On $q-$asymptotics for some $q-$difference-differential equations with Fuchsian and irregular singularities

Alberto Lastra[*†], Stéphane Malek[†], Javier Sanz[*]

[*] Universidad de Valladolid, Spain
[†] Université de Lille 1, France

Bedlewo 2011, August 8th

Notations

$\mathbb{N} = \{0, 1, 2, ...\}$

$\mathbb{C}[[z]]$ Formal complex power series

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

$D(0, r)$ denotes the open disc with center 0 and radius $r > 0$.

Given $V \subset \mathbb{C}$ and $q \in \mathbb{C}$, we consider $q-$spirals

$$Vq^{\mathbb{Z}} = \{vq^h : v \in V,\ h \in \mathbb{Z}\}, \quad Vq^{\mathbb{N}} = \{vq^h : v \in V,\ h \in \mathbb{N}\}.$$

Cauchy problem for some PDE's, by S. Malek

S. Malek studied PDE's of the form

$$t^{2r_2}\partial_t^{r_2}(z\partial_z)^{r_1}\partial_z^S u(t,z) = F(t,z,\partial_t,\partial_z)u(t,z) \quad (1)$$

where $S, r_1, r_2 \in \mathbb{N}$ and F is some suitable differential operator with polynomial coefficients.

Cauchy problem for some PDE's, by S. Malek

S. Malek studied PDE's of the form

$$t^{2r_2}\partial_t^{r_2}(z\partial_z)^{r_1}\partial_z^S u(t,z) = F(t,z,\partial_t,\partial_z)u(t,z) \tag{1}$$

where $S, r_1, r_2 \in \mathbb{N}$ and F is some suitable differential operator with polynomial coefficients.

For given initial data

$$(\partial_z^j \hat{u})(t,0) = \hat{u}_j(t) \in \mathbb{C}[[t]], \tag{2}$$

$0 \le j \le S-1$, he constructed formal power series solutions of (1)+(2) of the form

$$\hat{u}(t,z) = \sum_{m\ge 0} \hat{u}_m(t) z^m/m!,$$

with coefficients in $\mathbb{C}[[t]]$.

Summability for $r_1 = 0$, Gevrey expansion for $r_1 > 0$

Assume the initial conditions are $1-$Borel summable with respect to t in some direction $d \in \mathbb{R}$.

Summability for $r_1 = 0$, Gevrey expansion for $r_1 > 0$

Assume the initial conditions are $1-$Borel summable with respect to t in some direction $d \in \mathbb{R}$.

For $r_1 = 0$, it was shown in [S. Malek, J. Dyn. Control Syst. 13 (2007), no. 3, 419–449] that the formal series solution $\hat{u}(t,z)$ is $1-$Borel summable with respect to t in the direction d if d is well chosen, as series with coefficients in the Banach space of holomorphic functions near the origin (in z) with the supremum norm.

Summability for $r_1 = 0$, Gevrey expansion for $r_1 > 0$

Assume the initial conditions are $1-$Borel summable with respect to t in some direction $d \in \mathbb{R}$.

For $r_1 = 0$, it was shown in [S. Malek, J. Dyn. Control Syst. 13 (2007), no. 3, 419–449] that the formal series solution $\hat{u}(t, z)$ is $1-$Borel summable with respect to t in the direction d if d is well chosen, as series with coefficients in the Banach space of holomorphic functions near the origin (in z) with the supremum norm.

For $r_1 \neq 0$, in [S. Malek, J. Dyn. Control Syst. 15 (2009), no. 2, 277–305] it was shown the existence of actual holomorphic solutions $u(t, z)$ which are Gevrey asymptotic of order larger than 1 to $\hat{u}(t, z)$ with respect to t in sectors with finite radius in well chosen directions $d \in \mathbb{R}$.

Summability for $r_1 = 0$, Gevrey expansion for $r_1 > 0$

Assume the initial conditions are $1-$Borel summable with respect to t in some direction $d \in \mathbb{R}$.

For $r_1 = 0$, it was shown in [S. Malek, J. Dyn. Control Syst. 13 (2007), no. 3, 419–449] that the formal series solution $\hat{u}(t,z)$ is $1-$Borel summable with respect to t in the direction d if d is well chosen, as series with coefficients in the Banach space of holomorphic functions near the origin (in z) with the supremum norm.

For $r_1 \neq 0$, in [S. Malek, J. Dyn. Control Syst. 15 (2009), no. 2, 277–305] it was shown the existence of actual holomorphic solutions $u(t,z)$ which are Gevrey asymptotic of order larger than 1 to $\hat{u}(t,z)$ with respect to t in sectors with finite radius in well chosen directions $d \in \mathbb{R}$.

