# INTRODUCTION TO DRINFELD'S ASSOCIATOR THEORY

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Abstract. The theory of associator was introduced in a very fundamental paper of Vladimir Drinfeld, by "universalizing" a construction coming from the monodromy of some differential equations. Drinfeld also proves the existence of an associator over the rationals using a very interesting group which has many connections with the *Esquisse d'un programme* of Grothendieck.

Main references are the article of Drinfel'd [Dri] and the book of Etingof-Schiffmann [ES]. See also [Bar] for a more categorical approach and [Sch] for the relationship between $GT$ and $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$.

## 1. Braids groups

Let $B_n$ be the group generated by $\{\sigma_i,\ 1 \leq i \leq n-1\}$ and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

There is a canonical morphism $B_n \to S_n$ induced by

$$\sigma_i \longmapsto (i, i+1)$$

The kernel of this morphism is the pure braid group $PB_n$. It is generated by $\{x_{ij},\ 1 \leq i < j \leq n\}$ where

$$x_{ij} = (\sigma_{j-1}\sigma_{j-2}\dots\sigma_{i+1})\sigma_i^2(\sigma_{j-1}\sigma_{j-2}\dots\sigma_{i+1})^{-1}$$

and relations

$$(a_{ijk}, x_{ij}) = (a_{ijk}, x_{ik}) = (a_{ijk}, x_{jk}) = 1 \qquad \text{where } a_{ijk} = x_{ij}x_{ik}x_{jk} \tag{1.1a}$$
$$(x_{ij}, x_{kl}) = (x_{il}, x_{jk}) = 1 \qquad \text{for } i < j < k < l \tag{1.1b}$$
$$(x_{ik}, x_{ij}^{-1}x_{jl}x_{ij}) = 1 \qquad \text{for } i < j < k < l \tag{1.1c}$$

**Remark 1.1.** $B_2$ is obviously the free group with one generator (i.e. $\mathbb{Z}$). Thus, every element of $PB_2$ is of the form $\sigma_1^{2m}$.

**Remark 1.2.** Every element of $PB_3$ can be written as $f(\sigma_1^2, \sigma_2^2)(\sigma_1\sigma_2)^{3n}$ where $f(X, Y)$ is an element of the free group with generators $X, Y$.

## 2. Braided monoidal category

A braided monoidal category is the categorical analog of an abelian monoid, where commutativity and associativity holds only up to isomorphism. More precisely, a category $\mathcal{C}$ is a *braided monoidal category* if there exists:

- A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
- A natural isomorphism $\alpha : (-\otimes-)\otimes- \longrightarrow -\otimes(-\otimes-)$ (the associativity constraint)
- A natural isomorphism $\beta : -\otimes- \longrightarrow -\otimes^{op}-$ (the commutativity constraint)

such that the following diagrams commute for all $A, B, C \in Obj(\mathcal{C})$ (Mac Lane's coherence condition):

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha\otimes 1"] \arrow[d, "\alpha_{A\otimes B,C,D}"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A,B\otimes C,D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "1\otimes\alpha"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C\otimes D}"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[r, "\beta_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha"] \arrow[dr, "\beta\otimes 1"'] & & & B \otimes (C \otimes A) \\
& (B \otimes A) \otimes C \arrow[r, "\alpha"'] & B \otimes (A \otimes C) \arrow[ur, "1\otimes\beta"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (A \otimes B) \otimes C \arrow[r, "\beta_{A\otimes B,C}"] & C \otimes (A \otimes B) \arrow[dr, "\alpha^{-1}"] & \\
A \otimes (B \otimes C) \arrow[ur, "\alpha^{-1}"] \arrow[dr, "1\otimes\beta"'] & & & (C \otimes A) \otimes B \\
& A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}"'] & (A \otimes C) \otimes B \arrow[ur, "\beta\otimes 1"'] &
\end{tikzcd}$$

**Remark 2.1.** Any braided monoidal category carries representation of the braid group as follow: let $V_1, \ldots, V_n$ be $n$ copies of the same $V \in Obj(\mathcal{C})$ and set

$$V^{\otimes n} = (\ldots((V_1 \otimes V_2) \otimes V_3) \ldots \otimes V_n$$

There is a morphism $B_n \longrightarrow Aut(V^{\otimes n})$ defined by:

$$\begin{aligned}
\sigma_1 &\longmapsto \beta_{V_1,V_2} \\
\sigma_2 &\longmapsto \alpha^{-1}_{V_1,V_3,V_2}\beta_{V_2,V_3}\alpha_{V_1,V_2,V_3} \\
\vdots \quad & \quad \vdots \qquad\qquad \vdots
\end{aligned}$$

**Remark 2.2.** If the commutativity constraint is involutive i.e. if $\beta_{U,V} = \beta^{-1}_{V,U}$, then the category is said to be *symmetric*. In this case, the preceding representation of $B_n$ factor through a representation of $S_n$.

