Quasiconformal deformation and cross-ratios

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Harmonic Analysis, Complex Analysis, Spectral Theory and all that
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Quasiconformal mappings

Definition: A mapping $f : \Omega \to \Omega'$ between open sets Ω, Ω' in $\mathbb{C}$ is called *quasiconformal* if

(i) f is an orientation-preserving homeomorphism;

(ii) f_x, f_y (in the sense of distribution) $\in L^1_{loc}(\mathbb{C})$;

(iii) Denote $\mu_f(z) = \dfrac{f_{\overline{z}}(z)}{f_z(z)}$ and $D_f(z) = \dfrac{1+|\mu_f(z)|}{1-|\mu_f(z)|}$, then

$$||\mu_f||_\infty < 1 \quad (\text{or equivalently } K(f) = \operatorname{ess\,sup} D_f(z) < \infty).$$

Fact: A quasiconfomal mapping is almost everywhere differentiable (w.r.t. Lebesgue meas.), but not necessarily everywhere differentiable.

Problem: Given a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ and a point z_0, determine whether f is differentiable or conformal at $z = z_0$ in terms of $\mu_f(z)$.

f is *conformal at* $z = z_0 \iff f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists and $\neq 0$.

Only consider the case $z_0 = 0$.

Confomality

Theorem 1.[Gutlyanskiĭ-Martio] Let $f : \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping. If

$$\iint_{|z|<1} \frac{|\mu_f(z)|^2}{1-|\mu_f(z)|^2} \frac{dx\,dy}{|z|^2} < \infty \tag{1}$$

and

$$\text{the limit } \lim_{r \searrow 0} \iint_{r<|z|<1} \frac{\mu_f(z)}{1-|\mu_f(z)|^2} \frac{dx\,dy}{z^2} \text{ exists,} \tag{2}$$

then f is conformal at $z = 0$.

Earlier results: Teichmüller, Wittich, Belinskiĭ, Lehto, assuming

$$\iint_{0<|z|<1} \frac{|\mu_f(z)|}{1-|\mu_f(z)|^2} \frac{dx\,dy}{|z|^2} < \infty \tag{3}$$

These theorems were proved in two steps: the differentiability of the absolute value $|f(z)|$, then the estimate the variation of $\arg \frac{f(z)}{z}$. (See also Brakalova)

Our approach unifies the two estimates in a single step.

Theorems

Definitions: For $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ with $z_1 \neq z_2$, let

$$\phi_{z_1,z_2}(z) = \frac{z_1}{z(z-z_1)(z-z_2)}$$

which is integrable over $\mathbb{C}$. For a measurable $\mu : \mathbb{C} \to \mathbb{C}$ with $||\mu(z)||_\infty < 1$,

$$J(\mu; z_1, z_2) = 2\left|\iint_{\mathbb{C}} \frac{\mu(z)\phi_{z_1,z_2}(z)}{1-|\mu(z)|^2}\, dx\, dy\right| + 2\iint_{\mathbb{C}} \frac{|\mu(z)|^2|\phi_{z_1,z_2}(z)|}{1-|\mu(z)|^2}\, dx\, dy.$$

Theorem 2. Let $f : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping and suppose that there exists $0 < \delta < 1$ such that

$$J(\mu_f; z_1, z_2) \to 0 \quad \text{when } z_1 \text{ and } z_2 \text{ tend to } 0 \text{ satisfying } 0 < |z_2| \le \delta |z_1|.$$

Then f is conformal at $z = 0$. Moreover there exists a constant $C > 0$ depending only on K such that

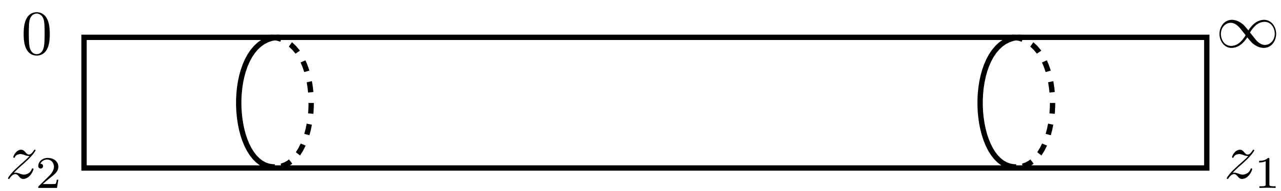
$$\left|\log \frac{f(z)}{z} - \log f'(0)\right|_{\mathrm{mod}\, 2\pi i\mathbb{Z}} \le C \liminf_{z_2 \to 0} J(\mu_f; z, z_2).$$

where $|w|_{\mathrm{mod}\, 2\pi i\mathbb{Z}} = \inf\{|w + 2\pi i n| : n \in \mathbb{Z}\}$.

