Fredholm determinants and (noncommutative) Painlevé II equation (... and others)

Marco Bertola

CRM-Mathematical Physics Laboratory.
Concordia University.

Joint with **M. Cafasso**. Based upon arXiv:1101.3997.
To appear in *Comm. Math. Phys.* and arXiv:1104.4940

Będlewo, August 9, 2011

Abstract

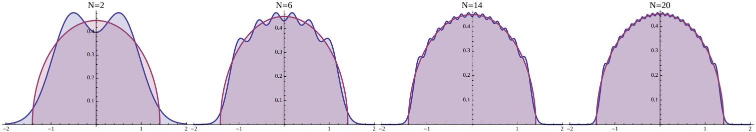
The connection between Fredholm determinants and Painleve' equations was observed in statistical mechanics and its most famous example is the Tracy-Widom distribution, connecting the distribution of the largest eigenvalue of a random matrix and the second Painleve' equation. I will briefly put into historical perspective the classification of ODEs of Painleve' and show how Fredholm determinants for matrix symbols are connected to a noncommutative version of the second Painleve' equation and a special solution which is pole free.

GUE gap probability

Consider an $N \times N$ Hermitean matrix with normal iid entries

$$\mathrm{d}\mu(M) := \frac{1}{C_N} \exp\left[-N \sum_{1\leqslant i\leqslant j\leqslant N} |M_{ij}|^2\right] \prod_{i=1}^{N} \mathrm{d}M_{ii} \prod_{1\leqslant i<j\leqslant N} \mathrm{d}\Re M_{ij}\, \mathrm{d}\Im M_{ij} \tag{1}$$

Here are some plots of the **density of eigenvalues** for different sizes of the random matrix M:



Gap probability

Tracy and Widom showed that the probability for the maximum eigenvalue λ_{max}

$$F_N(x) := \mathbb{P}(\lambda_{max} < x) \tag{2}$$

has the following limit

$$F_2(s) := \lim_{N\to\infty} F_N\left(\sqrt{2} + \frac{\sqrt{2}s}{2N^{\frac{2}{3}}}\right) = \det\left(\mathrm{Id} - K_{Ai,s}\right) \tag{3}$$

where $K_{\mathrm{Ai},s}$ is the integral operator with kernel

$$K_{\mathrm{Ai},s}(x,y) := \frac{\mathrm{Ai}(x+s)\mathrm{Ai}'(y+s) - \mathrm{Ai}(y+s)\mathrm{Ai}'(x+s)}{x-y} : L^2(\mathbb{R}_+) \circlearrowleft \tag{4}$$

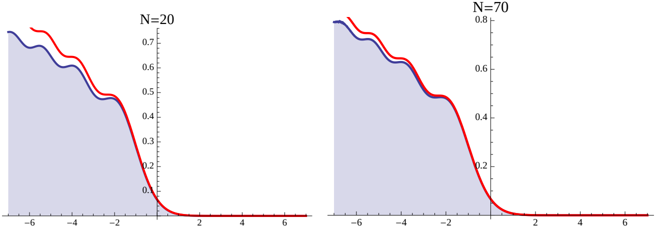


Figure: Comparison between the actual density and the Airy density (in red)

Gap probabilities and Painlevé.

GUE gap probability $\longrightarrow F_2(s)$ [Tracy-Widom '94];

$$F_2(s) = \det(Id - K_{\mathrm{Ai},s}) \quad \text{on } L^2(\mathbb{R}_+, \mathrm{d}x) \tag{5}$$

Tracy and Widom showed that

$$F_2(s) = \exp\left(-\int_s^\infty (x-s)u(x)^2 dx\right), \qquad u^2(s) = -\partial_s^2 \ln F_2(s) \tag{6}$$

$$u''(s) = 2u(s)^3 + su(s), \quad u(s) \sim \mathrm{Ai}(s), \quad s \longrightarrow +\infty. \tag{7}$$

This special solution to Painlevé II was studied by Hastings and McLeod and has the essential property that

The HMcL solution to PII has no poles on the real axis $s \in \mathbb{R}$.

Namely, the Fredholm determinant $F_2(s)$ never vanishes (is positive) for real values of s.

Gap probabilities and Painlevé.

GUE gap probability $\longrightarrow F_2(s)$ [Tracy-Widom '94];

$$F_2(s) = \det(Id - K_{\mathrm{Ai},s}) \quad \text{on } L^2(\mathbb{R}_+, \mathrm{d}x) \tag{5}$$

Tracy and Widom showed that

$$F_2(s) = \exp\left(-\int_s^\infty (x-s)u(x)^2 dx\right), \qquad u^2(s) = -\partial_s^2 \ln F_2(s) \tag{6}$$

$$u''(s) = 2u(s)^3 + su(s), \quad u(s) \sim \mathrm{Ai}(s), \quad s \longrightarrow +\infty. \tag{7}$$

This special solution to Painlevé II was studied by Hastings and McLeod and has the essential property that

The HMcL solution to PII has no poles on the real axis $s \in \mathbb{R}$.

Namely, the Fredholm determinant $F_2(s)$ never vanishes (is positive) for real values of s.

The kernel has the alternative representation

$$K_{\mathrm{Ai},s}(x,y) = \int_{R_+} \mathrm{Ai}(x+z+s)\mathrm{Ai}(y+z+s)\,\mathrm{d}z \tag{8}$$

which shows it to be the **square** of the following convolution operator

$$\begin{aligned} \mathcal{A}i_s :\ & L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+) \\ & f(y) \mapsto (\mathcal{A}i_s f)(x) := \int_{\mathbb{R}_+} \mathrm{Ai}(x+y+2s) f(y)\,\mathrm{d}y \end{aligned} \tag{9}$$

$$K_{\mathrm{Ai},s} = \mathcal{A}i_{\frac{s}{2}}^2 \tag{10}$$

GOE and Painlevé XXXIV

A similar procedure for **real-symmetric** matrices produces the GOE gap probability $\longrightarrow F_1(s)$: the original definition is in terms of Fredholm determinant of a matrix operator; [Ferrari and Spohn '05] showed that

$$F_1(s) = \det(Id - \mathcal{A}i_{s/2}) \quad \text{on } L^2(\mathbb{R}_+, \mathrm{d}x) \tag{11}$$

It was known since the work of Tracy and Widom that

$$F_1(s) = \exp\left(-\frac{1}{2}\int_s^\infty u(x)dx\right)\left(F_2(s)\right)^{\frac{1}{2}} \tag{12}$$

A similar representation for F_1 yields

$$F_1(s) \;= \exp\left(-\int_s^\infty (x-s)w(x)dx\right) \;, \qquad w(s) = -\partial_s^2 \ln F_1(s) \tag{13}$$

and now $w(s)$ solves a derivative version of Painlevé XXXIV [Clarkson et al. '99]

$$w'''(s) \;= 12w(s)w'(s) + 2w(s) + sw'(s), \quad w(s) \sim -\frac{1}{2}\mathrm{Ai}'(s), \quad s \longrightarrow +\infty. \tag{14}$$

