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APPLICATIONES MATHEMATICAE
XIII, 3 (1973)

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EXISTENCE OF SOLUTIONS FOR PRESTRESSED SHALLOW SHELLS

0. Introduction. In this paper we are concerned with a non-linear problem in the theory of shallow shells of an arbitrary shape. The paper is a continuation of previous works [11] and [12] and the results obtained give further insight into the mathematical structure of non-linear differential equations describing the stress state and the deflections of prestressed shallow shells under action of forces. This problem is studied by the use of functional analysis methods on appropriate Hilbert spaces.

Consider a thin elastic shallow shell of arbitrary shape under internal and external loads. The basic non-linear mathematical model for a thin elastic shallow shell under simultaneous action of lateral, internal and boundary forces has centered around the system of equations (see [6] and [18])

$$\frac{D}{h}\Delta^2 w = L(w, F) + k_1 \frac{\partial^2 F}{\partial x_1^2} + k_2 \frac{\partial^2 F}{\partial x_2^2} + q_1(x) \quad \text{in } \Omega, \tag{0.1}$$

$$\frac{1}{E}\Delta^2 F = -\frac{1}{2} L(w, w) - k_1 \frac{\partial^2 w}{\partial x_1^2} - k_2 \frac{\partial^2 w}{\partial x_2^2} + p_0(x) \quad \text{in } \Omega, \tag{0.2}$$

where $w(x)$ and $F(x)$ are real-valued functions of the independent variable $x = (x_1, x_2)$ defined on a domain Ω whose boundary is $\partial\Omega$; $\Omega \subset E_2$, the 2-dimensional Euclidean space; $w(x)$ represents the deflection of the shell from its initial position and $F(x)$ represents the so-called Airy stress function by means of which all stress components can be found in terms of the second derivatives of it. D is the flexural rigidity, E is the Young modulus, h is the thickness of the shell, the function $p_0(x)$ is a measure of the stresses which can occur in the undeflected shell independently of external loads, for example, prior stresses built into the shell or produced by heating the shell unevenly, the function $q_1 \equiv q_0(x) + q^*(x)$ is a measure of external forces, $k = (k_1, k_2)$ is the curvature of the shell, Δ^2 is the bi-harmonic operator, and

$$L(f, g) = f_{x_1x_1} g_{x_1x_1} + f_{x_2x_2} g_{x_2x_2} - 2 f_{x_1x_2} g_{x_1x_2}. \tag{0.3}$$

As we have said, we allow forces to be applied to the edge of the shell. If the boundary $\partial\Omega$ is smooth, such edge stresses are described by boundary conditions of the sort $F_{ss} = h_1(x)$, $F_{ns} = h_2(x)$ on $\partial\Omega$, where subscripts n and s represent derivatives in the normal and tangential directions, respectively.

1. Auxiliary problem. Let the function $F^0(x)$ be a solution of the following auxiliary problem

$$\Delta^2 F^0 = Ep_0(x) \quad \text{in } \Omega, \tag{1.1}$$

$$k_1 F^0_{x_1x_1} + k_2 F^0_{x_2x_2} = q_0(x) \quad \text{in } \Omega, \tag{1.2}$$

$$F^0_{ss} = h_1(x), \quad F^0_{ns} = h_2(x) \quad \text{on } \partial\Omega. \tag{1.3}$$

This system of equations can be interpreted physically as follows: we seek after such a stress function produced in the shell under the action of $p_0(x)$, $q_0(x)$, $h_1(x)$ and $h_2(x)$, when the shell is not allowed to deflect, i. e., $w(x) \equiv 0$. Such a case can be realized in engineering practise by enclosing the shell in a rigid casing. Since we are interested in a general case, i. e., when the shell is allowed to deflect from its unstressed configuration, and this occurs only when the magnitude of the applied forces is great enough, it will be convenient to introduce a parameter into the problem which is a measure of the strength of these applied forces. To this end, we think of $p_0(x)$, $q_0(x)$, $h_1(x)$ and $h_2(x)$, as all depending linearly on a real parameter λ_0. The parameter λ_0 can be positive or negative, this depends on the applied forces. Then the function $F^0(x)$ also depends linearly on λ_0. To emphasize the fact, we write $\lambda_0 F^0(x)$ instead of $F^0(x)$. Finally, we define $f \overset{\text{df}}{=} F - \lambda_0 F^0$, the stress function increment produced when the shell is allowed to deflect. By introducing the function $f(x)$ into (0.1) and (0.2), we obtain the following system of equations:

$$\frac{D}{h}\Delta^2 w = L(w, f) + \lambda_0 L(w, F^0) + k_1 \frac{\partial^2 f}{\partial x_1^2} + k_2 \frac{\partial^2 f}{\partial x_2^2} + q^*(x), \tag{1.4}$$

$$\frac{1}{E}\Delta^2 f = -\frac{1}{2} L(w, w) - k_1 \frac{\partial^2 w}{\partial x_1^2} - k_2 \frac{\partial^2 w}{\partial x_2^2}. \tag{1.5}$$

In this form the description of all prior internal and edge stresses is concentrated in the single function $F^0(x)$ and in the parameter λ_0. On smooth sections of the edge $\partial\Omega$ the modified stress function $f(x)$ will satisfy homogeneous boundary conditions, i. e., $f_{ss} = 0$ and $f_{ns} = 0$. These conditions are implied by the Dirichlet conditions

$$f = f_{x_1} = f_{x_2} = 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

In what follows we shall assume that the shell is clamped at its edge. This requirement corresponds to the boundary conditions for the deflection function $w(x)$:

(1.7) $$w = w_{x_1} = w_{x_2} = 0 \quad \text{on } \partial\Omega.$$

2. Outline of the procedure. We shall be mainly concerned with the existence and uniqueness of real-valued solutions of (0.1) and (0.2) with boundary conditions (1.6) and (1.7). We focus our attention on the following problems:

(a) Under what restrictions on the initial data does the boundary value problem have a real-valued solution?

(b) Is the solution, if it exists, unique?

Before we give answer to these questions, we first simplify the form of (1.4) and (1.5) by introducing new functions into these equations. In order to eliminate unimportant parameters E, D and h from (1.4) and (1.5), we introduce γw, $Dh^{-1}f$, $Dh^{-1}F^0\max(P)$, γk_i and $D\gamma\sigma q$ instead of w, f, F^0, k_i and q^*, respectively. Here $\gamma^2 = 2D/Eh$, and

$$\max(P) \equiv$$

$$\max\left[\sup_{\Omega}\left(\frac{2|k_1||\nabla(1-\zeta)F^0|}{\mu_1^{1/2}}, \frac{2|k_2||\nabla(1-\zeta)F^0|}{\mu_1^{1/2}}\right), \sqrt{\operatorname{mes}\Omega}\, 2\,|\Delta(1-\zeta)F^0|\right],$$

$$\sigma = h(D\gamma)^{-1}\sup_{\Omega}|q^*|.$$

After introducing the proposed quantities into (1.4) and (1.5), we obtain

(2.1) $$\Delta^2 w = L(w, f) + \lambda L(w, F^0) + \{k, f\} + \sigma q \quad \text{in } \Omega,$$

(2.2) $$\Delta^2 f = -L(w, w) - 2\{k, w\} \quad \text{in } \Omega,$$

where we use the notation $\{f, g\} \equiv f_1 g_{x_1x_1} + f_2 g_{x_2x_2}$, $\lambda = \lambda_0 \max(P)$.