Reason: presence of small divisors introduced by the Fuchsian operator $(z\partial_z)^{r_1}$.

Analogue for q-difference-differential equations

As a $q-$analogue of (1)+(2), we consider the $q-$difference-differential equation

$$((z\partial_z+1)^{r_1}(t\sigma_q)^{r_2}+1)\partial_z^S\hat{X}(t,z)=\sum_{k=0}^{S-1}b_k(z)(t\sigma_q)^{m_{0,k}}(\partial_z^k\hat{X})(t,zq^{-m_{1,k}})$$

with given initial conditions

$$(\partial_z^j\hat{X})(t,0)=\hat{X}_j(t)\in\mathbb{C}[[t]].$$

Analogue for q-difference-differential equations

As a $q-$analogue of (1)+(2), we consider the $q-$difference-differential equation

$$((z\partial_z+1)^{r_1}(t\sigma_q)^{r_2}+1)\partial_z^S\hat{X}(t,z)=\sum_{k=0}^{S-1}b_k(z)(t\sigma_q)^{m_{0,k}}(\partial_z^k\hat{X})(t,zq^{-m_{1,k}})$$

with given initial conditions

$$(\partial_z^j\hat{X})(t,0)=\hat{X}_j(t)\in\mathbb{C}[[t]].$$

S and $m_{0,k}$,$m_{1,k}$, $0\leq k\leq S-1$, are nonnegative integers,

$q\in\mathbb{C}$ with $|q|>1$,

σ_q is the dilation operator defined by $(\sigma_q\hat{X})(t,z)=\hat{X}(qt,z)$,

$b_k(z)=\sum_{s\in I_k}b_{ks}z^s$ are polynomial in z, where $I_k\subset\mathbb{N}$,

the map $(t,z)\mapsto(q^{m_{0,k}}t,zq^{-m_{1,k}})$ is volume shrinking, i.e., $m_{0,k}<m_{1,k}$,

and $r_2\geq 1$, while $r_1\geq 0$.

Analogue for q-difference-differential equations

As a $q-$analogue of (1)+(2), we consider the $q-$difference-differential equation

$$((z\partial_z+1)^{r_1}(t\sigma_q)^{r_2}+1)\partial_z^S\hat{X}(t,z)=\sum_{k=0}^{S-1}b_k(z)(t\sigma_q)^{m_{0,k}}(\partial_z^k\hat{X})(t,zq^{-m_{1,k}})$$

with given initial conditions

$$(\partial_z^j\hat{X})(t,0)=\hat{X}_j(t)\in\mathbb{C}[[t]].$$

S and $m_{0,k}$,$m_{1,k}$, $0\le k\le S-1$, are nonnegative integers,

$q\in\mathbb{C}$ with $|q|>1$,

σ_q is the dilation operator defined by $(\sigma_q\hat{X})(t,z)=\hat{X}(qt,z)$,

$b_k(z)=\sum_{s\in I_k}b_{ks}z^s$ are polynomial in z, where $I_k\subset\mathbb{N}$,

the map $(t,z)\mapsto(q^{m_{0,k}}t,zq^{-m_{1,k}})$ is volume shrinking, i.e., $m_{0,k}<m_{1,k}$,

and $r_2\ge 1$, while $r_1\ge 0$.

This problem will be called $(P1)$.

Formal solution

Proposition

$(P1)$ *has a unique formal power series solution of the form*

$$\hat{X}(t,z) = \sum_{h\geq 0} \hat{X}_h(t)\frac{z^h}{h!},$$

where $\hat{X}_h(t) \in \mathbb{C}[[t]]$, $h \geq 0$.

Formal solution

Proposition

$(P1)$ *has a unique formal power series solution of the form*

$$\hat{X}(t,z) = \sum_{h\geq 0} \hat{X}_h(t)\frac{z^h}{h!},$$

where $\hat{X}_h(t) \in \mathbb{C}[[t]]$, $h \geq 0$.

Objective: to construct actual holomorphic solutions of this problem that are asymptotically represented by $\hat{X}(t,z)$ in a precise sense.