Miura transformation

$$w(s) \;= \frac{1}{2}u^2(s) - \frac{1}{2}u'(s) \tag{15}$$

Convolution operators and their squares

We thus see that given a convolution operator $\mathcal{C}_s : \mathcal{L}^2((s,\infty))$, there is a relationship

$$\underset{\text{(KdV)}}{\det\left[Id + \mathcal{C}_s\right]} \quad \overset{\text{Miura}}{\longleftrightarrow} \quad \underset{\text{(mKdV)}}{\det\left[Id - \mathcal{C}_s^2\right]} \tag{16}$$

Goal

1. To relate any convolution operator on $L^2(\mathbb{R}_+)$ (with matrix symbol) and its square to an appropriate Riemann–Hilbert problem (**Its-Izergin-Korepin-Slavnov**).
2. Relate the two Fredholm determinants in eq. (16) via a noncommutative version of Miura's transformation.
3. Interesting (possibly!) example: noncommutative Painlevé II and XXXIV.

Convolution operators and their squares

We thus see that given a convolution operator $\mathcal{C}_s : \mathcal{L}^2((s,\infty))$, there is a relationship

$$\underset{\text{(KdV)}}{\det\left[Id + \mathcal{C}_s\right]} \quad \overset{\text{Miura}}{\longleftrightarrow} \quad \underset{\text{(mKdV)}}{\det\left[Id - \mathcal{C}_s^2\right]} \tag{16}$$

Goal

1. To relate any convolution operator on $L^2(\mathbb{R}_+)$ (with matrix symbol) and its square to an appropriate Riemann–Hilbert problem (**Its-Izergin-Korepin-Slavnov**).
2. Relate the two Fredholm determinants in eq. (16) via a noncommutative version of Miura's transformation.
3. Interesting (possibly!) example: noncommutative Painlevé II and XXXIV.

A short reminder about Fredholm determinants

Given an integral operator $\mathcal{K} : L^2(X, \mathrm{d}x) \to L^2(X, \mathrm{d}x)$ then

$$(\mathcal{K}f)(x) = \int_X K(x,y) f(y)\, \mathrm{d}y \tag{17}$$

$$\det(\mathrm{Id} - z\mathcal{K}) = 1 + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!} \int_{X^n} \det\left[K(x_j, x_k)\right]_{j,k\leqslant n} \mathrm{d}x_1 \dots \mathrm{d}x_n. \tag{18}$$

The series defines an entire function of z as long as $\mathcal{K}$ is **trace-class**. For sufficiently small z (less than the spectral radius of $\mathcal{K}$) then the following can be used equivalently

$$\ln \det(Id - z\mathcal{K}) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}\mathcal{K}^n \tag{19}$$

If $\mathcal{K}$ is not trace-class but Hilbert-Schmidt (or in some other trace-ideal [Simon]) then one can define a **regularized** Fredholm determinant (**Carleman determinant**)

$$\det_2(\mathrm{Id} - z\mathcal{K}) = 1 + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!} \int_{X^n} \det\left[(1-\delta_{ij})K(x_j, x_k)\right]_{j,k\leqslant n} \mathrm{d}x_1 \dots \mathrm{d}x_n. \tag{20}$$

A short reminder about Fredholm determinants

Given an integral operator $\mathcal{K} : L^2(X,\,\mathrm{d}x) \to L^2(X,\,\mathrm{d}x)$ then

$$(\mathcal{K}f)(x) = \int_X K(x,y)f(y)\,\mathrm{d}y \tag{17}$$

$$\det(\mathrm{Id} - z\mathcal{K}) = 1 + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!} \int_{X^n} \det\left[K(x_j, x_k)\right]_{j,k\leqslant n} \mathrm{d}x_1 \ldots \mathrm{d}x_n. \tag{18}$$

The series defines an entire function of z as long as $\mathcal{K}$ is **trace-class**. For sufficiently small z (less than the spectral radius of $\mathcal{K}$) then the following can be used equivalently

$$\ln\det(Id - z\mathcal{K}) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}\mathcal{K}^n \tag{19}$$

If $\mathcal{K}$ is not trace-class but Hilbert-Schmidt (or in some other trace-ideal [Simon]) then one can define a **regularized** Fredholm determinant (**Carleman determinant**)

$$\det_2(\mathrm{Id} - z\mathcal{K}) = 1 + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!} \int_{X^n} \det\left[(1-\delta_{ij})K(x_j, x_k)\right]_{j,k\leqslant n} \mathrm{d}x_1 \ldots \mathrm{d}x_n. \tag{20}$$

IIKS (Its-Izergin-Korepin-Slavnov) theory in a nutshell

Let $N : L^2(\Sigma, \mathbb{C}^n)$ with kernel given by ("**integrable form**")

$$N(\lambda, \mu) := \frac{\mathbf{f}^T(\lambda)\mathbf{g}(\mu)}{\lambda - \mu} \qquad \mathbf{f}^T(\lambda)\mathbf{g}(\lambda) \equiv \mathbf{0}\ , \quad \mathbf{f}, \mathbf{g} : \Sigma \to Mat(q \times n) \tag{21}$$

The **resolvent** operator is also of integrable form:

$$\mathcal{R}(\lambda, \mu) = N \circ (\mathrm{Id} - N)^{-1}(\lambda, \mu) = \frac{\mathbf{f}^T(\lambda)\Theta^T(\lambda)\Theta^{-T}(\mu)\mathbf{g}(\mu)}{\lambda - \mu} \tag{22}$$

where $\Theta(\lambda)$ is the $q \times q$ matrix bounded solution of the following Riemann–Hilbert problem

$$\begin{aligned} \Theta(\lambda)_+ &= \Theta(\lambda)_- \left(\mathbf{1}_q - 2i\pi \mathbf{f}(\lambda)\mathbf{g}^T(\lambda)\right) \\ \Theta(\lambda) &= \mathbf{1}_q + \mathcal{O}(\lambda^{-1})\ , \quad \lambda \to \infty \end{aligned} \tag{23}$$

Furthermore the solution of the RHP (23) exists **if and only if** $\det(\mathrm{Id} - N) \neq 0$.

