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Exponential growth and extinction in age structured populations incorporating environmental stochasticity

We study different strategies to ascertain growth or extinction in Leslie type matrix models for age structured populations subjected to environmental stochasticity [1]. We think of a population described at time n by vector $\mathbf{X}_n = (x_n^1, ..., x_n^N)^T$ and living in an ambient in which there are s different environmental states. The vital rates corresponding to each one of these environments are given by the Leslie matrices $\mathbf{L}_\alpha \in \mathbb{R}^{N \times N}$, $\alpha = 1, ..., s$ in such a way that, for each α, $\mathbf{L}_\alpha$ contains the fertility and survival rates of the population in environment α. The environmental variation is characterized by a sequence of random variables τ_n, that we will consider to be an irreducible and aperiodic Markov chain, with state space $\{1, ..., s\}$ in such a way that τ_{n+1} describes for the environmental condition for the system between times n and $n+1$. Thus, the model reads

$$\mathbf{X}_{n+1} = \mathbf{L}_{\tau_{n+1}} \mathbf{X}_n \tag{1}$$

where $\mathbf{X}_0 \geq \mathbf{0}$ is a fixed (non random) non-zero vector. Moreover, we assume that the set of matrices of vital rates meets a certain technical condition (ergodic set).

The most important parameter concerning the behavior of (1) is the so called stochastic growth rate (s.g.r.) defined as $a := \lim_{n\to\infty} \log \|\mathbf{X}_n\| / n$, with probability one [2]. Therefore, $a > 0$ implies that every realization grows asymptotically with rate e^a, and $a < 0$ implies that the population goes extinct with probability one. However, even in very simple situations, it is not possible to calculate a analytically. In order to find a useful way to study these models, the so called "lognormal approximation" has been proposed [2]. It consists in assuming that the distribution of population size has a lognormal distribution. In this way an approximate s.g.r. $\hat{a}$ can be defined. The validity of this approximation has only been tested numerically and in very specific situations [3]. Moreover, in principle the approximation does not allow one to calculate $\hat{a}$ analytically.

In the first place, this work examines both numerically and theoretically, the validity of the lognormal approximation, finding the range of situations in which it can be considered that it works well. Moreover, we build different bounds for a and for $\hat{a}$, and analyze the conditions under which each bound works best. This is used to give necessary-sufficient conditions for the explosion and the extinction of the population. The results are applied to the case of a population structured in juveniles and adults living in an ambient with a "good" and a "bad" environment.

References

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