As far as my knowledge of the literature on direct methods in the non-linear theory of elastic shallow shells is concerned only some symmetric problems or problems without the influence of prestressing loads on the stress state in the shell were considered (cf. [19] and [20]). In this paper we take into account prior stresses built into the shell as well as arbitrary shape of the region Ω bounded by a piecewise smooth curve $\partial\Omega$. The elastic structure can be loaded by very general external forces. Strict assumptions on the "arbitrarniess" of the load function will be formulated later. There are several methods to tackle the problem (2.1), (2.2), (1.6) and (1.7), one of which is that presented in [11] and [12], which we shall here apply in a somewhat modified version. The original classical problem

(0.1) and (0.2) is reduced to a generalized boundary-value problem and then imbedded into a one-parameter family of similar boundary problems described by an operator equation, in general non-linear, of the form

$$u - \nu T_Q(u) = 0, \quad \nu \in [0,1] \tag{2.3}$$

defined on a suitable Sobolev space. We shall make use of functional analysis methods and some imbedding theorems of Sobolev as well as of a simplified version of a fixed-point theorem of continuous mappings which implies the existence of at least one solution of (2.3). The problem of smoothness of the existing solution of (2.3) will also be analyzed in view of the initial data. This leads to interdependence of generalized and classical solutions. It should be emphasized that generalized solutions represent, in our case, a practical value since various physical quantities such as, for instance, the stress tensor or the deflection function are continuous functions under very weak conditions imposed on the initial data. However, the existence of classical solutions, i. e., strict solutions, requires much stronger assumptions on the initial data.

Derivation of *a priori* estimates for certain tri-linear integral forms constitutes the chief technical difficulty. In dealing with the terms involving $F^0(x)$, we make use of a modified version of a so-called cut-off function of Hopf [5], which was first applied in considering Navier-Stokes equations. In a previous paper [12] we have used another cut-off function, namely, a function proposed by Ladyzhenskaya [7], but the function of Hopf seems to be simpler and there is little difficulty to adjust it to our requirements. As it was already said, this paper deals with the existence and uniqueness of solutions of the problem (2.1), (2.2), (1.6) and (1.7). The non-uniqueness problem we hope to consider in another paper.

3. Function spaces and fundamental inequalities. We shall first introduce a number of function spaces and exhibit a few integro-differential inequalities.

(a) $L^p(\Omega)$ is a Lebesgue space consisting of real functions $u(x)$ efinedd on Ω and such that $\int_\Omega |u|^p dx < \infty$ for fixed p, $1 \leqslant p < \infty$. $L^p(\Omega)$ is a Banach space with the norm $\|u\|^p = \int_\Omega |u|^p dx$.

(b) $L^2(\Omega)$ is a Hilbert space, complete with respect to the norm $\|u\| = (u, u)_{0,\Omega}^{1/2}$, where $(u, v)_{0,\Omega} = \int_\Omega uv dx$.

(c) $\boldsymbol{L}^2(\Omega) = \{L^2(\Omega)\}^2$ is the cartesian double product of the space of scalar-valued square summable functions defined in the domain $\Omega \subset E^2$.

(d) $H_p^{(l)}(\Omega)$ is a Sobolev space consisting of real functions $u(x)$ having the property that $u(x)$ and all its first and second generalized derivatives are square Lebesgue inegrable on Ω.

(e) $H_2^{(l)}(\Omega)$ is a Hilbert space whose norm is defined by means of the inner product

$$(3.1) \qquad (u, v)_{l,\Omega} = \int\limits_{\Omega} \sum_{a=0}^{l} \sum_{(a)} D^a u D^a v dx.$$

(f) $H_{0,2}^{(2)}(\Omega)$ is the completion of $\mathscr{C}_0^{\infty}(\Omega)$ under the norm of $H_2^{(2)}(\Omega)$. Thus $H_{0,2}^{(2)}$ is a closed subspace of $H_2^{(2)}(\Omega)$. Hence it can be considered as a Hilbert space with the inner product (3.1) with $l = 2$.

(g) $\boldsymbol{H}_{0,2}^{(2)} = \{H_{0,2}^{(2)}(\Omega)\}^2$ is similarly defined as $\boldsymbol{L}^2(\Omega)$. We shall now introduce a few integro-differential inequalities which play an important role in the theory of function spaces. Elements of the foregoing spaces are characterized by various types of differential properties, so that if one space is a part of another (in the abstract set sense), the group of properties characterizing the first space will involve properties characteristic for elements of the second space. To take a familiar case, knowledge of some inequalities valid for elements of a given space, for instance, $H_2^{(l)}(\Omega)$, can lead to conclusions as regards the boundedness, continuity or even differentiability of the function itself. Further, by comparing the same function as an element of two distinct spaces, we obtain what is called the *imbedding operation*. The most important imbeddings, as we know, are those which are completely continuous. In most cases we shall be concerned with functions in the Hilbert space $\boldsymbol{H}_{0,2}^{(2)}(\Omega)$. Each function belonging to $\boldsymbol{H}_{0,2}^{(2)}(\Omega)$ can be regarded as a function of compact support defined on the whole Euclidean space E^2, if we extend the function by setting it equal to zero outside the domain Ω. All inequalities given in the sequel can be easily proved for functions of compact support and then generalized to functions defined on the domain Ω, which are not of compact support, provided only that the boundary $\partial\Omega$ is sufficiently smooth (see [9] and [16]). Moreover, since the smooth functions are dense in $H_{0,2}^{(l)}(\Omega)$, all inequalities valid for functions in $\mathscr{C}_0^{\infty}(\Omega)$ remain automatically true for any element belonging to the space $H_{0,2}^{(l)}(\Omega)$; this is a consequence of continuity of the norm.

LEMMA 1. *For any $u \in H_{0,2}^{(2)}$, we have*

$$(3.2) \qquad \int\limits_{\Omega} u^2 dx \leqslant \frac{1}{\mu_1} \|\nabla u\|_{L^2} \leqslant \frac{1}{\mu_1^2} \|\Delta u\|_{L^2},$$

where u_1 is the smallest eigenvalue of the operator $-\Delta$ in Ω with zero boundary conditions.

The foregoing inequality is, in general, not valid for unbounded domains Ω.

LEMMA 2. *Let Ω be a bounded domain with boundary $\partial\Omega$ of class $\mathscr{C}^2$ Then*

$$\|u\|_{H_2^{(2)}} \leqslant C(\Omega)\ \|\Delta u\|_{L^2} \quad \text{for every } u \in H_2^{(2)} \cap H_{0,2}^{(1)}. \tag{3.3}$$

Simple proofs of these lemmas can be found in [7] and [9].

It is well known that the functions $u \in H_2^{(2)}$, $\Omega \subset E^2$, are continuous and we have the estimate

$$\sup_{x\in\Omega} |u(x)| \leqslant C(\Omega)\|u\|_{H_2^{(2)}}, \tag{3.4}$$

where $C(\Omega)$ depends only on Ω.

In the space $\boldsymbol{H}_{0,2}^{(2)}$ we also make use of the norm

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}_{0,2}^{(2)}} = \int_\Omega [(\Delta u_1)^2 + (\Delta u_2)^2]dx, \qquad \text{where } \boldsymbol{u} = (u_1, u_2). \tag{3.5}$$

To show that (3.5) actually defines a norm on $\boldsymbol{H}_{0,2}^{(2)}(\Omega)$, it is enough to show that $\{\int_\Omega (\Delta\boldsymbol{u})^2 dx\}$ is equivalent to a more conventional norm on $\boldsymbol{H}_{0,2}^{(2)}$. Let $\|u\|_1$ denote $\{\int_\Omega (\Delta u)^2 dx\}^{1/2}$ and $\|u\|_2$ the more conventional norm, namely, $\{\int_\Omega [u^2 + |\nabla u|^2 + |\nabla\nabla u|^2]\}^{1/2}$, where ∇u denotes a vector function $\{\partial_i u\}$ or a tensor function $\{\partial_i u_j\}$; depending on whether $u_{1/2}$ is a scalar or a vector, $|\cdot|$ denotes a Euclidean length $(\sum_{i=1}^2 |\partial_i u|^2)^{1/2}$ for a scalar $u(x)$ or $(\sum_{i,j=1}^2 |\partial_i u_j|^2)^{1/2}$ for a vector $\boldsymbol{u}(x)$. The norm $\|\cdot\|_2$ is produced by the inner product (3.1). The formal differential operator ∇ has to be understood in a generalized sense.