Formal $q-$Laplace and $q-$Borel transforms

Definition

The formal $q-$Borel transform of order 1 of a series $\hat{f}(t)=\sum_{n\ge 0} f_n t^n \in \mathbb{C}[[t]]$ is defined as

$$\hat{\mathcal{B}}_q \hat{f}(\tau)=\sum_{n\ge 0}\frac{f_n}{q^{n(n-1)/2}}\tau^n.$$

Definition

The formal $q-$Laplace transform of order 1 of $\hat{g}(\tau)=\sum_{n\ge 0} g_n \tau^n \in \mathbb{C}[[\tau]]$ is defined as

$$\hat{\mathcal{L}}_q \hat{g}(t)=\sum_{n\ge 0} q^{n(n-1)/2} g_n t^n.$$

Transformed problem

$$((z\partial_z+1)^{r_1}(t\sigma_q)^{r_2}+1)\partial_z^S\hat{X}(t,z)=\sum_{k=0}^{S-1}b_k(z)(t\sigma_q)^{m_{0,k}}(\partial_z^k\hat{X})(t,zq^{-m_{1,k}})$$

$$\hat{X}(t,z)=\hat{\mathcal{L}}_q(\hat{W})(\tau,z)\quad\Bigg\uparrow\quad \hat{\mathcal{L}}_q(\tau\hat{W})(t)=t\hat{\mathcal{L}}_q\hat{W}(qt)=(t\sigma_q)\hat{\mathcal{L}}_q\hat{W}(t)$$

$$((z\partial_z+1)^{r_1}\tau^{r_2}+1)\partial_z^S\hat{W}(\tau,z)=\sum_{k=0}^{S-1}b_k(z)\tau^{m_{0,k}}(\partial_z^k\hat{W})(\tau,zq^{-m_{1,k}})\qquad(3)$$

Equivalence of the formal problems

$\hat{X}(t,z)=\sum_{h\geq 0}\hat{X}_h(t)\frac{z^h}{h!}$, where $\hat{X}_h\in\mathbb{C}[[t]]$ for every $h\geq 0$, satisfies $(P1)$ $\Longleftrightarrow$

$\hat{W}(\tau,z)=\sum_{h\geq 0}\hat{\mathcal{B}}_q\hat{X}_h(\tau)\frac{z^h}{h!}$ satisfies (3) with initial conditions

$$\hat{W}_j(\tau)=\hat{\mathcal{B}}_q\hat{X}_j,\ 0\leq j\leq S-1. \quad (4)$$

Equivalence of the formal problems

$\hat{X}(t,z)=\sum_{h\geq 0}\hat{X}_h(t)\frac{z^h}{h!}$, where $\hat{X}_h\in\mathbb{C}[[t]]$ for every $h\geq 0$, satisfies $(P1)$ $\Longleftrightarrow$
$\hat{W}(\tau,z)=\sum_{h\geq 0}\hat{\mathcal{B}}_q\hat{X}_h(\tau)\frac{z^h}{h!}$ satisfies (3) with initial conditions

$$\hat{W}_j(\tau)=\hat{\mathcal{B}}_q\hat{X}_j,\ 0\leq j\leq S-1. \tag{4}$$

(3)+(4) will be called $(P2)$.

In other words:

$\hat{W}(\tau,z)=\sum_{h\geq 0}\hat{W}_h(\tau)\frac{z^h}{h!}$, with $\hat{W}_h\in\mathbb{C}[[\tau]]$ for every $h\geq 0$, satisfies $(P2)$
$\Longleftrightarrow$ $\hat{X}(t,z)=\sum_{h\geq 0}\hat{\mathcal{L}}_q\hat{W}_h(t)\frac{z^h}{h!}$ satisfies $(P1)$ with $\hat{X}_j(t)=\hat{\mathcal{L}}_q\hat{W}_j(t)$ for $0\leq j\leq S-1$.

Assumptions on the initial conditions

Under certain technical assumptions on q and V, we suppose the initial data W_j, $0 \le j \le S-1$ are holomorphic in $Vq^{\mathbb{Z}}$ such that: there exists a constant $K_0 > 0$ with

$$\sup_{x \in V} |W_j(xq^l)| \le K_0 |q|^{\frac{1}{4}l^2} (\frac{1}{T_{0,j}})^l \frac{1}{1+l^2}$$

and

$$\sup_{x \in V} |W_j(xq^{-l})| \le K_0 (T_{0,j})^l \frac{1}{1+l^2}$$

for all $0 \le j \le S-1$, all $l \ge 0$.