Comparing two conditions $\quad 0 < |z_2| \ll |z_1| \ll 1$

Thm 1 $\quad \text{p.v.} \iint \frac{\mu(z)}{1-|\mu(z)|^2} \frac{dx\,dy}{z^2}$ exists $\qquad \mathbb{C}^* \simeq \mathbb{C}/2\pi i\mathbb{Z}$

Thm 2 $\quad \iint \frac{\mu(z)\phi_{z_1,z_2}}{1-|\mu(z)|^2}\, dx\,dy \to 0 \ (z_1, z_2 \to 0) \qquad \widehat{\mathbb{C}} \setminus \{0, z_1, z_2, \infty\}$



$$\phi_{z_1,z_2} = \frac{z_1}{z(z-z_1)(z-z_2)} = -\frac{1}{z^2(1-\frac{z}{z_1})(1-\frac{z_2}{z})} \sim -\frac{1}{z^2} \text{ when } |z_2| \ll |z| \ll |z_1|$$

Assumption of Thm 1 implies Assumption of Thm 2

In fact, with the stronger condition $\iint \frac{|\mu(z)|}{1-|\mu(z)|^2} \frac{dx\,dy}{|z|^2} < \infty$, this is immediate. But we need more care when the absolute value is not present.

Cauchy's criterion and cross-ratio

Let $f : \mathbb{C} \to \mathbb{C}$ is an ori-preserving homeo with $f(0) = 0$. $0 < \delta_1 \le 1$.

f is conformal at $z = 0 \iff$

$\forall \varepsilon > 0 \;\; \exists r > 0$ such that if $0 < |z_1| < r$ and $0 < |z_2| \le \delta_1 |z_1|$, then

$$\left| \log \frac{f(z_1)}{z_1} - \log \frac{f(z_2)}{z_2} \right|_{\mathrm{mod}\, 2\pi i\mathbb{Z}} < \varepsilon.$$

Let $\mathrm{Cr}(z_1, z_2, z_3, z_4) = \frac{z_2 - z_3}{z_1 - z_3} \cdot \frac{z_1 - z_4}{z_2 - z_4}$. Then the left hand side is:

$$\left| \log \tfrac{f(z_2)}{f(z_1)} - \log \tfrac{z_2}{z_1} \right|_{\mathrm{mod}\, 2\pi i\mathbb{Z}} = |\log \underbrace{\mathrm{Cr}(f(z_1), f(z_2), 0, \infty)}_{\zeta_2} - \log \underbrace{\mathrm{Cr}(z_1, z_2, 0, \infty)}_{\zeta_1}|_{\mathrm{mod}\, 2\pi i\mathbb{Z}}.$$

Let $\Omega = \mathbb{C} \smallsetminus \{0, 1\}$. Its Poincaé metric $\sim \frac{|dz|}{|z| \log \frac{1}{|z|}}$ near 0.

For any $L > 0$, there exist constants $C_1 > 0$ and $0 < \delta_1 < 1$ such that if $\zeta_1, \zeta_2 \in \Omega = \mathbb{C} \smallsetminus \{0, 1\}$ satisfy $|\zeta_1| < \delta_1$ and $d_\Omega(\zeta_1, \zeta_2) \le L$, then

$$|\log \zeta_1 - \log \zeta_2|_{\mathrm{mod}\, 2\pi i\mathbb{Z}} \le C_1 d_\Omega(\zeta_1, \zeta_2) \cdot \log \frac{1}{|\zeta_1|}.$$

