Semi-formal Stokes phenomenon for non-linear systems of ordinary differential equations

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Nonlinear semi-formal systems

Throughout, let a system $z^{r+1}\,x' = \check{g}(z,x)$ be given, where the *Poincaré rank* r is a positive integer, $x = (x_1, \ldots, x_\nu)^T$ is a vector of dimension $\nu \geq 1$, and

$$\check{g}(z,x) \;=\; \sum_{|p|\geq 1} g_p(z) x^p \;=\; G(z)\,x \;+\; \sum_{|p|\geq 2} g_p(z) x^p\,.$$

As usual, $p = (p_1, \ldots, p_\nu)$ is a multi-index, $|p| := p_1 + \ldots + p_\nu$ denotes the *length of* p, and $x^p := x_1^{p_1} \cdot \ldots \cdot x_\nu^{p_\nu}$. We emphasize that we assume (formally) $\check{g}(z,0) = 0$, so $x(z) \equiv 0$ is a (formal) solution.

While the power series for $\check{g}(z,x)$ may diverge for every $x \neq 0$, we require the coefficients $g_p(z)$ to be holomorphic functions in a fixed disc $\mathcal{D}_\rho$ of radius $\rho > 0$ about the origin.

Normalizations

Assumptions: The corresponding linear system $z^{r+1}\,x' = G(z)\,x$ has a formal fundamental solution $\hat{X}(z) = \hat{F}(z)\,z^L\,e^{Q(z)}$, where

- $\hat{F}(z)$ is a formal matrix power series whose constant term is the identity matrix.
- $Q(z) = \operatorname{diag}\,[q_1(z), \ldots, q_\nu(z)]$ is a diagonal matrix of, not necessarily distinct, polynomials in the variable $w := z^{-1}$ without constant terms.
- $L = \Lambda + N$, with a diagonal matrix $\Lambda = \operatorname{diag}\,[\lambda_1, \ldots, \lambda_\nu]$ and a nilpotent matrix N that commutes with both Λ and $Q(z)$.

The formal series $\hat{F}(z)$ is known to be multi-summable in every non-singular multi-direction $d = (d_1, \ldots, d_r)$. Its corresponding sum shall be denoted as $F(z;d)$, and we set

$$X(z;d) \;:=\; F(z;d)\,z^L\,e^{Q(z)}\,.$$

Semi-formal solutions

Every system of the form considered here has a semi-formal solution of the form

$$\check{x}(z,c) \;=\; \sum_{|p|\geq 1} x_p(z)\, c^p \;=\; X(z)\, c \;+\; \sum_{|p|\geq 2} x_p(z)\, c^p\,,$$

where $X(z)$ is a fundamental solution of the corresponding linear system, and for every multi-index $p \neq 0$

$$z^{r+1}\, x_p'(z) \;=\; G(z)\, x_p(z) \;+\; \sum_{2\leq|q|\leq|p|} x_{qp}(z)\, g_q(z)\,,$$

with $x_{qp}(z)$ depending upon coefficients $x_q(z)$ with $|q| < |p|$ only.

Every other semi-formal solution is of the form $\check{x}(z, v(c))$, with suitable

$$v(c) \;=\; \sum_{|p|\geq 1} v_p\, c^p \;=\; V\, c \;+\; \sum_{|p|\geq 2} v_p\, c^p\,.$$

Semi-formal normal solutions

Set

$$q(z,p) \;=\; \sum_{j=1}^{\nu} p_j\, q_j(z)\;, \qquad \lambda(p) \;=\; r \;+\; \sum_{j=1}^{\nu} p_j\,(\lambda_j - r)\;.$$

Theorem 1. *For every non-singular multi-direction d there exists a unique semi-formal solution*

$$\check{x}(z,c;d) \;=\; \sum_{|p|\geq 1} x_p(z;d)\, c^p$$

whose linear part is the normal solution $X(z;d)$ of the corresponding linear equation. The coefficients $x_p(z;d)$ all are logarithmic-exponential expressions of type $(q(z,p),\lambda(p))$ which are recursively obtained by means of the identity

$$x_p(z;d) \;=\; X(z;d)\int X^{-1}(z;d)\Big[\sum_{2\leq|q|\leq|p|} x_{qp}(z;d)\, g_q(z)\Big]\,\frac{dz}{z^{r+1}}\;.$$

Stokes functions

Given two non-singular multi-directions d and $\tilde{d}$, there exists a unique formal expression $\check{v}(c;\tilde{d},d) = \sum_p v_p(\tilde{d},d)\, c^p$ for which

$$\check{x}(z,c;d) \;=\; \check{x}(z,\check{v}(c;\tilde{d},d);\tilde{d})\,.$$

This $\check{v}(c;\tilde{d},d)$ will be referred to as the (formal) *Stokes function* corresponding to d and $\tilde{d}$.

Theorem 2. *For any normalized system, all Stokes functions are invertible with respect to formal composition, and for any three non-singular multi-directions d, $\tilde{d}$, $\hat{d}$ we have $v(c;d,d) = c$, i. e. is the identity element of $\mathbb{G}(c)$, while $\check{v}(c;\tilde{d},d)$ and $\check{v}(c;d,\tilde{d})$ are inverse to one another. Moreover,*

$$\check{v}(c;\hat{d},d) \;=\; \check{v}(\check{v}(c;\tilde{d},d);\hat{d},\tilde{d})\,.$$

Altogether, this shows that the the Stokes functions form a subgroup of the group $\mathbb{G}(c)$.

This talk is based on a manuscript that may be downloaded via my homepage. Also compare the following

References

[1] Werner Balser. Formal solutions of non-linear systems of ordinary differential equations. In *The Stokes phenomenon and Hilbert's 16th problem (Groningen, 1995)*, pages 25–49. World Sci. Publ., River Edge, NJ, 1996.

[2] Werner Balser. Existence and structure of complete formal solutions of non-linear meromorphic systems of ordinary differential equations. *Asymptot. Anal.*, 15(3-4):261–282, 1997.