IIKS (Its-Izergin-Korepin-Slavnov) theory in a nutshell

Let $N : L^2(\Sigma, \mathbb{C}^n)$ with kernel given by ("**integrable form**")

$$N(\lambda, \mu) := \frac{\mathbf{f}^T(\lambda)\mathbf{g}(\mu)}{\lambda - \mu} \qquad \mathbf{f}^T(\lambda)\mathbf{g}(\lambda) \equiv \mathbf{0} \ , \quad \mathbf{f}, \mathbf{g} : \Sigma \to Mat(q \times n) \tag{21}$$

The **resolvent** operator is also of integrable form:

$$\mathcal{R}(\lambda, \mu) = N \circ (\mathrm{Id} - N)^{-1}(\lambda, \mu) = \frac{\mathbf{f}^T(\lambda)\Theta^T(\lambda)\Theta^{-T}(\mu)\mathbf{g}(\mu)}{\lambda - \mu} \tag{22}$$

where $\Theta(\lambda)$ is the $q \times q$ matrix bounded solution of the following Riemann–Hilbert problem

$$\begin{aligned} \Theta(\lambda)_+ &= \Theta(\lambda)_- \left(\mathbf{1}_q - 2i\pi\mathbf{f}(\lambda)\mathbf{g}^T(\lambda)\right) \\ \Theta(\lambda) &= \mathbf{1}_q + \mathcal{O}(\lambda^{-1}) \ , \quad \lambda \to \infty \end{aligned} \tag{23}$$

Furthermore the solution of the RHP (23) exists **if and only if** $\det(\mathrm{Id} - N) \neq 0$.

Why is this helpful?

- The RHP typically has jumps which are conjugated to constant jumps by entire matrices $\Rightarrow$ the solution of the RHP solves an ODE with meromorphic coefficients;
- The deformation of the kernel w.r.t. parameters is (typically) **isomonodromic** $\Rightarrow$ use Jimobo-Miwa-Ueno theory of isomonodromic deformations;
- the Fredholm determinant is (in interesting cases) the **isomonodromc tau function** of JMU;
- derive ODEs (PDEs) for the Fredholm determinant (Painlevé property).

Goal

To show that Fredholm determinants of convolution (and possibly other) kernels without integrable form are *equal to* Fred. dets. of **integrable** kernels. Derive ODE/PDEs/Painlevé property, Lax representation etc.

Equivalence of determinants

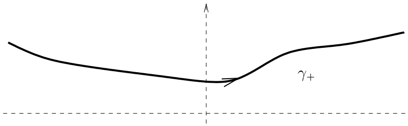
Let $\mathcal{C}$ be the **matrix** convolution operator on $L^2(\mathbb{R}_+)$ with symbol

$$\mathbf{C}_s(z) := \mathbf{C}(z+s) = -i\int_{\gamma_+} \mathrm{e}^{i(z+s)\mu}\mathbf{r}_0(\mu)\,\mathrm{d}\mu \tag{24}$$

$$\mathbf{r}(\mu,s) := \mathrm{e}^{i\mu s}\mathbf{r}_0(\mu)\ , \qquad \mathbf{r_0}(\mu) := E_1(\mu)E_2^T(\mu) \tag{25}$$

$$\mathrm{e}^{i\mu s/2}E_j(\mu) \in L^2 \cap L^\infty(\gamma_+, Mat(r\times p)) \tag{26}$$

Here γ_+ is a (collection of) contour(s) in the upper half plane.



Theorem (B.-Cafasso, 2011)

The two Fredholm determinants below (exist!) are equal

$$\det\left[\mathrm{Id}_{L^2(\mathbb{R}_+,\mathbb{C}^r)} + \mathcal{C}_s\right] = \det\left[\mathrm{Id}_{L^2(\gamma_+,\mathbb{C}^p)} + \mathcal{K}_s\right] \tag{27}$$

with $\mathcal{K}_s : L^2(\gamma_+, \mathbb{C}^p) \circlearrowleft$ *having kernel*

$$\mathcal{K}_s(\lambda,\mu) = \frac{\mathrm{e}^{\frac{i(\lambda+\mu)s}{2}} E_1^T(\lambda) E_2(\mu)}{\lambda+\mu} . \tag{28}$$

We shall study kernels of the form $\mathcal{K}$.

Sketch of Proof

By Paley–Wiener theorem, Fourier transform isomorphically maps

$$\mathcal{T} : L^2(\mathbb{R}_+, \mathbb{C}^r) \cong \mathcal{H}^2_r := \mathcal{H}^2 \otimes \mathbb{C}^r \tag{29}$$

with $\mathcal{H}^2$ the Hardy space of the upper half plane.

$$\begin{aligned}\psi(x) := (\mathcal{C}\varphi)(x) = \int_0^\infty C(x+y)\varphi(y)\,\mathrm{d}y = -i\int_0^\infty \mathrm{d}y \int_{\gamma_+} \mathrm{d}\xi\, \mathrm{e}^{i(x+y)\xi}\mathbf{r}(\xi)\varphi(y) = \\ = -i\sqrt{2\pi}\int_{\gamma_+}\mathrm{d}\xi\,\mathrm{e}^{ix\xi}\mathbf{r}(\xi)(\mathcal{T}\varphi)(\xi)\end{aligned}$$

Then, Fourier transforming the function ψ...

$$\begin{aligned}(\mathcal{T}\psi)(\lambda) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{i\lambda x}\psi(x)\,\mathrm{d}x = -i\int_0^\infty \mathrm{d}x\,\mathrm{e}^{i\lambda x}\int_{\gamma_+}\mathrm{d}\xi\,\mathrm{e}^{ix\xi}\mathbf{r}(\xi)(\mathcal{T}\varphi)(\xi) = \\ = \int_{\gamma_+}\mathrm{d}\xi\,\frac{\mathbf{r}(\xi)}{\lambda+\xi}(\mathcal{T}\varphi)(\xi) = \int_{\gamma_+}\mathrm{d}\xi\,\frac{\mathbf{r}(\xi)}{\lambda+\xi}(\mathcal{T}\varphi)(\xi)\,.\end{aligned} \tag{30}$$

Sketch of Proof

By Paley–Wiener theorem, Fourier transform isomorphically maps

$$\mathcal{T} : L^2(\mathbb{R}_+, \mathbb{C}^r) \cong \mathcal{H}^2_r := \mathcal{H}^2 \otimes \mathbb{C}^r \tag{29}$$

with $\mathcal{H}^2$ the Hardy space of the upper half plane.