Need to consider the variation of cross-ratio under a qc-mapping

Grötzsch-type inequality for cross-ration variation

Lemma. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal mapping and z_1, z_2, z_3, z_4 distinct points in $\widehat{\mathbb{C}}$, and put $z'_j = f(z_j)$ $(j = 1, 2, 3, 4)$. Then

$$d_\Omega(\mathrm{Cr}(z_1, z_2, z_3, z_4), \mathrm{Cr}(z'_1, z'_2, z'_3, z'_4)) \leq \log \overline{K}_f(z_1, z_2, z_3, z_4),$$

where

$$\overline{K}_f(z_1, z_2, z_3, z_4) = \frac{\displaystyle\sup_{\theta \in \mathbb{R}} \iint_{\mathbb{C}} \frac{\left|1 + e^{i\theta}\mu(z)\frac{\phi(z)}{|\phi(z)|}\right|^2}{1 - |\mu(z)|^2} |\phi(z)| dx\, dy}{\displaystyle\iint_{\mathbb{C}} |\phi(z)| dx\, dy}$$

with $\mu = \mu_f$ and $\phi(z) = \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$ (omit $(z-z_j)$ if $z_j = \infty$).

Moreover

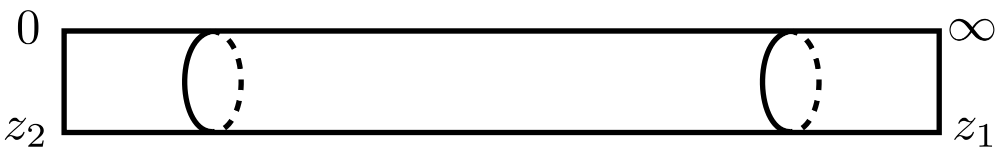
$$\overline{K}_f(z_1, z_2, 0, \infty) = 1 + \frac{J(\mu_f; z_1, z_2)}{J_*(z_1, z_2)},$$

where $J_*(z_1, z_2) = \displaystyle\iint_{\mathbb{C}} |\phi_{z_1, z_2}(z)| dx\, dy$ and recall that

$$J(\mu; z_1, z_2) = 2\left|\iint_{\mathbb{C}} \frac{\mu(z)\phi_{z_1,z_2}(z)}{1 - |\mu(z)|^2} dx\, dy\right| + 2\iint_{\mathbb{C}} \frac{|\mu(z)|^2 |\phi_{z_1,z_2}(z)|}{1 - |\mu(z)|^2} dx\, dy$$

with $\phi_{z_1,z_2}(z) = \frac{z_1}{z(z-z_1)(z-z_2)}$.

Proof of Theorem 2



$$\phi_{z_1,z_2} \sim -\frac{1}{z^2} \ (|z_2| \ll |z| \ll |z_1|)$$

$$J_*(z_1,z_2) = \iint_{\mathbb{C}} |\phi_{z_1,z_2}(z)|dx\,dy \sim 2\pi \log \frac{|z_1|}{|z_2|}$$

For $\zeta_1 = \mathrm{Cr}(z_1, z_2, 0, \infty)$, $\zeta_2 = \mathrm{Cr}(f(z_1), f(z_2), 0, \infty)$

$$\left| \log \frac{f(z_1)}{z_1} - \log \frac{f(z_2)}{z_2} \right|_{\mathrm{mod}\, 2\pi i \mathbb{Z}} = |\log \zeta_1 - \log \zeta_2|_{\mathrm{mod}\, 2\pi i \mathbb{Z}}$$

$$\le C_1 \left(\log \frac{1}{|\zeta_1|} \right) d_\Omega(\zeta_1, \zeta_2)$$

$$\le C_1 \left(\log \frac{|z_1|}{|z_2|} \right) \log \overline{K}_f = C_1 \left(\log \frac{|z_1|}{|z_2|} \right) \log \left(1 + \frac{J(\mu_f; z_1, z_2)}{J_*(z_1, z_2)} \right)$$

$$\le C_1 \left(\log \frac{|z_1|}{|z_2|} \right) \frac{J(\mu_f; z_1, z_2)}{J_*(z_1, z_2)} \le C J(\mu_f; z_1, z_2).$$

About Grötzsch-type inequality

For distinct points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$, $\exists \tau \in \mathbb{H}$ and $\exists p_\tau : \mathbb{E}_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau \to \widehat{\mathbb{C}}$, which is a holomorphic branched double cover, branched over z_1, z_2, z_3, z_4.