By definition, $\|u\|_1 \leqslant \|u\|_2$ for all $u \in H_{0,2}^{(2)}(\Omega)$. On the other hand, as it follows from Lemma 2, we have

$$\|u\|_2 \leqslant C(\Omega)\|u\|_1 \tag{3.6}$$

for all $u \in H_2^{(2)} \cap H_{0,2}^{(1)}$, provided the boundary $\partial\Omega$ has bounded first and second derivatives. C is a constant dependent only on Ω. Since $H_{0,2}^{(2)} \subset H_2^{(2)} \cap H_{0,2}^{(1)}$, it follows that $\|u\|_2 \leqslant C(\Omega)\|u\|_1$ for all $u \in H_{0,2}^{(2)}(\Omega)$. Hence $\|u\|_1 < \|u\|_2 \leqslant C(\Omega)\|u\|_1$ for all $u \in H_{0,2}^{(2)}(\Omega)$. But this means that the norms are equivalent.

4. Behaviour of a function in $\boldsymbol{H}_{0,2}^{(2)}$ near $\partial\Omega$. The following lemmas are concerned with the behaviour of a function $\boldsymbol{u} \in \boldsymbol{H}_{0,2}^{(2)}(\Omega)$ in a neighbourhood of the boundary $\partial\Omega$. We assume that $\partial\Omega$ is sufficiently smooth.

Let $\omega(\delta)$ denote a boundary strip of $\partial\Omega$ with width δ, i. e., a set of points x whose distance from $\partial\Omega$ is less than δ; to be precise we write

$$(*)\qquad \omega(\delta) = \{x \mid x \in \Omega, \operatorname{dist}(x, \partial\Omega) < \delta\}.$$

Since the boundary $\partial\Omega$ is smooth, there exists a $\delta > 0$ such that 2δ is smaller than any of the radii of curvature at points of $\partial\Omega$ and also so small that all points on a normal line of length 2δ originating from an arbitrary point P of $\partial\Omega$ are closer to P than to any other boundary point. Then the normal vectors of length δ sweep out a ring-like region having a coordinate system composed of the curve coordinate of $\partial\Omega$ and the distance s along a normal, where $0 \leqslant s \leqslant \delta$. We shall now prove a lemma which is a generalization of Leray's theorem [8].

LEMMA 3. *Let $\omega(\delta)$ be the boundary strip defined by $(*)$ and let $u \in H_{0,2}^{(2)}$. Then the inequality*

$$(4.1)\qquad \left\| \frac{\nabla u}{s} \right\|_{\omega(\delta_1)} \leqslant C(\delta) \|\nabla\nabla u\|_{\omega(\delta_1)}$$

holds true for an arbitrary δ_1, $0 \leqslant \delta_1 \leqslant \delta$.

Proof. It suffices to consider the case where $u \in \mathscr{C}_0^\infty(\Omega)$. We first bound the integral

$$\int_0^\delta \left| \frac{\nabla u}{s} \right|^2 ds,$$

where s is the usual normal distance of a point from the boundary $\partial\Omega$. Integration by parts, the fact that $u \in \mathscr{C}_0^\infty(\Omega)$, and Schwarz's inequality yield

$$\int_0^\delta \left| \frac{\nabla u}{s} \right|^2 ds = -\frac{|\nabla u|^2}{s}\Big|_0^\delta + \int_0^\delta \frac{|\nabla u|\,|\nabla\nabla u|}{s} ds$$

$$\leqslant 2\left(\int_0^\delta \frac{|\nabla u|^2}{s^2} ds\right)^{1/2}\left(\int_0^\delta |\nabla\nabla u|^2 ds\right)^{1/2},$$

and hence

$$(4.2)\qquad \int_0^\delta \frac{|\nabla u|^2}{s^2} ds \leqslant 4\int_0^\delta |\nabla\nabla u|^2 ds$$

or

$$(4.3)\qquad \left\| \frac{\nabla u}{s} \right\|_\delta \leqslant 2\,\|\nabla\nabla u\|_\delta .$$

We now wish to bound

$$\int_{\omega(\delta_1)} \frac{|\nabla u|^2}{s^2} dx.$$

We have immediately from (4.2)

$$\int_{\omega(\delta_1)} \frac{|\nabla u|^2}{s^2} dx \leqslant C(\delta) \int_{\omega(\delta_1)} |\nabla\nabla u|^2 dx. \tag{4.4}$$

We shall now construct a cut-off function which equals one on $\partial\Omega$, vanishes outside $\omega(\delta)$ and satisfies certain other requirements to be enumerated in Lemma 4.

LEMMA 4. *For any prescribed $\varepsilon > 0$, there exists a real function with the following properties:*

(a) *ζ is defined in the neighbourhood $\omega(\delta)$ of $\partial\Omega$ and has continuous derivatives up to the second order, which are bounded.*

(b) *$\zeta = 1$ on $\partial\Omega$ and $\zeta = 0$ outside $\omega(\delta)$.*

(c) *$|\nabla\zeta| = 0$ on $\partial\Omega$ and outside $\omega(\delta)$.*

(d) *$|\zeta| \leqslant \varepsilon/s$ and $|\nabla\zeta/s| \leqslant \varepsilon/s$ throughout $\omega(\delta)$, where s is the usual normal distance of a point x from $\partial\Omega$.*

Proof. We define the function $\zeta(s)$ by

$$\zeta(x) = \frac{\int_b^1 \frac{1}{t}\psi\left(\frac{s}{t}\right)dt}{\int_h^1 t^{-1}dt} = \frac{1}{\ln b^{-1}} \int_b^1 \frac{1}{t}\psi\left(\frac{s}{t}\right)dt, \tag{4.5}$$

where

$$\psi(s) = \begin{cases} 1 & \text{for } s \leqslant 0, \\ \left[1-\left(\frac{s}{\delta}\right)^2\right]^2 & \text{for } 0 \leqslant s \leqslant \delta, \\ 0 & \text{for } s \geqslant \delta, \end{cases}$$

and b, $0 < b < 1$, is a certain constant yet to be determined. It is not difficult to notice that $|\psi| \leqslant 1$ for an arbitrary s, and (4.5) yields $|\zeta| \leqslant 1$. We also have $\psi(0) = 1$, $\psi'(0) = \psi''(0) = 0$ and $\zeta(0) = 1$, $\zeta'(0) = \zeta''(0) = 0$. The definition of $\psi(s)$ yields $\zeta = 0$ and $\zeta' = 0$ outside $\omega(\delta)$. Condition (d) is proved as follows:

First we make the substitution $y = s/t$ in (4.5) to obtain

$$\zeta(x) = \frac{1}{\ln b^{-1}} \int_b^1 \frac{1}{t}\psi\left(\frac{s}{t}\right)dt = \frac{1}{\ln b^{-1}} \int_s^{s/b} \psi(y)\frac{dy}{y}.$$

By using the fact that $\psi(y)$ is a continuous function and y^{-1} is monotonic, we arrive at the following equality:

$$\frac{1}{\ln b^{-1}}\int_{s}^{s/b}\frac{1}{y}\psi(y)\,dy = \frac{1}{\ln b^{-1}}\left[\frac{1}{s}\int_{s}^{\xi}\psi(y)\,dy+\frac{b}{s}\int_{\xi}^{s/b}\psi(y)\,dy\right].$$

Hence

$$|\zeta|\leqslant\frac{1}{s}\,\frac{1+b}{\ln b^{-1}}\int_{0}^{\delta}|\psi(y)|\,dy.$$