$$\begin{aligned}\psi(x) := (\mathcal{C}\varphi)(x) = \int_0^\infty C(x+y)\varphi(y)\,\mathrm{d}y = -i\int_0^\infty \mathrm{d}y \int_{\gamma_+} \mathrm{d}\xi\, \mathrm{e}^{i(x+y)\xi}\mathbf{r}(\xi)\varphi(y) = \\ = -i\sqrt{2\pi}\int_{\gamma_+}\mathrm{d}\xi\, \mathrm{e}^{ix\xi}\mathbf{r}(\xi)(\mathcal{T}\varphi)(\xi)\end{aligned}$$

$$\begin{aligned}(\mathcal{T}\psi)(\lambda) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{i\lambda x}\psi(x)\,\mathrm{d}x = -i\int_0^\infty \mathrm{d}x\,\mathrm{e}^{i\lambda x}\int_{\gamma_+}\mathrm{d}\xi\,\mathrm{e}^{ix\xi}\mathbf{r}(\xi)(\mathcal{T}\varphi)(\xi) = \\ = \int_{\gamma_+}\mathrm{d}\xi\,\frac{\mathbf{r}(\xi)}{\lambda+\xi}(\mathcal{T}\varphi)(\xi) = \int_{\gamma_+}\mathrm{d}\xi\,\frac{\mathbf{r}(\xi)}{\lambda+\xi}(\mathcal{T}\varphi)(\xi)\,.\end{aligned} \tag{30}$$

$$(\mathcal{T}\psi)(\lambda) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{i\lambda x}\psi(x)\,\mathrm{d}x = -i\int_0^\infty \mathrm{d}x\,\mathrm{e}^{i\lambda x}\int_{\gamma_+}\mathrm{d}\xi\,\mathrm{e}^{ix\xi}\mathbf{r}(\xi)(\mathcal{T}\varphi)(\xi) =$$

$$= \int_{\gamma_+}\mathrm{d}\xi\,\frac{\mathbf{r}(\xi)}{\lambda+\xi}(\mathcal{T}\varphi)(\xi) = \int_{\gamma_+}\mathrm{d}\xi\,\frac{\mathbf{r}(\xi)}{\lambda+\xi}(\mathcal{T}\varphi)(\xi)\;. \tag{31}$$

We note that for a function in $\mathcal{H}_r^2$ like $f(\mu) := \mathcal{T}\varphi(\mu)$, the evaluation at a point $\xi \in \mathbb{C}_+$ can be written as

$$f(\xi) = \int_{\mathbb{R}} f(\mu)\frac{\mathrm{d}\mu}{2i\pi(\mu-\xi)} \quad \text{(Cauchy's theorem)}, \tag{32}$$

which is Cauchy's theorem. Thus

$$(\mathcal{T}\psi)(\lambda) = \frac{1}{2i\pi}\int_{\gamma_+}\mathrm{d}\xi\,\frac{\mathbf{r}(\xi)}{\lambda+\xi}\int_{\mathbb{R}}\frac{\mathrm{d}\mu}{\mu-\xi}(\mathcal{T}\varphi)(\mu) \tag{33}$$

We shall thus define

$$\widehat{\mathcal{K}}^T := \mathcal{T}^{-1}\mathcal{C}\mathcal{T} \tag{34}$$

$$(\mathcal{T}\psi)(\lambda) = \widehat{\mathcal{K}}^T(\mathcal{T}\varphi)(\lambda) = \frac{1}{2i\pi}\int_{\gamma_+}\mathrm{d}\xi\,\frac{\mathbf{r}(\xi)}{\lambda+\xi}\int_{\mathbb{R}}\frac{\mathrm{d}\mu}{\mu-\xi}(\mathcal{T}\varphi)(\mu) \tag{35}$$

(the reason for the transposition is solely for later convenience) with kernel given by

$$\widehat{\mathcal{K}}f(\lambda) = \int_{\mathbb{R}}\mathrm{d}\mu\int_{\gamma_+}\mathrm{d}\xi\,\frac{\mathbf{r}^T(\xi)}{\lambda-\xi}\,\frac{f(\mu)}{2i\pi(\mu+\xi)} \tag{36}$$

We use the factorization of $\mathbf{r}$:

$$\widehat{\mathcal{K}}f(\lambda) = \int_{\mathbb{R}}\mathrm{d}\mu\int_{\gamma_+}\mathrm{d}\xi\,\frac{E_2(\xi)}{\lambda-\xi}\,\frac{E_1^T(\xi)f(\mu)\,\mathrm{d}\mu}{2i\pi(\mu+\xi)} = \mathcal{C}_2\circ\mathcal{C}_1 f(\lambda)\ . \tag{37}$$

Both $\mathcal{C}_j$ are Hilbert Schmidt in $L^2(\mathbb{R}\cup\gamma_+,\mathbb{C}^{r+p})$ because

$$\int_{\gamma_+}|\mathrm{d}\xi|\int_{\mathbb{R}}|\mathrm{d}\mu|\,\frac{\mathrm{Tr}\,\left(E_j^\dagger(\xi)E_j(\xi)\right)}{|\xi\pm\mu|^2} < +\infty \tag{38}$$

Thus $\widehat{\mathcal{K}}:\mathcal{H}_r^2\to\mathcal{H}_r^2$ is trace class, so is $\mathcal{C}$ and their determinants are the same.

We shall thus define

$$\widehat{\mathcal{K}}^T := \mathcal{T}^{-1}\mathcal{C}\mathcal{T} \tag{34}$$

$$(\mathcal{T}\psi)(\lambda) = \widehat{\mathcal{K}}^T(\mathcal{T}\varphi)(\lambda) = \frac{1}{2i\pi}\int_{\gamma_+}\mathrm{d}\xi\,\frac{\mathbf{r}(\xi)}{\lambda+\xi}\int_{\mathbb{R}}\frac{\mathrm{d}\mu}{\mu-\xi}(\mathcal{T}\varphi)(\mu) \tag{35}$$

(the reason for the transposition is solely for later convenience) with kernel given by

$$\widehat{\mathcal{K}}f(\lambda) = \int_{\mathbb{R}}\mathrm{d}\mu\int_{\gamma_+}\mathrm{d}\xi\,\frac{\mathbf{r}^T(\xi)}{\lambda-\xi}\,\frac{f(\mu)}{2i\pi(\mu+\xi)} \tag{36}$$

We use the factorization of $\mathbf{r}$:

$$\widehat{\mathcal{K}}f(\lambda) = \int_{\mathbb{R}}\mathrm{d}\mu\int_{\gamma_+}\mathrm{d}\xi\,\frac{E_2(\xi)}{\lambda-\xi}\,\frac{E_1^T(\xi)f(\mu)\,\mathrm{d}\mu}{2i\pi(\mu+\xi)} = \mathcal{C}_2\circ\mathcal{C}_1 f(\lambda)\ . \tag{37}$$

Both $\mathcal{C}_j$ are Hilbert Schmidt in $L^2(\mathbb{R}\cup\gamma_+,\mathbb{C}^{r+p})$ because

$$\int_{\gamma_+}|\mathrm{d}\xi|\int_{\mathbb{R}}|\mathrm{d}\mu|\,\frac{\mathrm{Tr}\left(E_j^\dagger(\xi)E_j(\xi)\right)}{|\xi\pm\mu|^2} < +\infty \tag{38}$$

Thus $\widehat{\mathcal{K}}:\mathcal{H}^2_r\to\mathcal{H}^2_r$ is trace class, so is $\mathcal{C}$ and their determinants are the same.