Assumptions on the initial conditions

Under certain technical assumptions on q and V, we suppose the initial data W_j, $0 \leq j \leq S-1$ are holomorphic in $Vq^{\mathbb{Z}}$ such that: there exists a constant $K_0 > 0$ with

$$\sup_{x \in V} |W_j(xq^l)| \leq K_0 |q|^{\frac{1}{4}l^2} (\frac{1}{T_{0,j}})^l \frac{1}{1+l^2}$$

and

$$\sup_{x \in V} |W_j(xq^{-l})| \leq K_0 (T_{0,j})^l \frac{1}{1+l^2}$$

for all $0 \leq j \leq S-1$, all $l \geq 0$.

The first inequalities express the property of W_j being of $q-$exponential growth on $Vq^{\mathbb{N}}$ (with order 2).

$q-$growth of the coefficients in $Vq^{\mathbb{N}}$

Theorem

There exists a unique solution of $(P2)$

$$(\tau, z) \mapsto W(\tau, z) = \sum_{h \geq 0} W_h(\tau) \frac{z^h}{h!}$$

which is holomorphic on $Vq^{\mathbb{Z}} \times \mathbb{C}$.

Moreover, for all $\rho > 0$*, there exist* $C, T > 0$ *(depending on the data) such that*

$$\sup_{x \in V, z \in D(0,\rho)} |W(xq^l, z)| \leq CK_0 |q|^{\frac{1}{2}l^2} (\frac{1}{T})^l,$$

and

$$\sup_{x \in V, z \in D(0,\rho)} |W(xq^{-l}, z)| \leq CK_0 T^l$$

for all $l \geq 0$.

Assumptions

In the Cauchy problem (P2) we consider initial conditions W_j which are holomorphic functions respectively defined in open sets containing the closed disc

$$\overline{D}_j = \{\tau : |\tau| \leq 1/(2(j+1)^{r_1/r_2})\}$$

for $0 \leq j \leq S-1$ (for the sake of brevity, we say that W_j is holomorphic in $\overline{D}_j$).

Assumptions

In the Cauchy problem (P2) we consider initial conditions W_j which are holomorphic functions respectively defined in open sets containing the closed disc

$$\overline{D}_j = \{\tau : |\tau| \leq 1/(2(j+1)^{r_1/r_2})\}$$

for $0 \leq j \leq S-1$ (for the sake of brevity, we say that W_j is holomorphic in $\overline{D}_j$).

Cauchy's integral formula for the derivatives allows us to obtain constants $A_j > 0$ such that for every $n \geq 0$ we have

$$\max_{\tau \in \overline{D}_j} |\partial^n W_j(\tau)| \leq A_j^n n!.$$

Result

Theorem

Suppose $W_j(\tau)$, $0 \leq j \leq S-1$, *are holomorphic functions in* $\overline{D}_j$ *such that there exist* $T_{0,j} > 0$ *and a constant* $K > 0$ *such that*

$$\max_{\tau \in \overline{D}_j} |\partial^n W_j(\tau)| \leq K \left(\frac{1}{T_{0,j}} \right)^n \frac{n!}{1+n^2}, \quad n \geq 0, \; j = 0, 1, ..., S-1.$$

Result

Theorem

Suppose $W_j(\tau)$, $0 \le j \le S-1$, *are holomorphic functions in* $\overline{D}_j$ *such that there exist* $T_{0,j} > 0$ *and a constant* $K > 0$ *such that*

$$\max_{\tau \in \overline{D}_j} |\partial^n W_j(\tau)| \le K \left(\frac{1}{T_{0,j}} \right)^n \frac{n!}{1+n^2}, \quad n \ge 0, \ j = 0, 1, ..., S-1.$$

Then there exists a unique formal solution of (P2), $W(\tau, z) = \sum_{h \ge 0} W_h(\tau) \frac{z^h}{h!}$, *where* W_h *is a holomorphic function in* $\overline{D}_h = \{\tau : |\tau| \le 1/(2(h+1)^{r_1/r_2})\}$, $h \ge S$.

Result

Theorem

Suppose $W_j(\tau)$, $0 \leq j \leq S-1$, are holomorphic functions in $\overline{D}_j$ such that there exist $T_{0,j} > 0$ and a constant $K > 0$ such that

$$\max_{\tau \in \overline{D}_j} |\partial^n W_j(\tau)| \leq K \left(\frac{1}{T_{0,j}}\right)^n \frac{n!}{1+n^2}, \quad n \geq 0, \ j = 0, 1, ..., S-1.$$

Then there exists a unique formal solution of (P2), $W(\tau, z) = \sum_{h \geq 0} W_h(\tau) \frac{z^h}{h!}$, where W_h is a holomorphic function in $\overline{D}_h = \{\tau : |\tau| \leq 1/(2(h+1)^{r_1/r_2})\}$, $h \geq S$.