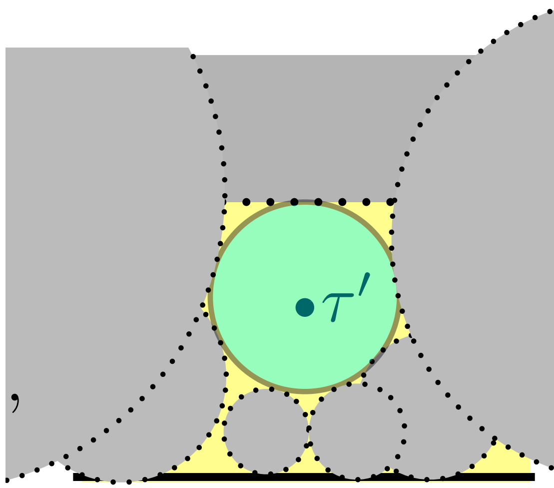
The correspondence $\mathbb{H} \ni \tau \mapsto \mathrm{Cr}(z_1, z_2, z_3, z_4) \in \Omega = \mathbb{C} \setminus \{0, 1\}$ is a universal covering map. (elliptic modular function)

Given a qc $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $z'_j = f(z_j)$, f can be lifted to $\tilde{f} : \mathbb{E}_\tau \to \mathbb{E}_{\tau'}$.

Torus version of Grötzsch-type ineq.

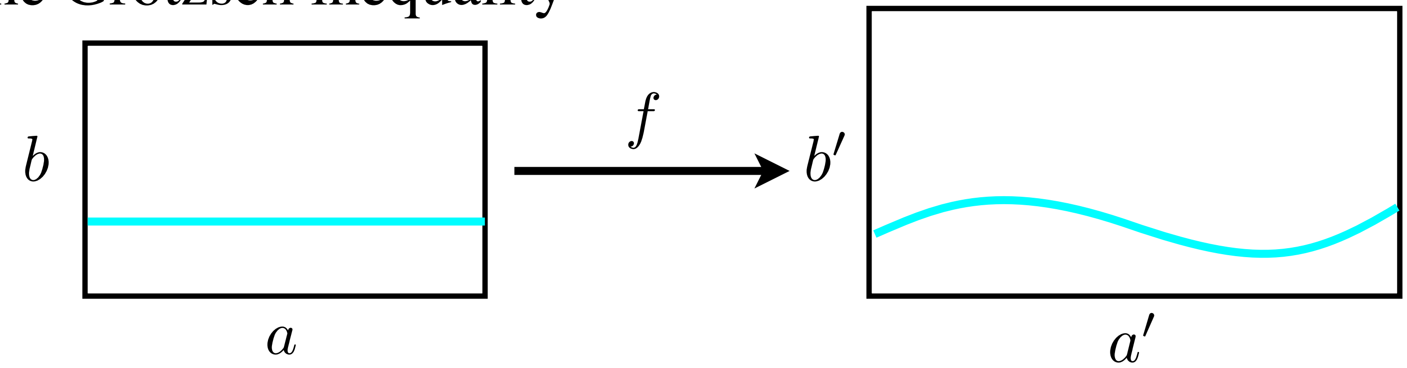
$$d_{\mathbb{H}}(\tau, \tau') \leq \log \overline{K}_{\tilde{f}} = \log \frac{\sup_{\theta \in \mathbb{R}} \iint_{\mathbb{E}_\tau} \frac{|1+e^{i\theta}\mu_{\tilde{f}}(z)|^2}{1-|\mu_{\tilde{f}}(z)|^2} dx\, dy}{\iint_{\mathbb{E}_\tau} dx\, dy}.$$

This comes from
$$\mathrm{Im}\tau \leq \mathrm{Im}\tau' \frac{\iint_{\mathbb{E}_\tau} \frac{|1+\mu|^2}{1-|\mu|^2} dxdy}{\iint_{\mathbb{E}_\tau} dxdy},$$



and the change of basis of $\mathbb{Z} + \mathbb{Z}\tau$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, which has an effect $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ and $\mu \rightsquigarrow e^{i\theta}\mu$.