We shall thus define

$$\widehat{\mathcal{K}}^T := \mathcal{T}^{-1}\mathcal{C}\mathcal{T} \tag{34}$$

$$(\mathcal{T}\psi)(\lambda) = \widehat{\mathcal{K}}^T(\mathcal{T}\varphi)(\lambda) = \frac{1}{2i\pi}\int_{\gamma_+}\mathrm{d}\xi\,\frac{\mathbf{r}(\xi)}{\lambda+\xi}\int_{\mathbb{R}}\frac{\mathrm{d}\mu}{\mu-\xi}(\mathcal{T}\varphi)(\mu) \tag{35}$$

(the reason for the transposition is solely for later convenience) with kernel given by

$$\widehat{\mathcal{K}}f(\lambda) = \int_{\mathbb{R}}\mathrm{d}\mu\int_{\gamma_+}\mathrm{d}\xi\,\frac{\mathbf{r}^T(\xi)}{\lambda-\xi}\,\frac{f(\mu)}{2i\pi(\mu+\xi)} \tag{36}$$

We use the factorization of $\mathbf{r}$:

$$\widehat{\mathcal{K}}f(\lambda) = \int_{\mathbb{R}}\mathrm{d}\mu\int_{\gamma_+}\mathrm{d}\xi\,\frac{E_2(\xi)}{\lambda-\xi}\,\frac{E_1^T(\xi)f(\mu)\,\mathrm{d}\mu}{2i\pi(\mu+\xi)} = \mathcal{C}_2\circ\mathcal{C}_1 f(\lambda)\ . \tag{37}$$

Both $\mathcal{C}_j$ are Hilbert Schmidt in $L^2(\mathbb{R}\cup\gamma_+,\mathbb{C}^{r+p})$ because

$$\int_{\gamma_+}|\mathrm{d}\xi|\int_{\mathbb{R}}|\mathrm{d}\mu|\,\frac{\mathrm{Tr}\,\left(E_j^\dagger(\xi)E_j(\xi)\right)}{|\xi\pm\mu|^2} < +\infty \tag{38}$$

Thus $\widehat{\mathcal{K}}:\mathcal{H}_r^2\to\mathcal{H}_r^2$ **is trace class, so is $\mathcal{C}$ and their determinants are the same.**

We now use

$$\det\left(Id_{\mathcal{H}_1} + \mathcal{C}_2 \circ \mathcal{C}_1\right) = \det\left(Id_{\mathcal{H}_2} + \mathcal{C}_1 \circ \mathcal{C}_2\right) \ . \tag{39}$$

with

$$(\mathcal{C}_1 \circ \mathcal{C}_2 f)(\mu) = \frac{E_1^T(\mu)}{2i\pi} \int_{\mathbb{R}} \mathrm{d}\xi \int_{\gamma_+} \mathrm{d}\lambda \frac{E_2(\lambda) f(\lambda)}{(\xi - \lambda)(\xi + \mu)} \tag{40}$$

$$\text{(Cauchy)} = E_1^T(\mu) \int_{\gamma_+} \mathrm{d}\lambda \frac{E_2(\lambda) f(\lambda)}{\lambda + \mu} =: (\mathcal{K} f)(\mu) \tag{41}$$

This ends the proof.

Resolvents

We want to find the (kernels of the) resolvent operators

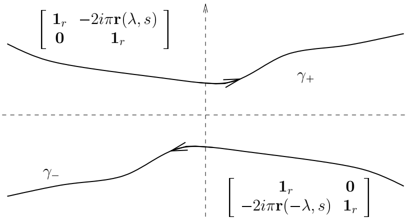
$$\mathcal{S} := -\mathcal{K} \circ (\mathrm{Id}_{\gamma_+} + \mathcal{K})^{-1} , \qquad \mathcal{R} := \mathcal{K}^2 \circ (\mathrm{Id}_{\gamma_+} - \mathcal{K}^2)^{-1} \tag{42}$$

Theorem (B.-Cafasso 2011)

$$\mathcal{S}(\lambda, \mu) = \frac{2\mu \left[E_1^T(\lambda), \mathbf{0}_{p\times r}\right] \Gamma^T(\lambda)\Gamma^{-T}(\mu) \begin{bmatrix} \mathbf{0}_{r\times p} \\ E_2(\mu) \end{bmatrix}}{\lambda^2 - \mu^2} \tag{43}$$

$$\mathcal{R}(\lambda, \mu) = \left[E_1^T(\lambda), \mathbf{0}_{p\times r}\right] \frac{\Xi^T(\lambda)\Xi^{-T}(\mu)}{\lambda - \mu} \begin{bmatrix} \mathbf{0}_{r\times p} \\ E_2(\mu) \end{bmatrix} \tag{44}$$

where $\Gamma(\lambda)$, $\Xi(\lambda)$ are $2r \times 2r$ matrix solutions of two (related) Riemann–Hilbert problems on $\gamma_+ \cup \gamma_-$ ($\gamma_- = -\gamma_+$) described below.



$$\mathbf{r}(\lambda,s) := \mathrm{e}^{i\lambda s}E_1(\lambda)E_2^T(\lambda)$$

Problem 1	**Problem 2**
$\Xi_+(\lambda) = \Xi_-(\lambda)M(\lambda)$ $\Xi(\lambda) = \mathbf{1}_{2r} + \frac{\Xi_1}{\lambda} + \dots$	$\Gamma_+(\lambda) = \Gamma_-(\lambda)M(\lambda)$ $\Gamma(\lambda) = \begin{bmatrix} \mathbf{1}_r & \mathbf{1}_r \\ -i\lambda\mathbf{1}_r & i\lambda\mathbf{1}_r \end{bmatrix}\left(\mathbf{1}_{2r} + \frac{Q\otimes\sigma_3}{\lambda} + \dots\right)$ $\Gamma(\lambda)\begin{bmatrix} \mathbf{1}_r & \mathbf{1}_r \\ -i\lambda\mathbf{1}_r & i\lambda\mathbf{1}_r \end{bmatrix}^{-1} = \mathcal{O}(1)\ \lambda \to 0$ $\Gamma(\lambda) = \widehat{\sigma}_1\Gamma(-\lambda)\widehat{\sigma}_1$

(45)

Idea of proof:
Reduce both $\mathcal{K}$ and $\mathcal{K}^2$ to integrable form

$$\mathcal{K}(\lambda,\mu) := \frac{E_1(\lambda)^T E_2(\mu)}{\lambda+\mu} = \frac{(\lambda-\mu)E_1(\lambda)^T E_2(\mu)}{\lambda^2-\mu^2} \tag{46}$$

so that it is of the IIKS form in the variable λ^2
For $\mathcal{K}^2$, setting $\gamma_- = -\gamma_+$ and $\tilde{f}(\lambda) := f(-\lambda)$:

$$\mathcal{K}^2(\lambda,\mu) = E_1^T(\lambda)\left(\int_{\gamma_+} \frac{E_2(\xi)E_1^T(\xi)\,\mathrm{d}\xi}{(\lambda+\xi)(\xi+\mu)}\right) E_2(\mu) = \tag{47}$$

$$= E_1^T(\lambda)\left(\int_{\gamma_-} \frac{\tilde{E}_2(\xi)\tilde{E}_1^T(\xi)}{(\lambda-\xi)(\xi-\mu)} d\xi\right) E_2(\mu) = (\mathcal{G}\circ\mathcal{F})(\lambda,\mu) \tag{48}$$