Differentiation of ζ with respect to s yields

$$\frac{\partial\zeta}{\partial s}=\frac{1}{\ln b^{-1}}\int_{b}^{1}\frac{1}{t^2}\psi'\left(\frac{s}{t}\right)dt=\frac{1}{s}\,\frac{1}{\ln b^{-1}}\int_{s}^{s/b}\psi'(y)\,dy,$$

and hence

$$\left|\frac{\partial\zeta}{\partial s}\right|\leqslant\frac{1}{s}\,\frac{1}{\ln b^{-1}}\int_{0}^{\delta}|\psi'|\,dy.$$

If we now choose correspondingly the constant b, and this is possible since $0<1/\ln b<\infty$ for $0<b<1$, then we obtain $|\zeta|\leqslant\varepsilon/s$ and $|\partial\zeta/\partial s|\leqslant\varepsilon/s$. The gradient of ζ satisfies the relation $\nabla\boldsymbol{\zeta}\cdot\boldsymbol{s}=\partial\zeta/\partial s$, where $\boldsymbol{s}$ is a unit vector in the s-direction. Hence we obtain $|\nabla\zeta|\leqslant\varepsilon/s$. This completes the proof of Lemma 4.

Now we give the statement and proof of the major lemma which provides effective bounds of the non-linear forms that appear in our problem.

LEMMA 5. *For any $\bar{\varepsilon}>0$ and any $\boldsymbol{u}\in\boldsymbol{H}_{0,2}^{(2)}(\Omega)$, we have the estimate*

$$\begin{aligned}&|L_2(u_1,u_1,\zeta F^0)|\\&\quad\equiv\left|\lambda\int_{\Omega}[(\zeta F^0)_{x_1}(u_{1x_1x_2}u_{1x_2}-u_{1x_1}u_{1x_2x_1})+\right.\\&\qquad\qquad\left.+(\zeta F^0)_{x_2}(u_{1x_1}u_{1x_1x_2}-u_{1x_1x_1}u_{1x_2})]\,dx\right|\\&\quad\leqslant\bar{\varepsilon}\,|\lambda|\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}_{0,2}^{(2)}}.\end{aligned}$$

Proof. By elementary inequalities we obtain

$$|L_2(u_1,u_1,\zeta F^0)|\leqslant 4|\lambda|\int_{\Omega}|\Delta\boldsymbol{u}|\,|\nabla\boldsymbol{u}|\,|\nabla(\zeta F^0)|\,dx.$$

Since $|\nabla(\zeta F^0)|$ vanishes outside $\omega(\delta)$, we have, using Lemma 4, Schwarz's inequality, and (4.4),

$$\int_{\Omega}|\Delta\boldsymbol{u}|\,|\nabla\boldsymbol{u}|\,|\nabla(\zeta F^0)|\,dx = \int_{\omega(\delta)}|\Delta\boldsymbol{u}|\,|\nabla\boldsymbol{u}|\,|\nabla(\zeta F^0)|\,dx$$

$$\leqslant \int_{\omega(\delta)}\frac{\varepsilon}{s}(|F^0|+|\nabla F^0|)\,|\nabla\boldsymbol{u}|\,|\Delta\boldsymbol{u}|\,dx \leqslant \varepsilon C_1\Big(\int_{\omega(\delta)}|\Delta\boldsymbol{u}|^2dx\Big)^{1/2}\Big(\int_{\omega(\delta)}\frac{|\nabla u|^2}{s^2}dx\Big)^{1/2}$$

$$\leqslant \varepsilon C_1\Big(\int_{\omega(\delta)}|\Delta\boldsymbol{u}|^2dx\Big)^{1/2}\Big(C_2\int_{\omega(\delta)}|\Delta\boldsymbol{u}|^2dx\Big)^{1/2} \leqslant \varepsilon C\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}^{(2)}_{0,2}} \leqslant \frac{\bar{\varepsilon}}{4}\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}^{(2)}_{0,2}},$$

letting $\bar{\varepsilon} = 4\varepsilon C$. Hence

$$|L_2(u_1, u_1, \zeta F^0)| \leqslant \bar{\varepsilon}\,|\lambda|\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}^{(2)}_{0,2}(\Omega)}$$

for any $\bar{\varepsilon} > 0$ and any $\boldsymbol{u} \in \boldsymbol{H}^{(2)}_{0,2}(\Omega)$. This completes the proof of Lemma 5.

In order to obtain effective estimates of solutions, we have to modify the original problem. This modification is achived by introducing a new function, to be defined below, and the use of the foregoing cut-off function ζ. To use a vector notation we write the modified solutions as $w(x) \equiv u_1(x)$ and $g(x) \equiv u_2(x)$, where $g(x) \overset{\mathrm{df}}{=} F(x)+\lambda_0(1-\zeta)F^0(x)$. If we take into account the previously derived properties of the cut-off function ζ, we can easily show that $g(x) = \partial g/\partial x = 0$ on $\partial\Omega$. The modified Airy's stress function $g(x) \equiv u_2(x)$ belongs to the Hilbert space $H^{(2)}_{0,2}(\Omega)$.

5. Formulation of the generalized problem. Suppose the vector function $\boldsymbol{u} = (u_1, u_2)$ is sufficiently smooth and satisfies the system of equations

$$(5.1)\qquad \Delta^2 u_2 = \nu\big[-L(u_1, u_1)-2\{k, u_1\}+\Delta^2\big((1-\zeta)F^0\big)\big],$$

$$(5.2)\qquad \Delta^2 u_1 = \nu[L(u_1, u_2)+\lambda_0 L(u_1, \zeta F^0)+\{k, u_2\}-\{k, (1-\zeta)F^0\}+\sigma q]$$

in Ω, where $0 \leqslant \nu \leqslant 1$. For $\nu = 1$, we obtain equations describing the stress state and deflection of a prestressed shallow shell, i. e., (1.8) and (1.9) with the function $g(x)$ instead of $f(x)$.

The following boundary conditions are assumed:

$$(5.3)\qquad \boldsymbol{u}(x) = \nabla\boldsymbol{u}(x) = 0 \quad \text{on } \partial\Omega.$$

Let $\boldsymbol{\eta} = (\eta_1, \eta_2) \in \mathscr{C}_0^\infty$. Multiplying (5.1) and (5.2) by η_2 and η_1 respectively, and integrating over the domain Ω, we obtain

$$(5.4)\qquad \int_\Omega \eta_1 \Delta^2 u_1 dx = \nu\Big[\int_\Omega \eta_1 L(u_1, u_1)dx + \lambda_0 \int_\Omega \eta_1 L(u_1, \zeta F^0)dx + \\ + \int_\Omega \eta_1\{k, u_2\}dx - \lambda_0 \int_\Omega \eta_1\{k, (1-\zeta)F^0\}dx + \sigma \int_\Omega \eta_1 dx\Big],$$

$$(5.5)\qquad \int_\Omega \eta_2 \Delta^2 u_2 dx = \nu\Big[-\int_\Omega \eta_2 L(u_1, u_2)dx - 2\int_\Omega \eta_2\{k, u_1\}dx + \\ + \lambda_0 \int_\Omega \eta_2 \Delta^2[(1-\zeta)F^0]dx\Big].$$

Since η is of class $\mathscr{C}_0^\infty$, (5.4) and (5.5) can be reduced, integrating them by parts twice, to the form

$$(5.6)\qquad L_1(\eta_1, u_1) = \nu\big[L_2(\eta_1, u_1, u_2) + \lambda_0 L_2(\eta_1, u_1, \zeta F^0) + \\ + \sigma L_4(\eta_1, q) + L_3(\eta_1, k, u_2) - \lambda_0 L_3\big(\eta_1, k, (1-\zeta)F^0\big)\big],$$

$$(5.7)\qquad L_1(\eta_2, u_2) = \nu\big[-L_2(\eta_2, u_1, u_1) - 2L_3(\eta_2, k, u_1) + \\ + \lambda_0 L_1\big(\eta_2, (1-\zeta)F^0\big)\big],$$

where

$$L_1(f, g) \equiv \int_\Omega \Delta f \Delta g dx,$$

$$L_3(f, g, k) \equiv L_3(f, g, k_1, k_2) \equiv \int_\Omega (k_1 f_{x_1} g_{x_1} + k_2 f_{x_2} g_{x_2})dx,$$

$$L_4(f, g) \equiv \int_\Omega f g dx.$$

Definition. Let Ω be a bounded domain and let $\nu = 1$. Then the vector function $\boldsymbol{u} = (u_1, u_2)$ is called a *generalized solution* of the problem (5.1), (5.2) and (5.3) if the following conditions are fulfilled:

(a) $\boldsymbol{u} \in \boldsymbol{H}_{0,2}^{(2)}(\Omega)$;

(b) $\boldsymbol{u} = (u_1, u_2)$ satisfies the integral identities (5.6) and (5.7) for an arbitrary test function $\boldsymbol{\eta} = (\eta_1, \eta_2)$.