## 3. PRO-UNIPOTENT COMPLETION OF A GROUP OF FINITE TYPE

Let $G$ be a group generated by $\{g_1, \ldots, g_n\}$ and relations $\{R_1, \ldots, R_p\}$. Let $\mathfrak{a}$ be the quotient of the complete free $k$-Lie algebra on generators $\{\gamma_1, \ldots, \gamma_n\}$ by relations

$$\log R_i(e^{\gamma_1}, \ldots, e^{\gamma_n}) = 0, \ \forall i = 1 \ldots p$$

Denote by $G(k)$ the Lie group of $\mathfrak{a}$, i.e. $G(k) = \exp(\mathfrak{a}) = \{e^a, \ a \in \mathfrak{a}\}$. It is called the *$k$-pro-unipotent completion* of $G$.

## 4. Infinitesimal braids relations

4.1. **Definition.** Let $\mathfrak{t}_n(k)$ be the $k$-Lie algebra generated by $\{t_{ij},\ 1 \leq i \neq j \leq n\}$ and relations

$$t_{ij} = t_{ji} \tag{4.1a}$$
$$[t_{ij}, t_{l,m}] = 0 \quad \text{for distincts } i,j,l,m \tag{4.1b}$$
$$[t_{ij}, t_{ik} + t_{jk}] = 0 \quad \text{for distincts } i,j,k \tag{4.1c}$$

They are called *infinitesimal braids relations.*

Set $deg(t_{ij}) = 1$, and denote by $\hat{\mathfrak{t}}_n(k)$ the degree completion of $\mathfrak{t}_n(k)$.

4.2. **Realization.** Let $\mathfrak{g}$ be a simple Lie algebra and $t \in S^2(\mathfrak{g})^{\mathfrak{g}}$. Assume that $t = \sum x \otimes y$, and set

$$t^{i,j} = \sum 1 \otimes \ldots \otimes x \otimes \ldots \otimes y \otimes \ldots 1$$

It can easily be checked that the $t^{i,j}$ satisfy the infinitesimal braids relations.

**Proposition 4.1.** *Let $\hbar$ be a formal variable. There is a unique Lie algebra morphism $\hat{\mathfrak{t}}_n \to U(\mathfrak{g})^{\otimes n}[[\hbar]]$ such that*

$$t_{ij} \longmapsto \hbar t^{i,j}$$

## 5. Associators

Let $\hat{\mathfrak{f}}_2(k)$ be the completed free $k$-Lie algebra in two variables, and $\hat{\mathfrak{F}}_2(k) = \exp(\hat{\mathfrak{f}}_2(k))$. An element $\Phi \in \hat{\mathfrak{F}}_2(k)$ will be called a $\lambda$-associator if it satisfies:

$$\Phi(B, A) = \Phi(A, B)^{-1}$$

$$e^{\lambda A/2}\Phi(C, A)e^{\lambda C/2}\Phi(B, C)e^{\lambda B/2}\Phi(A, B) = 1$$

where $A + B + C = 1$, and

$$\Phi(t_{12}, t_{23} + t_{24})\Phi(t_{13} + t_{23}, t_{34}) = \Phi(t_{23}, t_{34})\Phi(t_{12} + t_{13}, t_{24} + t_{34})\Phi(t_{12}, t_{23})$$

in $\exp(\hat{\mathfrak{t}}_4)$.

Denote by $M_\lambda(k)$ the set of $\lambda$-associators over $k$, and set $M(k) = \bigcup_{\lambda \in k^*} M_\lambda(k)$.

Associators gives a universal way for constructing braided monoidal categories. Let $\mathfrak{g}$ be a simple Lie algebra, $t \in S^2(\mathfrak{g})^{\mathfrak{g}}$ and $\hbar$ a formal variable. If $(\Phi, \lambda)$ is an associator, then elements $\Phi(\hbar t^{1,2}, \hbar t^{2,3})$ and $e^{\hbar\lambda t^{1,2}}$ induces a braided monoidal category structure on $U(\mathfrak{g})[[\hbar]]$-mod. (Equivalently, $(U(\mathfrak{g})[[\hbar]], \Delta_0, \Phi(\hbar t^{1,2}, \hbar t^{2,3}), e^{\hbar\lambda t^{1,2}})$ is a quasitriangular quasi bialgebra.)

## 6. The Grothendieck-Teichmüller group

The main question which motivates the definition of the GT group is: how to change the associativity and commutativity constraints in a braided monoidal category in such a way that the result is again a braided monoidal category ?