Moreover, there exist $T_1, X_1 > 0$ such that

$$\sup_{\tau \in \overline{D}_h} |\partial^n W_h(\tau)| \leq C_1 \left(\frac{1}{T_1}\right)^n \left(\frac{1}{X_1}\right)^h n! h! (h+1)^{r_1 n / r_2} |q|^{-h^2/2},$$

for every $n, h \geq 0$, where C_1 is a positive constant (depending on S,q,$b_k(z)$,$m_{1,k}$, for $0 \leq k \leq S-1$ and $T_{0,j}$, for $0 \leq j \leq S-1$).

What we know about (P2)

Let W_h be the initial data in the Cauchy problem (P2), and suppose they are subject to the hypotheses in the previous Theorems. Then, we have a sequence of functions $\{W_h\}_{h\geq 0}$, holomorphic in $Vq^{\mathbb{Z}} \cup D_h$ for each $h \geq 0$, and such that the series

$$W(\tau, z) = \sum_{h\geq 0} W_h(\tau)\frac{z^h}{h!}$$

defines a holomorphic function on $Vq^{\mathbb{Z}} \times \mathbb{C}$ which solves the Cauchy problem.

What we know about (P2)

Let W_h be the initial data in the Cauchy problem (P2), and suppose they are subject to the hypotheses in the previous Theorems. Then, we have a sequence of functions $\{W_h\}_{h\geq 0}$, holomorphic in $Vq^{\mathbb{Z}} \cup D_h$ for each $h \geq 0$, and such that the series

$$W(\tau, z) = \sum_{h\geq 0} W_h(\tau)\frac{z^h}{h!}$$

defines a holomorphic function on $Vq^{\mathbb{Z}} \times \mathbb{C}$ which solves the Cauchy problem.

Moreover, we know that

$$\sup_{x\in V} |W_h(xq^l)| \leq K_0 C' |q|^{\frac{l^2}{2}} |q|^{\frac{-h^2}{4}} h! (\frac{1}{T})^l (\frac{1}{X})^h$$

for all $l, h \geq 0$.

$q-$Laplace transform for functions of $q-$exponential growth I

Let $q \in \mathbb{C}$ such that $|q| > 1$.

Let V be an open and bounded set in $\mathbb{C}^*$ and $D(0, \rho_0)$ a disc such that $V \cap D(0, \rho_0) \neq \emptyset$.

Let $\phi : Vq^{\mathbb{N}} \cup D(0, \rho_0) \to \mathbb{C}$ be a holomorphic function which satisfies the following estimates: there exist $C, M > 0$ such that

$$|\phi(xq^m)| \leq M|q|^{m^2/2}C^m$$

for all $m \geq 0$, all $x \in V$.

ϕ is said to have $q-$exponential growth of order 1 in $Vq^{\mathbb{N}}$.

$q-$Laplace transform for functions of $q-$exponential growth I

Let $q \in \mathbb{C}$ such that $|q| > 1$.

Let V be an open and bounded set in $\mathbb{C}^*$ and $D(0, \rho_0)$ a disc such that $V \cap D(0, \rho_0) \neq \emptyset$.

Let $\phi : Vq^{\mathbb{N}} \cup D(0, \rho_0) \to \mathbb{C}$ be a holomorphic function which satisfies the following estimates: there exist $C, M > 0$ such that

$$|\phi(xq^m)| \leq M|q|^{m^2/2}C^m$$

for all $m \geq 0$, all $x \in V$.

ϕ is said to have $q-$exponential growth of order 1 in $Vq^{\mathbb{N}}$.

Let Θ be the Theta Jacobi function defined in $\mathbb{C}^*$ by

$$\Theta(x) = \sum_{n \in \mathbb{Z}} q^{-n(n-1)/2} x^n.$$

$q-$Laplace transform for functions of $q-$exponential growth II

Let $\delta > 0$ and $\lambda \in V \cap D(0, \rho_0)$. We denote by

$$\mathcal{R}_{\lambda,q,\delta} = \{t \in \mathbb{C}^*/|1 + \frac{\lambda}{tq^k}| > \delta, \forall k \in \mathbb{Z}\}, \quad \mathcal{T}_{\lambda,q,\delta,r_1} = \mathcal{R}_{\lambda,q,\delta} \cap D(0, r_1).$$