In connection with the generalized boundary-value problem we consider the behaviour of elements $u \in H_{0,2}^{(2)}$ on the boundary $\partial\Omega$. If the boundary $\partial\Omega$ is sufficiently smooth and $u \in H_{0,2}^{(2)}$, then u together with its first derivatives tends to zero in the $L^2(\partial\Omega)$-norm as we approach the boundary. To make this more precise, let $u \in H_2^{(2)}(\Omega)$. Then we can define for such u its trace $\mathrm{tr}(u)$ on the boundary $\partial\Omega$. For $u \in \mathscr{C}^2(\overline{\Omega})$, $\mathrm{tr}(u)$ is simply the restriction of u on $\partial\Omega$. In this case we have

$$(\mathrm{A})\qquad \Big(\int_{\partial\Omega} |\mathrm{tr}(u)|^2 dl\Big)^{1/2} \leqslant C\|u\|_{H_2^{(2)}(\Omega)},$$

where the constant C is independent of $u(x)$. Hence, $u \to \mathrm{tr}(u)$ is a bounded linear transformation from $\mathscr{C}^2$ into $L^2(\partial\Omega)$. Since $\mathscr{C}^2$ is dense in $H_2^{(2)}(\Omega)$, we can extend the transformation in a unique manner by continuity to the whole of $H_2^{(2)}(\Omega)$. This defines the trace on the boundary of a function $u \in H_2^{(2)}$ as an element of $L^2(\partial\Omega)$. Since $H_{0,2}^{(2)} \subset H_2^{(2)}$, inequality (A) holds true for any u in $H_{0,2}^{(2)}$. But we know that the test functions are dense in $H_{0,2}^{(2)}$, therefore, any element $u \in H_{0,2}^{(2)}$ can be approximated by functions $\varphi \in \mathscr{C}_0^\infty(\Omega)$. Let $\{\varphi_n\}$ be a sequence of test functions such that $\|\varphi_n - u\|_{H_{0,2}^{(2)}} \to 0$, $n \to \infty$. Then we have

$$\text{(B)} \qquad \Big(\int_{\partial\Omega} |\mathrm{tr}(u)|^2 dl\Big)^{1/2} \leqslant C\|\varphi_n - u\|_{H_{0,2}^{(2)}} \to 0.$$

Hence $\mathrm{tr}(u) = 0$ on $\partial\Omega$ as an element of $L^2(\partial\Omega)$. The argument holds true for first derivatives of u as well. Shortly speaking, u and ∇u belonging to $H_{0,2}^{(2)}$ vanish at the boundary $\partial\Omega$ in the trace sense, i. e., almost everywhere on $\partial\Omega$.

Definition. A vector function $\boldsymbol{u} = (u_1, u_2)$ is called a *classical solution* if

(a) $\boldsymbol{u} \in \mathscr{C}^{*4}(\Omega) \cap \mathscr{C}^{*1}(\bar{\Omega})$,

(b) $\boldsymbol{u}$ satisfies (5.1) and (5.2) for $\nu = 1$ pointwise in Ω,

(c) $\boldsymbol{u}$ and $\nabla\boldsymbol{u}$ vanish pointwise at the boundary $\partial\Omega$.

THEOREM 1. *Norms of all possible modified generalized solutions* $\boldsymbol{u}$ *are uniformly bounded, and the estimate*

$$\text{(5.8)} \qquad \|\boldsymbol{u}\|_{H_{0,2}^{(2)}} \leqslant \nu C(\Omega)(\lambda + \sigma)$$

holds true.

Proof. To prove (5.8), let us multiply (5.1) by u_2, (5.2) by u_1, integrate over Ω, use Lemma 4 and, finally, add them side by side. We obtain

$$\begin{aligned}\int_\Omega (|\Delta u_1|^2 + |\Delta u_2|^2)\,dx \leqslant \nu\Bigg\{&|\lambda_0|\bar{\varepsilon}C_0(\Omega)\int_\Omega (|\Delta u_1|^2 + |\Delta u_2|^2)\,dx + \\ &+ \sup_\Omega\left(\frac{|k_1|}{\mu_1}, \frac{|k_2|}{\mu_1}\right)\int_\Omega (|\Delta u_1|^2 + |\Delta u_2|^2)\,dx + \\ &+ C_1(\Omega)|\lambda_0|\left[\sup_\Omega\left(\frac{|k_1||\nabla(1-\zeta)F^0|}{\mu_1^{1/2}}, \frac{|k_2||\nabla(1-\zeta)F^0|}{\mu_1^{1/2}}\right) + \right. \\ &\left. + \sup_\Omega\left(|\Delta(1-\zeta)F^0|\sqrt{\mathrm{mes}\,\Omega}\right)\right]\Big(\int_\Omega (|\Delta u_1|^2 + |\Delta u_2|^2)\,dx\Big)^{1/2} + \\ &+ \sigma C_2(\Omega)\Big(\int_\Omega (|\Delta u_1|^2 + |\Delta u_2|^2)\,dx\Big)^{1/2}\Bigg\}.\end{aligned}$$

Letting $\bar{\varepsilon} = 1/4|\lambda_0|C_0(\Omega)$, $\sup\limits_{\Omega}(|k_1|/\mu_1, |k_2|/\mu_1) \leqslant 1/4$ and taking into account that $1/2 \leqslant 1-\nu/2$, we have

$$\Big(\int\limits_{\Omega} |\Delta \boldsymbol{u}|^2 dx\Big)^{1/2} \leqslant \nu\big(\lambda_0 \max(P) + \sigma\big)C(\Omega)$$

or

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^{(2)}_{0,2}} \leqslant \nu C(\Omega)(\lambda+\sigma).$$

It is well known from Sobolev's theorem [16] that the functions $\boldsymbol{u}$ in $\boldsymbol{H}^{(2)}_2$ are continuous functions of x if the dimension of the space Ω is not greater than 3; moreover, the functions $\boldsymbol{u}$ obey the inequality

$$|\boldsymbol{u}| \leqslant C(\Omega)\|\boldsymbol{u}\|_{\boldsymbol{H}^{(2)}_2}. \tag{5.9}$$

Hence

$$\sup_{\Omega}|\boldsymbol{u}(x)| \leqslant C(\Omega)(\lambda+\sigma).$$

Let us introduce the following inner product in the Hilbert space $\boldsymbol{H}^{(2)}_{0,2}$:

$$(\boldsymbol{u}, \boldsymbol{\eta}) \overset{\mathrm{df}}{=} \int\limits_{\Omega}(\Delta u_1 \Delta\eta_1 + \Delta u_2 \Delta\eta_2)\,dx. \tag{C}$$

One can easily check that definition (C) is correct; all exioms of an inner product are satisfied.