If $\mathcal{C}$ is a braided monoidal category, and if $V_1, V_2, V_3 \in Obj(\mathcal{C})$, then every $b \in B_3$ induces an isomorphism

$$(V_1 \otimes V_2) \otimes V_3 \longrightarrow (V_{\sigma(1)} \otimes V_{\sigma(2)}) \otimes V_{\sigma(3)}$$

where $\sigma$ is the image of $b$ by the canonical projection $B_3 \to S_3$. Thus, every $\phi \in PB_3$ induces an automorphism $\tilde{\phi}$ of $(V_1 \otimes V_2) \otimes V_3$, and then induces an isomorphism

$$\alpha \circ \tilde{\phi} : (V_1 \otimes V_2) \otimes V_3 \longrightarrow V_1 \otimes (V_2 \otimes V_3)$$

In the same way, every element of $PB_2$ defines a new commutativity constraint. Then, this action reads:

$$\beta \longmapsto \beta^{2m+1}$$

$$\alpha \longmapsto \alpha f(\beta^2, \alpha^{-1}(1 \otimes \beta^2)\alpha)(\beta\alpha^{-1}(1 \otimes \beta)\alpha)^{3n}$$

Mac Lane's coherence conditions impose that $n = 0$ and relations which are equivalent to:

$$f(X,Y) = f(Y,X)^{-1} \tag{6.1}$$

$$f(X_3,X_1)X_3^m f(X_2,X_3)X_2^m f(X_1,X_2)X_1^m = 1 \tag{6.2}$$

for $X_1X_2X_3 = 1$, and a relation on $\phi$ which can be expressed using the generators of $PB_4$:

$$f(x_{12},x_{23}x_{24})f(x_{13}x_{23},x_{34}) = f(x_{23},x_{34})f(x_{12}x_{13},x_{24}x_{34})f(x_{12},x_{23}) \tag{6.3}$$

The set $\underline{GT}$ of all pairs ($\lambda = 2m+1, f$) satisfying these relations has a natural semi-group structure. Unfortunaly, $\underline{GT}$ is almost trivial. But if $\mathcal{C}$ is $k[[\hbar]]$-linear, these relations make sense even if $\lambda \in k^*$ and $f$ belongs to the $k$-pro-unipotent completion $\mathfrak{F}_2(k)$ of the free group, and it's easily seen that an element $(f,\lambda)$ is invertible iff $\lambda \neq 0$, which leads to the definition of the Grothendieck-Teichmüller group $GT(k)$.

## 7. $GT(k)$-torsor structure of $M(k)$

It follows from the above section that $GT(k)$ acts on the set of Lie associator by

$$(f,\lambda)\cdot(\Phi(A,B),\mu) = (f(\Phi(A,B)e^A\Phi(A,B)^{-1},e^B)\Phi(A,B),\mu\lambda) \tag{7.1}$$

**Theorem 7.1.** *For each Lie associator $(\Phi,\lambda) \in M(k)$, the map $\alpha_n : B_n(k) \to \exp(\hat{\mathfrak{t}}_n) \rtimes S_n$ mapping $\sigma_i$ to*

$$\Phi(t^{1i}+\cdots+t^{i-1i},t^{ii+1})^{-1}(i,i+1)e^{\lambda t^{ii+1}/2}\Phi(t^{1i}+\cdots+t^{i-1i},t^{ii+1})$$

*is a group isomorphism.*

*Proof.* This formula is analog to the formula defining a representation of the braid group from a braided monoidal category, and thus $\alpha_n$ is a group morphism.

Thus, $\alpha_n$ induces a morphism $PB_n(k) \to \exp(\hat{\mathfrak{t}}_n)$ and therefore a Lie algebra morphism

$$\alpha_n^* : \mathrm{Lie}(PB_n(k)) \longrightarrow \hat{\mathfrak{t}}_n$$

The Lie algebra $\mathrm{Lie}(PB_n(k))$ is generated by $\xi_{ij},\ 1 \leq i < j \leq n$ and relations obtained from (1.1) by setting $x_{ij} = e^{\xi_{ij}}$. Denote by $\mathrm{grLie}(PB_n(k))$ the associated graded of $\mathrm{Lie}(PB_n(k))$ and by $x \to [x]$ the canonical projection $\mathrm{Lie}(PB_n(k)) \to \mathrm{gr}_1\mathrm{Lie}(PB_n(k))$. The presentation of $PB_n$ and $\mathfrak{t}_n$ implies that there is a morphism $\mu_n : \mathfrak{t}_n \to \mathrm{gr}\,\mathrm{Lie}(PB_n(k))$ defined by $t_{ij} \mapsto [\xi_{ij}]$, which is surjective as $\mathrm{grLie}(PB_n(k))$ is generated in degree 1. The morphism $\alpha_n^*$ takes $\xi_{ij}$ to $\lambda t_{ij} + \{\text{higher degree terms}\}$, and thus the associated graded morphism

$$\mathrm{gr}\,\alpha_n^* : \mathrm{grLie}(PB_n(k)) \to \mathfrak{t}_n$$

is such that $\mathrm{gr}\,\alpha_n^* \circ \mu_n$ is bijective.