It is now manifested as the **composition** of two integrable kernels between $L^2(\gamma_-) \leftrightarrow L^2(\gamma_+)$. Then one uses the identity (need to verify both $\mathcal{F},\mathcal{G}$ of trace class)

$$\det(Id - \mathcal{G}\circ\mathcal{F}) = \det\left(Id - \left[\begin{array}{c|c} 0 & \mathcal{F} \\ \hline \mathcal{G} & 0 \end{array}\right]\right) \tag{49}$$

etc. etc. $\square$

Relationships between problems 1 and 2

Proposition (B.-Cafasso 2011)

- Ξ *exists* $\Rightarrow \Gamma$ *exists; moreovoer*

$$\Gamma(\lambda) = \left[\begin{array}{cc} \mathbf{1}_r & \mathbf{1}_r \\ -i\lambda\mathbf{1}_r - 2\beta_1 & i\lambda\mathbf{1}_r - 2\beta_2 \end{array}\right]\Xi(\lambda)\ , \quad \Xi_1 = \alpha_1 \otimes \sigma_3 + \beta_2 \otimes \sigma_2 \tag{50}$$

- Ξ *exists* $\Leftrightarrow \Gamma$ *exists* **and**

$$\det \Gamma_{11}(0) \neq 0 \text{ where } \Gamma(\lambda) := \left[\begin{array}{c|c} \Gamma_{11}(\lambda) & \Gamma_{12}(\lambda) \\ \hline \Gamma_{21}(\lambda) & \Gamma_{22}(\lambda) \end{array}\right] \tag{51}$$

The logic behind the proposition

Ξ exists iff $\det(\mathrm{Id} - \mathcal{K}^2) \neq 0$, but

$$\det(\mathrm{Id} - \mathcal{K}^2) = \det(\mathrm{Id} - \mathcal{K}) \det(\mathrm{Id} + \mathcal{K}) \tag{52}$$

and thus Ξ may fail to exist because either determinants $\det(\mathrm{Id} \pm \mathcal{K}) = 0$. On the other hand for the existence of Γ it is sufficient $\det(\mathrm{Id} + \mathcal{K}) \neq 0$

Relationships between problems 1 and 2

Proposition (B.-Cafasso 2011)

- Ξ *exists* $\Rightarrow \Gamma$ *exists; moreovoer*

$$\Gamma(\lambda) = \left[\begin{array}{cc} \mathbf{1}_r & \mathbf{1}_r \\ -i\lambda\mathbf{1}_r - 2\beta_1 & i\lambda\mathbf{1}_r - 2\beta_2 \end{array}\right]\Xi(\lambda)\ , \quad \Xi_1 = \alpha_1 \otimes \sigma_3 + \beta_2 \otimes \sigma_2 \tag{50}$$

- Ξ *exists* $\Leftrightarrow \Gamma$ *exists* **and**

$$\det \Gamma_{11}(0) \neq 0 \textit{ where } \Gamma(\lambda) := \left[\begin{array}{c|c} \Gamma_{11}(\lambda) & \Gamma_{12}(\lambda) \\ \hline \Gamma_{21}(\lambda) & \Gamma_{22}(\lambda) \end{array}\right] \tag{51}$$

The logic behind the proposition

Ξ exists iff $\det(\mathrm{Id} - \mathcal{K}^2) \neq 0$, but

$$\det(\mathrm{Id} - \mathcal{K}^2) = \det(\mathrm{Id} - \mathcal{K}) \det(\mathrm{Id} + \mathcal{K}) \tag{52}$$

and thus Ξ may fail to exist because either determinants $\det(\mathrm{Id} \pm \mathcal{K}) = 0$. On the other hand for the existence of Γ it is sufficient $\det(\mathrm{Id} + \mathcal{K}) \neq 0$

Matrix Miura relation

The two solutions have expansions

$$\Xi(\lambda, s) = \mathbf{1}_{2r} + \frac{V(s)\otimes\sigma_3 + U(s)\otimes\sigma_2}{\lambda} + \dots\ , \tag{53}$$

$$\Gamma(\lambda) = \begin{bmatrix} \mathbf{1}_r & \mathbf{1}_r \\ -i\lambda\mathbf{1}_r & i\lambda\mathbf{1}_r \end{bmatrix} \left(\mathbf{1}_{2r} + \frac{Q(s)\otimes\sigma_3}{\lambda} + \dots\right) \tag{54}$$

$$\partial_s V(s) = -2iU^2(s) \quad Q(s) = V(s) - iU(s) \tag{55}$$

Matrix Miura relation

$$\partial_s Q = -2iU^2(s) - i\partial_s U(s)\ . \tag{56}$$

Fredholm determinants and RHPs: variational formulæ

Let us denote by

$$\tau_\Gamma := \det\left[\mathrm{Id} + \mathcal{K}\right] , \qquad \tau_\Xi := \det\left[\mathrm{Id} - \mathcal{K}^2\right] \tag{57}$$

and let ∂ denote any variation of the symbol $\mathbf{r}(\lambda) := E_1(\lambda) E_1^T(\lambda)$. Then

Theorem (B.-Cafasso 2011)

The variational formulæ hold

$$\partial \ln \tau_\Gamma = \frac{1}{2} \int_{\gamma_+ \cup \gamma_-} \mathrm{Tr}\left(\Gamma_-^{-1} \Gamma'_- \partial M M^{-1}\right) \frac{\mathrm{d}\lambda}{2i\pi} \tag{58}$$

$$\partial \ln \tau_\Xi = \int_{\gamma_+ \cup \gamma_-} \mathrm{Tr}\left(\Xi_-^{-1} \Xi'_- \partial M M^{-1}\right) \frac{\mathrm{d}\lambda}{2i\pi} \tag{59}$$

$$M := \mathbf{1} + 2i\pi\left(\mathbf{r}(\lambda) \otimes \sigma_+ \chi_{\gamma_+} + \mathbf{r}(-\lambda) \otimes \sigma_- \chi_{\gamma_-}\right) \tag{60}$$

Furthermore the respective problems have solutions if and only if $\tau_\Gamma \neq 0$ ($\tau_\Xi \neq 0$).