We shall now show that every integral appearing in equations (5.6) and (5.7) defines a continuous linear functional on the space $\boldsymbol{H}^{(2)}_{0,2}(\Omega)$. By the fact that second order generalized derivatives of $\boldsymbol{u}$ are in $\boldsymbol{L}^2(\Omega)$, and the same is true with the test functions $\boldsymbol{\eta}$, we obtain

$$\begin{aligned}
&\Big|\int\limits_{\Omega}[\eta_1(u_{1x_1x_1}u_{2x_2x_2}+u_{1x_2x_2}u_{2x_1x_1}-2u_{1x_1x_2}u_{2x_1x_2})]\,dx\Big| \\
&\qquad \leqslant \sup_{\Omega}|\eta_1|\int\limits_{\Omega}|u_{1x_1x_1}u_{2x_2x_2}+u_{1x_2x_2}u_{2x_1x_1}-2u_{1x_1x_2}u_{2x_1x_2}|\,dx \\
&\qquad \leqslant \sup_{\Omega}|\eta_1|\Big(\int\limits_{\Omega}(|u_{1x_1x_1}|^2+|u_{1x_2x_2}|^2+2|u_{1x_1x_2}|^2)\,dx\Big)^{1/2}\times \\
&\qquad\qquad \times\Big(\int\limits_{\Omega}(|u_{2x_1x_1}|^2+|u_{2x_2x_2}|^2+2|u_{2x_1x_2}|^2)\,dx\Big)^{1/2},
\end{aligned}$$

where we have used Hölder's inequality. Hence, it immediately follows the inequality

$$\Big|\int\limits_{\Omega}\eta_1 L(u_1, u_2)\,dx\Big| \leqslant \sup_{\Omega}|\eta_1|\,\|u_1\|_{H^{(2)}_{0,2}}\|u_2\|_{H^{(2)}_{0,2}}.$$

Making use of relation (5.9), we can write

$$\left|\int_\Omega \eta_1 L(u_1, u_2)\,dx\right| \leqslant C(\Omega)\,\|\eta_1\|_{H^{(2)}_{0,2}}\|u_1\|_{H^{(2)}_{0,2}}\|u_2\|_{H^{(2)}_{0,2}}, \tag{5.10}$$

where the constant C is not necessary the same in the foregoing inequalities. However, it is independent of the elements of the space $H^{(2)}_{0,2}(\Omega)$. Likewise we can show the following inequality:

$$\left|\int_\Omega \eta_2 L(u_1, u_1)\,dx\right| \leqslant C(\Omega)\,\|\eta_2\|_{H^{(2)}_{0,2}}\|u_1\|^2_{H^{(2)}_{0,2}}. \tag{5.11}$$

Combining inequalities (5.10) and (5.11) together, we obtain

$$\begin{aligned} |I_1| &\equiv \left|\int_\Omega \big(\eta_2 L(u_1, u_1) - \eta_1 L(u_1, u_2)\big)\,dx\right| \\ &\leqslant C(\Omega)\,\|\boldsymbol{\eta}\|_{\boldsymbol{H}^{(2)}_{0,2}}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^{(2)}_{0,2}} \end{aligned} \tag{5.12}$$

for arbitrary $\boldsymbol{\eta}$ and $\boldsymbol{u}$ in $\boldsymbol{H}^{(2)}_{0,2}(\Omega)$.

Inequality (5.12) shows that the integral

$$\int_\Omega \big(\eta_2 L(u_1, u_1) - \eta_1 L(u_1, u_2)\big)\,dx$$

is a continuous linear functional with respect to $\boldsymbol{\eta}$, defined on the Hilbert space $\boldsymbol{H}^{(2)}_{0,2}$, for fixed but otherwise arbitrary $\boldsymbol{u}$ in $\boldsymbol{H}^{(2)}_{0,2}(\Omega)$. By a well-known Riesz-Fischer Representation Theorem there exists an operator T_2: $\boldsymbol{H}^{(2)}_{0,2} \to \to H^{(2)}_{0,2}$, in general non-linear, such that

$$(T_2\boldsymbol{u}, \boldsymbol{\eta}) = -\int_\Omega \big(\eta_2 L(u_1, u_1) - \eta_1 L(u_1, u_2)\big)\,dx. \tag{5.13}$$

In a simple way one can estimate the integrals

$$I_2 \equiv L_2(\eta_1, u_1, \zeta F^0) + L_3(\eta_1, k, u_2) \quad \text{and} \quad I_3 \equiv -2L_3(\eta_2, k, u_1)$$

to obtain

$$|I_2| \leqslant C(\Omega)\,\|\boldsymbol{\eta}\|_{\boldsymbol{H}^{(2)}_{0,2}}\|\boldsymbol{u}\|_{\boldsymbol{H}^{(2)}_{0,2}} \quad \text{and} \quad |I_3| \leqslant C(\Omega)\,\|\boldsymbol{\eta}\|_{\boldsymbol{H}^{(2)}_{0,2}}\|\boldsymbol{u}\|_{\boldsymbol{H}^{(2)}_{0,2}},$$

where the constants C, in general distinct for each inequality, depend on the domain Ω, the function $F^0(x)$, the curvature k and the cut-off function but not on the elements $\boldsymbol{u}$ in $\boldsymbol{H}^{(2)}_{0,2}(\Omega)$. Thus we again obtained the fact that I_2 as well as I_3 are continuous linear functionals with respect to $\boldsymbol{\eta}$ in $\boldsymbol{H}^{(2)}_{0,2}(\Omega)$. Hence, there exists an operator T_1: $\boldsymbol{H}^{(2)}_{0,2} \to \boldsymbol{H}^{(2)}_{0,2}$ such that

$$(T_1\boldsymbol{u}, \boldsymbol{\eta}) = I_2 + I_3. \tag{5.14}$$

Finally, there exists an element $Q \in H_{0,2}^{(2)}(\Omega)$ such that

$$(Q, \eta) = L_1(\eta_2, (1-\zeta)F^0) - L_3(\eta_1, k, (1-\zeta)F^0) + L_4(\eta_1, q) \tag{5.15}$$

for $q \in L^2(\Omega)$ and an arbitrary η in $H_{0,2}^{(2)}(\Omega)$. However, there is no need to require that $q \in L^2(\Omega)$ as far as the existence of solutions is concerned. The element q need not be of class $L^2(\Omega)$ in order that $\int_\Omega q\eta_1 dx$ be a continuous linear functional. It can also be a so-called generalized function, e. g., Dirac's delta function concentrated on some smooth curve lying in Ω. For such q, the integral $\int_\Omega q\eta_1 dx$ actually defines a linear functional on $H_{0,2}^{(2)}(\Omega)$, due to inequality (A).

6. Compactness of operators T_1 and T_2 and equivalence of equations. We now prove that the operators T_1 and T_2 are completely continuous, and (5.1) and (5.2) are equivalent to one operator equation to be derived in this section.

LEMMA 6. *The operators T_1 and T_2 are completely continuous, i. e., compact on the Hilbert space $H_{0,2}^{(2)}(\Omega)$.*

Proof. Since they are continuous as proved before, it remains to show that they map every bounded set $\{u\} \subset H_{0,2}^{(2)}$ into a compact set. But, by a theorem of Rothe [14], every Sobolev space $H_p^{(l)}(\Omega)$ is reflexive, thus the closed ball $\|u\| \leqslant \gamma$, $\gamma > 0$, is weakly compact, and hence, by a theorem of Tsitlanadze [17], strong continuity implies complete continuity. Since the space $H_{0,2}^{(2)}(\Omega)$ is Hilbert, the converse is also true. It is, therefore, necessary and sufficient to prove that, for any given sequence $\{u_n\} \subset H_{0,2}^{(2)}(\Omega)$ convergent weakly to u_0, the sequence $T_i u_n$ also converges strongly to $T_i u_0$ for $i = 1, 2$ as $n \to \infty$.