The Grötzsch inequality



$$a' \le \int_0^a |f_x| dx = \int_0^a |f_z + f_{\bar z}| dx = \int_0^a |1+\mu||f_z| dx$$

$$\begin{aligned}(a'b)^2 = \left(\int_0^b a' dy\right)^2 &\le \left(\iint_R |1+\mu||f_z| dxdy\right)^2 \\ &\le \iint_R \underbrace{(1-|\mu|^2)|f_z|^2}_{\text{Jacobian } f} dxdy \iint_R \frac{|1+\mu|^2}{1-|\mu|^2} dxdy \\ &= a'b' \iint_R \frac{|1+\mu|^2}{1-|\mu|^2} dxdy\end{aligned}$$

Hence $\quad \dfrac{b}{a} \le \dfrac{b'}{a'} \dfrac{\iint_R \frac{|1+\mu|^2}{1-|\mu|^2} dxdy}{\iint_R dxdy} \quad \left(\le \dfrac{b'}{a'} \dfrac{\iint_R \frac{1+|\mu|}{1-|\mu|} dxdy}{\iint_R dxdy} \le \dfrac{b'}{a'} K(f)\right).$

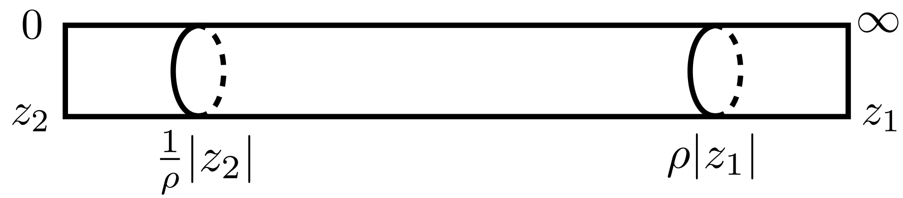
For $f : \mathbb{E}_\tau \to \mathbb{E}_{\tau'}$, $\mathrm{Im}\tau \le \mathrm{Im}\tau' \dfrac{\iint_{\mathbb{E}_\tau} \frac{|1+\mu|^2}{1-|\mu|^2} dxdy}{\iint_{\mathbb{E}_\tau} dxdy}.$

About the comparison of the integrals

$$\text{p.v.}\iint \frac{\mu(z)}{1-|\mu(z)|^2}\,\frac{dx\,dy}{z^2}$$

$$\iint \frac{\mu(z)\phi_{z_1,z_2}}{1-|\mu(z)|^2}\,dx\,dy$$

$$\phi_{z_1,z_2} = \frac{z_1}{z(z-z_1)(z-z_2)}$$



Let $p > 2$ and $p > s > 0$. For $\mu \in L^\infty(\mathbb{C})$ with $||\mu||_\infty < 1$, define

$$I_{p,s}(\mu; r) = \iint_{\mathbb{C}} \frac{|\mu(z)|^p}{(1-|\mu(z)|^2)^p}\,\frac{dx\,dy}{|z|^2\left(1+\frac{|z|}{r}\right)^s} \to 0 \ (r \to 0).$$

$$\left|\iint_{\{|z|<\frac{1}{\rho}|z_2|\}} \frac{\mu(z)\phi(z)}{1-|\mu(z)|^2}dx\,dy\right| \le \iint_{\{|z|<\frac{1}{\rho}|z_2|\}} \left|\frac{\mu(z)}{(1-|\mu(z)|^2)|z|^{\frac{2}{p}}\left(1+\frac{|z|}{r}\right)^{\frac{s}{p}}}\right| \cdot \left||z|^{\frac{2}{p}}\left(1+\frac{|z|}{r}\right)^{\frac{s}{p}}\phi(z)\right| dx\,dy$$

$$\le I_{p,s}(\mu; r)^{\frac{1}{p}} \left(\iint_{\{|z|<\frac{1}{\rho}|z_2|\}} |z|^{2q-2}\left(1+\frac{|z|}{r}\right)^{s(q-1)} |\phi(z)|^q dx\,dy\right)^{\frac{1}{q}}$$

$$\le I_{p,s}(\mu; r)^{\frac{1}{p}} \left(\frac{(1+\rho)^{s(q-1)}}{(1-\rho)^q} \iint_{\{|\zeta|<\frac{1}{\rho}\}} \frac{|\zeta|^{q-2}}{|\zeta-1|^q} d\xi\,d\eta\right)^{\frac{1}{q}}.$$

etc

$C^{1+\alpha}$-conformality

Similar estimates can give:

Theorem 3. Let $f : \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping and suppose that for some $\alpha > 0$,

$$I(r) = \iint_{\{z:|z|<r\}} \frac{|\mu(z)|}{1 - |\mu(z)|^2} \frac{dx\,dy}{|z|^2} \text{ is finite and has order } O(r^\alpha)\ (r \searrow 0).$$

Then for any $0 < \beta < \frac{\alpha}{2+\alpha}$, f is $C^{1+\beta}$-conformal at 0, i.e.

$$f(z) = f(0) + f'(0)z + \varepsilon_f(z) \quad \text{with } \varepsilon_f(z) = O(|z|^{1+\beta}) \text{ as } z \to 0.$$

Known results:

Lehto-Virtanen: $\varepsilon_f(r) \leq O\left(\frac{1}{\log \frac{1}{r}}\right)^{1/2(K+2)}$

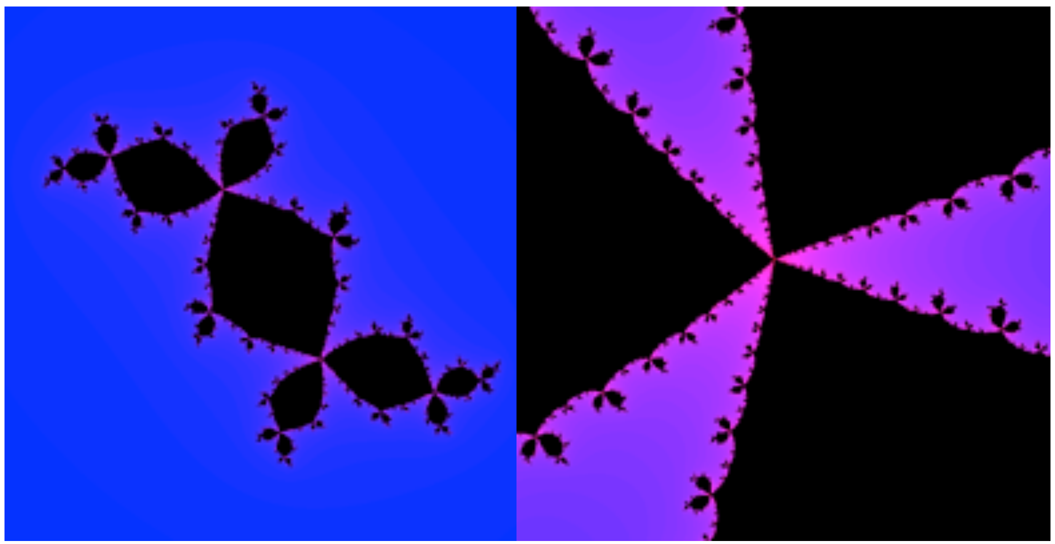
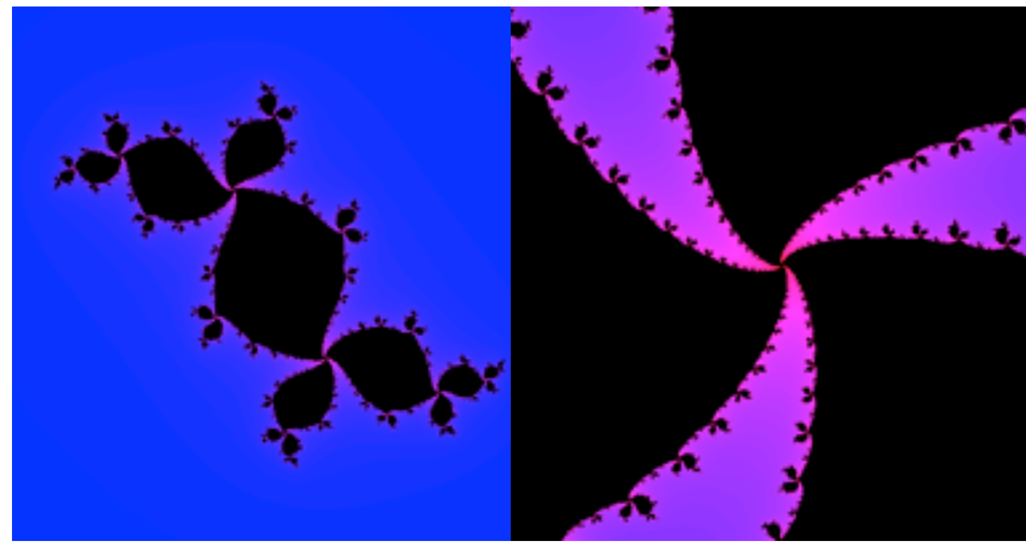
Schatz assuming $\iint \frac{|\mu(z)|}{|z|^p} dx\,dy < \infty$ $(p > 2)$

