden ciągnik i pewna liczba przyczep zlokalizowanych w punktach załadowczym i wyładowczym. Celem pracy jest podanie opisu probabilistycznego i propozycja wskaźnika wydajności systemu. Zakłada się, że czasy załadunku, jazdy, wyładunku i powrotu są zmiennymi losowymi niezależnymi o znanych rozkładach. Analizę stanu systemu, zdefiniowanego przez liczby przyczep gotowych do drogi w punktach załadowczym i wyładowczym, sprowadza się do analizy pewnego włożonego łańcucha Markowa. Dla tego łańcucha została znaleziona macierz prawdopodobieństw przejścia. Wskaźnik wydajności systemu zdefiniowano korzystając z charakterystyki czasu czekania ciągnika.