It follows that $\mu_n$ is bijective, and then that so is $\mathrm{gr}\,\alpha_n^*$. As both Lie algebras are complete and separated, $\alpha_n^*$ is an isomorphism. □

**Theorem 7.2.** *The action of $GT(k)$ on $M(k)$ is free and transitive.*

*Proof.* Let $(\Phi_1,\mu_1),(\Phi_2,\mu_2) \in M(k)$. As the action of $\hat{F}_2 \times k^*$ on $M(k)$ is free and transitive, there exists a unique $f$ such that $(f,\lambda)\cdot(\Phi_1,\mu_1) = (\Phi_2,\mu_2)$ with $\lambda = \mu_2/\mu_1$. Thus, it's enough to prove that $(f,\lambda) \in GT(k)$. It can be done by applying the above morphism to each relation, and it's easily shown that the fact that both sides have the same image follows from the fact that $\Phi_2,\Phi_2$ are associators. □

## 8. Existence of rationals associators

Let $GT_1(k) = \{(1,f) \in GT(k)\}$ and $M_1(k)$ be the set of 1-associators. By identifying $GT_1(k)$ (resp. $M_1(k)$) with the quotient of $GT(k)$ (resp. $M(k)$) by the natural action of $k^*$, one see that the action of $GT_1(k)$ on $M_1(k)$ is free and transitive. There is also an action of $GT(k)$ on $M_1(k)$ which is free but not transitive. If $M(k) \neq \emptyset$, there is a morphism $\nu : GT(k) \longrightarrow k^*$ mapping $(\lambda,f)$ to $\lambda$, which is obviously surjective.

Thus, in this case, the following sequence makes sense and is exact

$$1 \to GT_1(k) \to GT(k) \xrightarrow{\nu} k^* \to 1$$

Let $\mathfrak{gt}(k)$ be the Lie algebra of $GT(k)$. If $M_1(k) \neq \emptyset$, one has an exact sequence

$$0 \to \mathfrak{gt}_1(k) \to \mathfrak{gt}(k) \xrightarrow{\nu^*} k \to 0$$

**Theorem 8.1.** *If the map $\nu^* : \mathfrak{gt}_k \to k$ is surjective, then $M(k) \neq \emptyset$.*

*Proof.* Every element $\Phi$ in $M_1(k)$ induces a morphism $\theta_\Phi : k^* \to GT(k)$ where $\theta_\Phi(k^*)$ is the stabilizer of the equivalence classe of $\Phi$ in $M_1(k) = M(k)/k^*$. Thus, every $(f, \lambda)$ in $\theta_\Phi(k^*)$ verifies

$$(f, \lambda) \cdot \Phi(A, B) = \Phi(\lambda A, \lambda B)$$

Write $f = \exp(\epsilon\psi(\ln X, \ln Y))$ and $\lambda = 1 + \epsilon s$, thus $(\psi, s) \in \mathfrak{gt}(k)$. We are looking for an element of the form $(\psi, 1) \in \mathfrak{gt}(k)$, as the existence of such an element is equivalent to the surjectivity of $\nu^*$. In this case, one has:

$$\Phi((1+\epsilon)A, (1+\epsilon)B) = \exp(\epsilon\psi(\ln(\Phi(A,B)e^A\Phi(A,B)^{-1}), \ln e^B))\Phi(A,B)$$

By linearizing with respect to $\epsilon$ and setting $t = 1 + \epsilon$, one get

$$\frac{d}{dt}\Phi(tA, tB)\bigg|_{t=1} = \psi(\Phi(A,B)A\Phi(A,B)^{-1}, B) \tag{8.1}$$

Conversly, suppose now given an element $(\psi, 1)$, it exists a unique $\Phi \in \hat{F}_2$ such that (8.1) is satisfied. By working degree by degree, it can be shown that $\Phi$ is a 1-associator.

□

**Corollary 8.2.** *If $M(\mathbb{C}) \neq \emptyset$, then $M(k) \neq \emptyset$ for all $k \subset \mathbb{C}$.*

*Proof.* If $M(\mathbb{C}) \neq \emptyset$, then one has the short exact sequence of Lie algebra

$$0 \to \mathfrak{gt}_1(\mathbb{C}) \to \mathfrak{gt}(\mathbb{C}) \xrightarrow{\nu} \mathbb{C} \to 0$$

which proves that $\nu^* : \mathfrak{gt}(\mathbb{Q}) \to \mathbb{Q}$ is surjective, and thus that $M_1(\mathbb{Q}) \neq \emptyset$. □

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