Suppose $u_n \to u_0$ weakly in $H_{0,2}^{(2)}(\Omega)$. Thus $\{u_n\}$ is bounded. From (5.13) and (5.14) we have

$$|(T_i u_n, \eta)| \leqslant C \|\eta\|_{H_{0,2}^{(2)}} \|u_n\|^i_{H_{0,2}^{(2)}}, \quad i = 1, 2.$$

Hence, for $\eta = T_i u_n$, we obtain

$$|(T_i u_n, T_i u_n)| \leqslant C \|T_i u_n\|_{H_{0,2}^{(2)}} \|u_n\|^i_{H_{0,2}^{(2)}}, \quad i = 1, 2,$$

i. e.,

$$\|T_i u_n\|_{H_{0,2}^{(2)}} \leqslant C \|u_n\|^i_{H_{0,2}^{(2)}}, \quad i = 1, 2,$$

implying $\{T_i u_n\}$ is likewise bounded. Since the closed ball $\|u\|_{H_{0,2}^{(2)}} \leqslant \gamma$, $\gamma > 0$, is weakly compact in $H_{0,2}^{(2)}$, every bounded set is weakly compact. Hence $\{T_i u_n\}$ is weakly compact in $H_{0,2}^{(2)}(\Omega)$. By Sobolev's imbedding theorem [16], the imbedding of $H_{0,2}^{(2)}(\Omega)$ in $\mathscr{C}^* \overset{\mathrm{df}}{=} \{\mathscr{C}(\Omega)\}^2$ is completely continuous. Therefore, $\{T_i \boldsymbol{u}_n\}$ is compact in $\mathscr{C}^*(\Omega)$. Thus any sequence $\{\boldsymbol{u}_n\} \subset \boldsymbol{H}_{0,2}^{(2)}$ with uniformly bounded norms has a subsequence which is strongly convergent in $\mathscr{C}^*$. Now we show that the operators T_1 and T_2 are completely continuous. As it is easily seen the integral $\int_\Omega f L(g, h)$ is symmetric, linear and continuous in f, g, h if they are smooth. By continuity this assertion can be extended to the entire space $\boldsymbol{H}_{0,2}^{(2)}$. By definition,

$$(T_2 \boldsymbol{u}, \boldsymbol{\eta}) = -\int_\Omega [\eta_2 L(u_1, u_1) - \eta_1 L(u_1, u_2)]\,dx$$
$$= -\int_\Omega [u_1 L(\eta_2, u_1) - u_1 L(\eta_1, u_2)]\,dx.$$

Hence

$$\begin{aligned}
&(T_2 \boldsymbol{u}_n - T_2 \boldsymbol{u}_0, \boldsymbol{\eta}) \\
&= -\int_\Omega [\eta_2 L(u_{n_1}, u_{n_1}) - \eta_1 L(u_{n_1}, u_{n2}) - \eta_2 L(u_{0_1}, u_{0_1}) + \eta_1 L(u_{0_1}, u_{0_2})]\,dx \\
&= -L_2(\eta_2, u_{n_1}, u_{n_1}) + L_2(\eta_1, u_{n_1}, u_{n_2}) + L_2(\eta_2, u_{0_1}, u_{0_1}) - L_2(\eta_1, u_{0_1}, u_{0_2}) \\
&= -L_2(\eta_2, u_{n_1}, u_{n_1}) + L_2(\eta_1, u_{n_1}, u_{n_2}) + L_2(\eta_2, u_{n_1}, u_{n_2}) - L_2(\eta_2, u_{n_1}, u_{n_2}) + \\
&\quad + L_2(\eta_2, u_{0_1}, u_{0_1}) - L_2(\eta_1, u_{0_1}, u_{0_2}) - L_2(\eta_2, u_{0_1}, u_{0_2}) + L_2(\eta_2, u_{0_1}, u_{0_2}) \\
&= L_2(u_{n_1}, \eta_2 - \eta_1, u_{n_2}) + L_2(u_{n_2} - u_{n_1}, u_{n_1}, \eta_2) + \\
&\qquad + L_2(u_{0_1}, \eta_2 - \eta_1, u_{0_2}) + L_2(u_{0_1} - u_{0_2}, \eta_2, u_{0_1}).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&|(T_2 \boldsymbol{u}_n - T_2 \boldsymbol{u}_0, \boldsymbol{\eta})| \\
&\quad \leqslant C\{\|u_{n_1}\|_{\mathscr{C}} \|\eta_2 - \eta_1\|_{H_{0,2}^{(2)}} \|u_{n_2}\|_{H_{0,2}^{(2)}} + \|u_{n_2} - u_{n_1}\|_{\mathscr{C}} \|u_{n_1}\|_{H_{0,2}^{(2)}} \|\eta_2\|_{H_{0,2}^{(2)}} + \\
&\qquad + \|u_{0_1}\|_{\mathscr{C}} \|\eta_2 - \eta_1\|_{H_{0,2}^{(2)}} \|u_{0_2}\|_{H_{0,2}^{(2)}} + \|u_{0_1} - u_{0_2}\|_{\mathscr{C}} \|\eta_2\|_{H_{0,2}^{(2)}} \|u_{0_1}\|_{H_{0,2}^{(2)}}\} \\
&\quad \leqslant C \|\boldsymbol{\eta}\|_{\boldsymbol{H}_{0,2}^{(2)}} \|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathscr{C}^*} \|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\boldsymbol{H}_{0,2}^{(2)}} \quad \text{for some } C > 0.
\end{aligned}$$

Putting $\boldsymbol{\eta} = T_2 \boldsymbol{u}_n - T_2 \boldsymbol{u}_0$, we obtain

$$\|T_2 \boldsymbol{u}_n - T_2 \boldsymbol{u}_0\|_{\boldsymbol{H}_{0,2}^{(2)}} \leqslant C(\Omega) \|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathscr{C}(\Omega)} \to 0$$

as $n \to \infty$. Thus the sequence $\{T_2 \boldsymbol{u}_n\}$ is strongly convergent in $\boldsymbol{H}_{0,2}^{(2)}$, and hence T_2 is completely continuous on $\boldsymbol{H}_{0,2}^{(2)}(\Omega)$. It can be proved likewise that

$$\|T_1 \boldsymbol{u}_n - T_1 \boldsymbol{u}_0\|_{\boldsymbol{H}_{0,2}^{(2)}} \leqslant C(\Omega) \|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\mathscr{C}(\Omega)} \to 0$$

as $n \to \infty$. Thus we have proved that both operators T_1 and T_2 are completely continuous, and hence $T \equiv T_1 + T_2$ is also completely continuous. A characteristic feature of generalized solutions is the fact that they are often expressible as solutions of an operator equation in a suitable Hilbert space.

LEMMA 7. *Equations* (5.1), (5.2) *and* (5.3) *are equivalent to one operator equation of the form*

$$u - \nu T u = 0, \tag{6.1}$$

where $\boldsymbol{u} = (u_1, u_2)$ *is a generalized solution, and* T *is completely continuous on* $\boldsymbol{H}_{0,2}^{(2)}(\Omega)$.