$q-$Laplace transform for functions of $q-$exponential growth II

Let $\delta > 0$ and $\lambda \in V \cap D(0, \rho_0)$. We denote by

$$\mathcal{R}_{\lambda,q,\delta} = \{t \in \mathbb{C}^* / |1 + \frac{\lambda}{tq^k}| > \delta, \forall k \in \mathbb{Z}\}, \quad \mathcal{T}_{\lambda,q,\delta,r_1} = \mathcal{R}_{\lambda,q,\delta} \cap D(0, r_1).$$

The $q-$Laplace transform of ϕ in the direction $\lambda q^{\mathbb{Z}}$ is defined by

$$\mathcal{L}_q^{\lambda}(\phi)(t) := \sum_{m \in \mathbb{Z}} \phi(q^m \lambda) / \Theta(\frac{q^m \lambda}{t})$$

for all $t \in \mathcal{T}_{\lambda,q,\delta,r_1}$, if $r_1 < |\lambda q^{1/2}|/C$.

$q-$Laplace transform for functions of $q-$exponential growth II

Let $\delta > 0$ and $\lambda \in V \cap D(0, \rho_0)$. We denote by

$$\mathcal{R}_{\lambda,q,\delta} = \{t \in \mathbb{C}^*/|1 + \frac{\lambda}{tq^k}| > \delta, \forall k \in \mathbb{Z}\}, \quad \mathcal{T}_{\lambda,q,\delta,r_1} = \mathcal{R}_{\lambda,q,\delta} \cap D(0, r_1).$$

The $q-$Laplace transform of ϕ in the direction $\lambda q^{\mathbb{Z}}$ is defined by

$$\mathcal{L}_q^{\lambda}(\phi)(t) := \sum_{m \in \mathbb{Z}} \phi(q^m \lambda)/\Theta(\frac{q^m \lambda}{t})$$

for all $t \in \mathcal{T}_{\lambda,q,\delta,r_1}$, if $r_1 < |\lambda q^{1/2}|/C$.

$\mathcal{L}_q^{\lambda}(\phi)(t)$ defines a bounded holomorphic function on $\mathcal{T}_{\lambda,q,\delta,r_1}$ whenever $r_1 < |\lambda q^{1/2}|/C$.

$q-$Laplace transform for functions of $q-$exponential growth III

Assume that the function ϕ has the following Taylor expansion

$$\phi(\tau) = \sum_{n\geq 0} \frac{f_n}{q^{n(n-1)/2}} \tau^n$$

on $D(0, \rho_0)$, where $f_n \in \mathbb{C}$, $n \geq 0$.

Then, there exist two constants $D, B > 0$ such that

$$|\mathcal{L}_q^{\lambda}(\phi)(t) - \sum_{m=0}^{n-1} f_m t^m| \leq DB^n |q|^{n(n-1)/2} |t|^n$$

for all $n \geq 1$, for all $t \in \mathcal{T}_{\lambda,q,\delta,r_1}$.

$\mathcal{L}_q^{\lambda}(\phi)$ admits the series $\sum_{m=0}^{\infty} f_m t^m$ as $q-$Gevrey asymptotic expansion of order 1.

$q-$Laplace transform for the coefficients of our solution

Every W_h verifies suitable estimates so as to admit $q-$Laplace transform.

$q-$Laplace transform for the coefficients of our solution

Every W_h verifies suitable estimates so as to admit $q-$Laplace transform.

Choose an integer $n(h)$ in such a way that $\lambda q^{n(h)} \in D_h$.

$q-$Laplace transform for the coefficients of our solution

Every W_h verifies suitable estimates so as to admit $q-$Laplace transform.

Choose an integer $n(h)$ in such a way that $\lambda q^{n(h)} \in D_h$.

The $q-$Laplace transform of W_h in the direction $\lambda q^{n(h)} q^{\mathbb{Z}} \equiv \lambda q^{\mathbb{Z}}$ is given by

$$\mathcal{L}_q^{\lambda q^{n(h)}}(W_h)(t) = \sum_{m\in\mathbb{Z}} \frac{W_h(q^m \lambda q^{n(h)})}{\Theta(\frac{q^m \lambda q^{n(h)}}{t})} = \sum_{m\in\mathbb{Z}} \frac{W_h(q^m \lambda)}{\Theta(\frac{q^m \lambda}{t})},$$

so that it deserves to be denoted by $\mathcal{L}_q^{\lambda}(W_h)(t)$.

$q-$Laplace transform for the coefficients of our solution

Every W_h verifies suitable estimates so as to admit $q-$Laplace transform.