Special case: for $\partial = \frac{\partial}{\partial s}$

Proposition

$$\partial_s \ln \tau_\Gamma = -\frac{1}{2} \operatorname*{res}_{\infty} \mathrm{Tr}\left(\Gamma^{-1}(\lambda)\Gamma'(\lambda) i\lambda\sigma_3 \otimes \mathbf{1}_r\right) \mathrm{d}\lambda = -2i\mathrm{Tr}\, Q(s) \tag{61}$$

$$\partial_s \ln \tau_\Xi = -\operatorname*{res}_{\infty} \mathrm{Tr}\left(\Xi^{-1}(\lambda)\Xi'(\lambda) i\lambda\sigma_3 \otimes \mathbf{1}_r\right) \mathrm{d}\lambda = -i\mathrm{Tr}\, V(s) \tag{62}$$

where the residues are understood as formal residues, or the coefficient of λ^{-1} in the expansion at infinity.

For $r = 1$ (scalar kernels) then one has the (standard, integrated) **Miura relation** between the determinants

$$(\partial_s \ln \tau_\Xi - 2\partial_s \ln \tau_\Gamma)^2 = -\partial_s^2 \ln \tau_\Xi \tag{63}$$

Noncommutative Painlevé II

In the study of noncommutative Toda equations, Retakh and Rubtsov defined it on a **noncommutative**, **associative unital** algebra $\mathcal{A}$ with **derivation** $\mathbf{D}$ and distinguished element $\mathrm{s} \in \mathbf{A}$ with the property

$$\mathbf{D}\mathbf{s} = \mathbf{1} \tag{64}$$

Then

NC-PII [Retakh-Rubtsov '10]

$$\mathbf{D}^2 U = 4\{\mathbf{s}, U\} + 8U^3 + \alpha \ , \quad \alpha \in Z(\mathcal{A}) \ . \tag{65}$$

They provided (matrix) solutions in terms of *quasideterminants* (i.e. Schur complements). Previously, attempts at defining noncommutative versions were in [Balandin Sokolov '98] but with s in the center.

Problem

No Lax representation was given (and no isomonodromic system).

NC-PII and Fredholm determinants

We consider the example of the **matrix Airy convolution kernel** on $L^2(\mathbb{R}_+, \mathbb{C}^r)$ defined as:

$$(\mathcal{A}i_{\vec{s}}\, f)(x) := \int_{\mathbb{R}_+} \mathbf{Ai}(x+y;\vec{s}) f(y)\,\mathrm{d}y \tag{66}$$

$$\mathbf{Ai}(x;\vec{s}) := \int_{\gamma_+} \mathrm{e}^{\theta(\mu)} C \mathrm{e}^{\theta(\mu)} \mathrm{e}^{ix\mu} \frac{\mathrm{d}\mu}{2\pi} = \left[c_{jk}\mathrm{Ai}(x+s_j+s_k)\right]_{j,k} \tag{67}$$

$$\theta := \frac{i\mu^3}{6}\mathbf{1}_r + \begin{bmatrix} is_1\mu & & & \\ & is_2\mu & & \\ & & \ddots & \\ & & & is_r\mu \end{bmatrix} = \frac{i\mu^3}{6}\mathbf{1} + i\mathbf{s}\mu \tag{68}$$

$$\mathbf{s} := \mathrm{diag}(s_1, s_2, \dots, s_r) \tag{69}$$

where $C \in Mat(r \times r, \mathbb{C})$ is an arbitrary constant.

Theorem

Suppose $C = C^\dagger$ is a Hermitean matrix; then the solution to Problem 1 for Ξ with

$$\mathbf{r}(\lambda) = -\frac{1}{2i\pi} \mathrm{e}^{\theta(\lambda)} C \mathrm{e}^{\theta(\lambda)} \tag{70}$$

exists for all values of $\vec{s} \in \mathbb{R}^r$ if and only if the eigenvalues of C are all in the interval $[-1,1]$. If C is an arbitrary complex matrix with singular values in $[0,1]$ then the solution still exists for all $\vec{s} \in \mathbb{R}^r$.

The singular values of a matrix are the square roots of the eigenvalues of $C^\dagger C$.
The matrix $\Xi(\lambda)$ has expansion at infinity

$$\Xi(\lambda; \mathbf{s}) = \mathbf{1}_{2r} + \frac{1}{\lambda}\left[\begin{array}{c|c} V(\mathbf{s}) & iU(\mathbf{s}) \\ \hline -iU(\mathbf{s}) & V(\mathbf{s}) \end{array}\right] + \mathcal{O}(\lambda^{-2}) \tag{71}$$

and the matrix $U(\mathbf{s})$ solves noncommutative PII. In particular this provides a linear auxiliary system for ncPII.

Lax (isomonodromic) pair

Lemma

The compatibility of the $2r \times 2r$ isomonodromy system

$$\partial_{s_j} \Psi(\lambda, \mathbf{s}) = \mathcal{S}_j(\lambda, \mathbf{s}) \Psi(\lambda, \mathbf{s}) \tag{72}$$

$$\mathcal{S}_j(\lambda, \mathbf{s}) = i\lambda\, \mathrm{e}_j \otimes \sigma_3 + i[V, \mathrm{e}_j] \otimes \mathbf{1} + \{U, \mathrm{e}_j\} \otimes \sigma_1 \tag{73}$$

$$\partial_\lambda \Psi(\lambda, \mathbf{s}) = A(\lambda, \mathbf{s}) \Psi(\lambda, \mathbf{s}) \tag{74}$$

$$A(\lambda, \mathbf{s}) := i\frac{\lambda^2}{2}\widehat{\sigma}_3 + \lambda U \otimes \sigma_1 - \frac{1}{2}\mathbf{D}U \otimes \sigma_2 + i(U^2 + \mathbf{s}) \otimes \sigma_3 \tag{75}$$

$$\mathbf{D} := \sum_{j=1}^{r} \partial_{s_j} \ , \quad \mathrm{e}_j := \mathrm{diag}(0, 0, \dots, 1, 0, \dots) \ , \quad \mathbf{s} := \mathrm{diag}(s_1, \dots, s_r) \tag{76}$$

is equivalent to NC-PII

$$\mathbf{D}^2 U = 4\mathbf{s}U + 4U\mathbf{s} + 8U^3 \ , \tag{77}$$

Proposition (Noncommutative Hastings–McLeod solution)

For any $C = [c_{ij}] \in Mat(r \times r, \mathbb{C})$ there is a unique solution of noncommutative PII

$$\mathbf{D}^2 U = 4\mathbf{s}U + 4U\mathbf{s} + 8U^3 \ , \quad \mathbf{s} := \mathrm{diag}(s_1, \dots, s_r) \ , \quad \mathbf{D} := \sum_{j=1}^{r} \frac{\partial}{\partial s_j} \tag{78}$$

with the asymptotics as follows: if $S := \frac{1}{r}\sum_{j=1}^{r} s_j \ \to +\infty$ and $\delta_j := s_j - S$, $j = 1, \dots, r$ are kept fixed, $|\delta_j| \leqslant m$, then

$$[U]_{k\ell} = -c_{k\ell}\mathrm{Ai}(s_k + s_\ell) + \mathcal{O}\left(\sqrt{S}\mathrm{e}^{-\frac{4}{3}(2S-2m)^{\frac{3}{2}}}\right) \tag{79}$$

If $C = C^\dagger$ then the solution is pole-free on $\mathbb{R}^r$ iff $\|C\| \leqslant 1$.
If $\|C\| \leqslant 1$ then the solution is pole-free on $\mathbb{R}^r$.