The operator T *is, in general, non-linear.*

Proof. As we have already shown, all integrals appearing in the integral identities defining generalized solutions are continuous linear functionals and can be written in the form

$$(\boldsymbol{u}, \boldsymbol{\eta}) = \int_{\Omega} (\Delta u_1 \Delta \eta_1 + \Delta u_2 \Delta \eta_2) dx, \tag{6.2}$$

$$-\nu(T\boldsymbol{u}, \boldsymbol{\eta}) = I_1 + I_2 + I_3 \tag{6.3}$$

and

$$\nu(\boldsymbol{Q}, \boldsymbol{\eta}) = I_4. \tag{6.4}$$

Combining (6.2), (6.3) and (6.4) together, and taking into account the integral identities (5.6) and (5.7), we obtain

$$(\boldsymbol{u} - \nu T u + \nu Q, \boldsymbol{\eta}) = 0 \tag{6.5}$$

for any element $\boldsymbol{\eta} \in \mathscr{C}_0^\infty(\Omega)$. Since the set $\mathscr{C}_0^\infty$ is dense in $\boldsymbol{H}_{0,2}^{(2)}$, the existence of generalized solutions $\boldsymbol{u}(x)$ is reduced to the problem of existence of solutions of the operator equation

$$\boldsymbol{u} - \nu(T\boldsymbol{u} - \boldsymbol{Q}) = 0, \quad 0 \leqslant \nu \leqslant 1, \tag{6.6}$$

in the space $\boldsymbol{H}_{0,2}^{(2)}(\Omega)$.

The operator T is completely continuous on $\boldsymbol{H}_{0,2}^{(2)}$, therefore, the operator $T_Q \overset{\text{df}}{=} T + \boldsymbol{Q}I$, where I is the identity operator, which assigns the function $T\boldsymbol{u} + \boldsymbol{Q}$ to each element $\boldsymbol{u}$, is also completely continuous. Equation (6.6) assumes the form

$$\boldsymbol{u} - \nu T_Q \boldsymbol{u} = 0, \quad 0 \leqslant \nu \leqslant 1. \tag{6.7}$$

To investigate the solvability of equation (6.7), we can apply a simplified version of the fixed-point theorem of continuous mappings given by Schaefer (cf. [15] and [4]). That theorem guaranties the existence of at least one solution of equation (6.7) if the operator T_Q is completely continuous and all norms of possible solutions $\boldsymbol{u}$ of (6.7) are bounded. But these properties of T_Q and $\boldsymbol{u}$ have been already proved.

7. Uniqueness of solution. The Schaefer theorem is particularly remarkable in that it can even be used to investigate problems for whose solutions there is no uniqueness theorem.

In our problem we can investigate the finer question of how many solutions can exist globally but only for small data.

THEOREM 2. *If λ, σ and Ω are such that*

$$C(\Omega)(\lambda+\sigma) < 3/8, \tag{7.1}$$

then there exists exactly one generalized solution to the problem (5.1), (5.2) *and* (5.3).

Proof. Suppose that, on the contrary, there were two generalized solutions $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$. Then the difference $\boldsymbol{w} = \boldsymbol{u}-\boldsymbol{v}$ would belong to the space $\boldsymbol{H}_{0,2}^{(2)}$ and $u_1 = v_1+w_1$, $u_2 = v_2+w_2$ would satisfy the identities

$$\Delta^2(v_1+w_1) = L(v_1+w_1, v_2+w_2)+\lambda L(v_1+w_1, F^0)+\{k, v_2+w_2\}+q, \tag{7.2}$$

$$\Delta^2(v_2+w_2) = -L(v_1+w_1, v_1+w_1)-2\{k, v_1+w_1\}. \tag{7.3}$$

Since

$$L(f+g, k+h) = L(f, k)+L(f, h)+L(g, k)+L(g, h),$$

so

$$\Delta^2 w_1 = L(w_1, w_2)+L(v_1, w_2)+L(v_2, w_1)+\{k, w_2\}, \tag{7.4}$$

$$\Delta^2 w_2 = -L(w_1, w_1)-2L(w_1, v_1)-2\{k, w_1\}. \tag{7.5}$$

Multiplying identity (7.4) by w_1, and (7.5) by w_2, adding side by side and integrating over Ω, we obtain

$$\int_\Omega (|\Delta w_1|^2+|\Delta w_2|^2)\,dx$$
$$= \int_\Omega v_2 L(w_1, w_1) - \int_\Omega v_1 L(w_1, w_2)\,dx - \int_\Omega -w_2\{k, w_1\}\,dx,$$

$$\Big|\int_\Omega (|\Delta w_1|^2+|\Delta w_2|^2)\,dx\Big| \tag{7.6}$$
$$\leqslant \sup_\Omega|v_2|\int_\Omega |L(w_1, w_1)|\,dx+\sup_\Omega|v_1|\int_\Omega |L(w_1, w_2)|\,dx+\int_\Omega |w_2|\,|\{k, w_1\}|\,dx$$
$$\leqslant (\sup_\Omega|v_2|+\sup_\Omega|v_1|+\sup_\Omega|k|)\int_\Omega (|\Delta w_1|^2+|\Delta w_2|^2)\,dx$$

since

$$\int_{\Omega} |L(w_1, w_2)|\, dx \leqslant \Big(\int_{\Omega} |\Delta w_1|^2 dx\Big)^{1/2} \Big(\int_{\Omega} |\Delta w_2|^2 dx\Big)^{1/2}$$

and

$$\int_{\Omega} |w_2\{k, w_1\}|\, dx \leqslant \frac{1}{4} \int_{\Omega} (|\Delta w_1|^2 + |\Delta w_2|^2)\, dx.$$

If λ, σ and Ω are such that condition (7.1) is fulfilled, then (7.6) implies that $\boldsymbol{w}$ vanishes, i. e., $\boldsymbol{u}$ and $\boldsymbol{v}$ coincide. Thus, for small data, the existing solution is unique in Ω.

Let us emphasize once more that the operator equation (6.7) has been derived for q in $L^2(\Omega)$. And the uniqueness theorem holds for such q. However, as we already mentioned before, the problem of existence can be solved for other q as well. An answer to this is given in the theorem that follows:

THEOREM 3. *If q is such that the integral $\int_{\Omega} q\eta_1 dx$ defines a linear functional, then the problem* (5.1), (5.2) *and* (5.3) *has at least one generalized solution.*

For such arbitrary q, in general, there does not exist a classical solution. However, generalized solutions are often more smooth than it is assumed in their definition. In the problem considered here one can show that the generalized solutions $\boldsymbol{u}(x)$ actually are of the class $H_2^{(4)}(\Omega)$. The result is based on the theory developed for $L^p(\Omega)$-functions (see [1]-[3]). The application of the $L^p(\Omega)$ approach to a non-linear problem in elasticity is given in [12].

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INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI PAN
ZAKŁAD TEORII OŚRODKÓW CIĄGŁYCH
WARSZAWA

Received on 16. 12. 1971

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O ISTNIENIU ROZWIĄZAŃ DLA POWŁOK O MAŁEJ WYNIOSŁOŚCI Z UWZGLĘDNIENIEM NAPRĘŻEŃ POCZĄTKOWYCH

STRESZCZENIE

W pracy zajęto się konkretnym zagadnieniem, występującym w teorii cienkich powłok sprężystych o małej wyniosłości, z uwzględnieniem naprężeń początkowych i brzegowych. Matematycznie jest to zagadnienie brzegowe dla układu dwóch równań różniczkowych cząstkowych nieliniowych rzędu czwartego z niejednorodnymi warunkami brzegowymi. Praca dotyczy przede wszystkim zagadnienia istnienia i jednoznaczności rozwiązań uogólnionych, w zależności od gładkości danych początkowych.

Wyjściowe zagadnienie sprowadza się do problemu uogólnionego, a następnie zanurza w jednoparametrowej rodzinie podobnych zagadnień brzegowych, opisanych nieliniowym równaniem operatorowym postaci (2.3), określonym na odpowiedniej przestrzeni Hilberta. Korzysta się z twierdzenia Schaefera o stałym punkcie przekształceń ciągłych i pokazuje, że równanie (2.3) ma co najmniej jedno rozwiązanie o danej przestrzeni Hilberta. Z równoważności równania (2.3) i układu równań uogólnionych wynika, że istnieje rozwiązanie uogólnione. Pokazano, że dla małych danych początkowych rozwiązanie to jest jedyne.