Choose an integer $n(h)$ in such a way that $\lambda q^{n(h)} \in D_h$.

The $q-$Laplace transform of W_h in the direction $\lambda q^{n(h)} q^{\mathbb{Z}} \equiv \lambda q^{\mathbb{Z}}$ is given by

$$\mathcal{L}_q^{\lambda q^{n(h)}}(W_h)(t) = \sum_{m\in\mathbb{Z}} \frac{W_h(q^m \lambda q^{n(h)})}{\Theta(\frac{q^m \lambda q^{n(h)}}{t})} = \sum_{m\in\mathbb{Z}} \frac{W_h(q^m \lambda)}{\Theta(\frac{q^m \lambda}{t})},$$

so that it deserves to be denoted by $\mathcal{L}_q^{\lambda}(W_h)(t)$.

This function is well defined and holomorphic in the set $\mathcal{T}_{\lambda q^{n(h)},q,\delta,r(h)} \equiv \mathcal{T}_{\lambda,q,\delta,r(h)}$, whenever $r(h) < |\lambda q^{n(h)} q^{1/2}|T$.

$q-$Laplace transform for the coefficients of our solution

Every W_h verifies suitable estimates so as to admit $q-$Laplace transform.

Choose an integer $n(h)$ in such a way that $\lambda q^{n(h)} \in D_h$.

The $q-$Laplace transform of W_h in the direction $\lambda q^{n(h)} q^{\mathbb{Z}} \equiv \lambda q^{\mathbb{Z}}$ is given by

$$\mathcal{L}_q^{\lambda q^{n(h)}}(W_h)(t) = \sum_{m\in\mathbb{Z}} \frac{W_h(q^m \lambda q^{n(h)})}{\Theta(\frac{q^m \lambda q^{n(h)}}{t})} = \sum_{m\in\mathbb{Z}} \frac{W_h(q^m \lambda)}{\Theta(\frac{q^m \lambda}{t})},$$

so that it deserves to be denoted by $\mathcal{L}_q^{\lambda}(W_h)(t)$.

This function is well defined and holomorphic in the set $\mathcal{T}_{\lambda q^{n(h)},q,\delta,r(h)} \equiv \mathcal{T}_{\lambda,q,\delta,r(h)}$, whenever $r(h) < |\lambda q^{n(h)} q^{1/2}|T$.

Objective: Show that these radii $r(h)$ can be taken independent of h, equal to $r_0 = |\lambda q^{1/2}|T/|q| = |\lambda q^{-1/2}|T$ for every $h \geq 0$, and obtain precise estimates for the corresponding $q-$asymptotic expansions.

Uniform $q-$asymptotic expansion for the coefficients

Let us assume that the function W_h has the following Taylor expansion at 0,

$$W_h(\tau) = \sum_{n \geq 0} \frac{f_{n,h}}{q^{n(n-1)/2}} \tau^n, \tag{5}$$

where $f_{n,h} \in \mathbb{C}$, $n, h \geq 0$, and $\tau \in \overline{D}_h$.

Uniform $q-$asymptotic expansion for the coefficients

Let us assume that the function W_h has the following Taylor expansion at 0,

$$W_h(\tau) = \sum_{n\geq 0} \frac{f_{n,h}}{q^{n(n-1)/2}} \tau^n, \tag{5}$$

where $f_{n,h} \in \mathbb{C}$, $n, h \geq 0$, and $\tau \in \overline{D}_h$.

Proposition

There exist constants $B(h), D(h) > 0$ such that

$$|\mathcal{L}_q^{\lambda}(W_h)(t) - \sum_{m=0}^{n-1} f_{m,h} t^m| \leq D(h) B(h)^n |q|^{n(n-1)/2} |t|^n \tag{6}$$

for all $n \geq 1$, for all $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$.

Uniform $q-$asymptotic expansion for the coefficients

Let us assume that the function W_h has the following Taylor expansion at 0,

$$W_h(\tau) = \sum_{n\geq 0} \frac{f_{n,h}}{q^{n(n-1)/2}}\tau^n, \tag{5}$$

where $f_{n,h} \in \mathbb{C}$, $n, h \geq 0$, and $\tau \in \overline{D}_h$.