McMullen assuming $\mathrm{Area}(B_r(0) \cap \mathrm{supp}\mu) = O(r^{2+\alpha})$

When $C^{1+\alpha}$-conformality is applied

Complex dynamics: iteration of rational functions $f(z)$ on $\widehat{\mathbb{C}}$
Chaotic set = Julia set J_f, usually fractals.

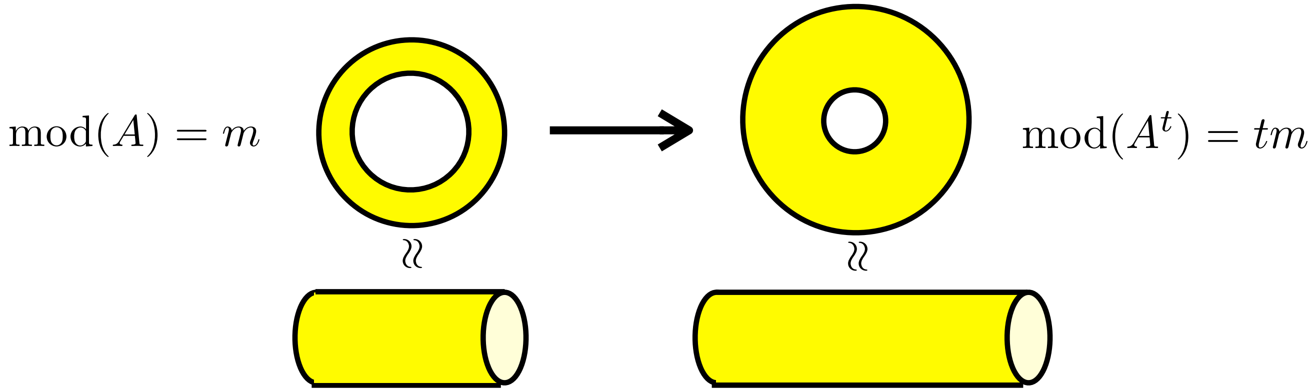
Conjugacies between such maps cannot be smooth in general, otherwise the multipliers of periodic points must coincide. Quasiconformal maps is the "right class" for the conjugacies between rational maps.

However there are cases where the combinatorics determines the "geometric structure of the Julia set, in the sense the conjugacy can be $C^{1+\alpha}$-conformal at certain point (e.g. critical points).

Such a phenomenon is explained using the theory of "Renormalizations". Feigenbaum-Tresser maps, Siegel disks, Hedgehogs for Cremer points.

Question about stretching deformation of cross-ratio

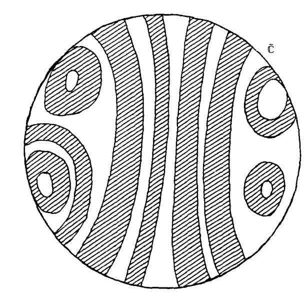


Problem: Suppose A an annulus and $\mathcal{A}$ is a collection of disjoint sub-annuli of A. For $A_i \in \mathcal{A}$, let A_i^t denote t-stretching of A_i. Apply t-stretching to all $A_i \subset A$ and obtain a new (measurable) conformal structure for A. Let A^t be A with this conformal structure. Show

$$mod(A^t) \sim \sum_{\substack{A_i \in \mathcal{A} \\ A_i \text{ is essential in A}}} t\ mod(A_i) \quad \text{as } t \to \infty.$$

If one can show this, it is expected that for four points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$, if there is a maximal segment J separating z_1, z_2 from z_3, z_4, then

$$\frac{1}{2\pi} \log CR(z_1^t, z_2^t, z_3^t, z_4^t) \sim t \text{ length}(J) \text{ as } t \to \infty,$$



where CR denotes a certain cross ratio.

Note: The above claim is true when $\mathcal{A}$ is finite.

This question is related to the boundary of the moduli space of rational maps

Thank you!