Proposition

There exist constants $B(h), D(h) > 0$ such that

$$|\mathcal{L}_q^{\lambda}(W_h)(t) - \sum_{m=0}^{n-1} f_{m,h}t^m| \leq D(h)B(h)^n|q|^{n(n-1)/2}|t|^n \tag{6}$$

for all $n \geq 1$, for all $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$.

$$B(h) = A_1(h+1)^{r_1/r_2}, \qquad D(h) = A_2(h+1)^{r_1/r_2}h!A_3^h|q|^{-h^2/4}$$

Conditions for our main result

Suppose $\hat{X}_j(t) = \sum_{m\geq 0} f_{m,j}t^m \in \mathbb{C}[[t]]$, $0 \leq j \leq S-1$, are given initial conditions for the Cauchy problem $(P1)$, and let

$$\hat{X}(t,z) = \sum_{h\geq 0} \hat{X}_h(t)\frac{z^h}{h!} = \sum_{h\geq 0}\sum_{m\geq 0} f_{m,h}t^m\frac{z^h}{h!}$$

be the only formal series solution of the problem.

Conditions for our main result

Suppose $\hat{X}_j(t) = \sum_{m\geq 0} f_{m,j}t^m \in \mathbb{C}[[t]]$, $0 \leq j \leq S-1$, are given initial conditions for the Cauchy problem $(P1)$, and let

$$\hat{X}(t,z) = \sum_{h\geq 0} \hat{X}_h(t)\frac{z^h}{h!} = \sum_{h\geq 0}\sum_{m\geq 0} f_{m,h}t^m\frac{z^h}{h!}$$

be the only formal series solution of the problem.

Suppose $\hat{X}_j(t)$, $0 \leq j \leq S-1$, are $q-$Gevrey of order 1, and that their formal $q-$Borel transforms of order 1, $W_j(\tau) = \hat{\mathcal{B}}_q\hat{X}_j(\tau)$, which are holomorphic functions around 0, satisfy the assumptions of the Theorems for $(P2)$.

Conditions for our main result

Suppose $\hat{X}_j(t) = \sum_{m\geq 0} f_{m,j} t^m \in \mathbb{C}[[t]]$, $0 \leq j \leq S-1$, are given initial conditions for the Cauchy problem $(P1)$, and let

$$\hat{X}(t,z) = \sum_{h\geq 0} \hat{X}_h(t) \frac{z^h}{h!} = \sum_{h\geq 0} \sum_{m\geq 0} f_{m,h} t^m \frac{z^h}{h!}$$

be the only formal series solution of the problem.

Suppose $\hat{X}_j(t)$, $0 \leq j \leq S-1$, are $q-$Gevrey of order 1, and that their formal $q-$Borel transforms of order 1, $W_j(\tau) = \hat{\mathcal{B}}_q \hat{X}_j(\tau)$, which are holomorphic functions around 0, satisfy the assumptions of the Theorems for $(P2)$.

Let

$$W(\tau,z) = \sum_{h\geq 0} W_h(\tau) \frac{z^h}{h!}$$

be the solution of the Cauchy problem $(P2)$, corresponding to the initial conditions W_j, $0 \leq j \leq S-1$.

Main result

Theorem

1) The function $X(t,z)=\sum_{h\geq 0}\mathcal{L}_q^{\lambda}(W_h)(t)\frac{z^h}{h!}$ *is holomorphic in* $\mathcal{T}_{\lambda,q,\delta,r_0}\times\mathbb{C}$*.*

Main result

Theorem

1) The function $X(t,z)=\sum_{h\geq 0}\mathcal{L}_q^{\lambda}(W_h)(t)\frac{z^h}{h!}$ *is holomorphic in* $\mathcal{T}_{\lambda,q,\delta,r_0}\times\mathbb{C}$.

2) The function $X(t,z)$ *solves the Cauchy problem* $(P1)$.

Main result

Theorem

1) The function $X(t,z) = \sum_{h\geq 0} \mathcal{L}_q^{\lambda}(W_h)(t)\frac{z^h}{h!}$ *is holomorphic in* $\mathcal{T}_{\lambda,q,\delta,r_0} \times \mathbb{C}$.

2) The function $X(t,z)$ *solves the Cauchy problem* $(P1)$.

3) If $r_1 \geq 1$, *given* $R > 0$ *there exist constants* $\tilde{C} > 0$, $\tilde{D} > 0$ *such that for every* $n \in \mathbb{N}$, $n \geq 1$, *one has*

$$\left| X(t,z) - \sum_{h\geq 0}\sum_{m=0}^{n-1} f_{m,h} t^m \frac{z^h}{h!} \right| \leq \tilde{C}\tilde{D}^n \Gamma(\frac{r_1}{r_2}(n+1))|q|^{n(n-1)/2}|t|^n \qquad (7)$$

for every $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$, $z \in D(0,R)$.