Theorem (Noncommutative Tracy-Widom)

Let $U(\mathrm{s})$ be the noncommutative Hastings–McLeod solution of above: then

$$\det\left(\mathrm{Id} - \mathcal{A}i^2_{\vec{s}}\right) = \exp\left[-4\int_S^\infty (t-S)\mathrm{Tr}\; U^2(t+\vec{\delta})dt\right] \tag{80}$$

where

$$S := \frac{1}{r}\sum_{j=1}^{r} s_j \;, \quad s_j = S + \delta_j \;, \quad t+\vec{\delta} := (t+\delta_1, \ldots, t+\delta_r)\;. \tag{81}$$

Corollary

The Fredholm determinant of the matrix Airy convolution kernel $\mathcal{A}i_{\vec{s}}$ satisfies

$$\det\left(Id + \mathcal{A}i_{\vec{s}}\right) = \exp\left[\int_S^\infty \mathrm{Tr}\;\left(U(t+\vec{\delta}) + 2(t-S)U^2(t+\vec{\delta})\right)\,\mathrm{d}t\right] \tag{82}$$

where $U(\vec{s})$ is the Hastings-McLeod family of solutions to noncommutative Painlevé II as above

The Fredholm determinant $\det(\mathrm{Id} + \mathcal{A}i_{\vec{s}})$ is also related to a noncommutative version of the Painlevé XXXIV equation ($' = \mathbf{D}$)

$$\begin{cases} W''' = 8i[W,\mathbf{s}]W + 8W + 8i[\mathbf{s},V] + 6i(W')^2 + 4\{W',\mathbf{s}\} \\ V' = W'W \end{cases} \Rightarrow \quad (83)$$

$$W^{iv} = 6i\{W'',W'\} + 8iW'[\mathbf{s},W] + 8i[W,\mathbf{s}W'] + 8i\mathbf{s}[W',W] + 4\{\mathbf{s},W''\} + 16W' \quad (84)$$

$$\mathbf{D}\ln\det(\mathrm{Id} + \mathcal{A}i_{\vec{s}}) = -2i\mathrm{Tr}\,W(\vec{s}) \quad (85)$$

The Airy process

This is a determinantal point field with configuration space

$$X = \mathbb{R} \times \{\tau_1 < \tau_2 < \cdots < \tau_n\} \simeq \mathbb{R} \times \{1, 2, \ldots, n\} \tag{86}$$

$$\begin{aligned}
A_{ij}(x,y) &:= \tilde{A}_{ij}(x,y) - B_{ij}(x,y)\ , 1 \leqslant i,j \leqslant n && (87)\\
\tilde{A}_{ij}(x,y) &:= \frac{1}{(2\pi i)^2} \int_{\gamma_{R_i}} \mathrm{d}\mu \int_{i\mathbb{R}} \mathrm{d}\lambda \frac{\mathrm{e}^{\theta(x,\mu)-\theta(y,\lambda)}}{\lambda + \tau_j - \mu - \tau_i} && (88)\\
\theta(x,\mu) &:= \frac{\mu^3}{3} - x\mu. && (89)\\
B_{ij}(x,y) &:= \chi_{\tau_i < \tau_j} \frac{1}{\sqrt{4\pi(\tau_j - \tau_i)}} \mathrm{e}^{\frac{(\tau_j-\tau_i)^3}{12} - \frac{(x-y)^2}{4(\tau_j-\tau_i)} - \frac{(\tau_j-\tau_i)(x+y)}{2}} && (90)
\end{aligned}$$

It represents a field of ∞'ly many particles undergoing **mutually avoiding Brownian motions**.

Multi-layer PolyNuclear Growth (PNG) model

The Airy process was introduced by Praehofer and Spohn in the study of the fluctuations around the top layer of the growth model.

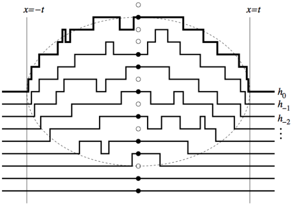
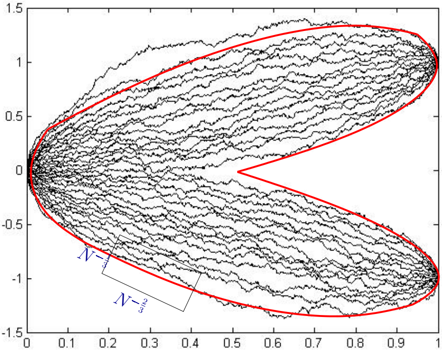


Figure: A snapshot of a multi-layer PNG configuration at time t. Asymptotic droplet is also marked. From Praehofer-Spohn, 2001

It also occurs in the study of fluctuations around the edge in the model of self-avoiding brownian motions in the limit $N \to \infty$



Simulation with $N = 30$ non-intersecting Brownian particles starting at $x = 0$ and ending at $x = 1, x = -1$. Courtesy of P.M. Roman, S. Delvaux.

$N \longrightarrow \infty$

Transition probability: $p_N(\Delta t, x, y) := Ce^{-N\frac{(x-y)^2}{2\Delta t}}$

For example the two–times Airy process has a matrix kernel

$$A(x,y) = \begin{bmatrix} \widetilde{A}_{11}(x,y) & \widetilde{A}_{12}(x,y) - B_{12}(x,y) \\ \widetilde{A}_{21}(x,y) & \widetilde{A}_{22}(x,y) \end{bmatrix} \tag{91}$$

One verifies that $A_{jj}(x,y) = K_{\mathrm{Ai}}(x,y)$ **does** have the IIKS form: however all the other (off-diagonal) entries **do not**.

Yet, we want to characterize the Fredholm determinants describing the **gap probabilities**; the simplest example of which is

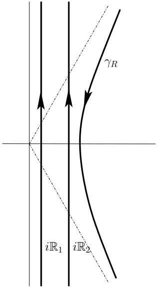
$$Pr\left(\begin{array}{c}\text{no particle in } (a,\infty) \text{ at time } \tau_1 \\ \text{no particle in } (b,\infty) \text{ at time } \tau_2 > \tau_1\end{array}\right) = \det\left(Id_{\mathbb{R}^2} - A(\bullet;\tau_1,\tau_2)_{\substack{(a,\infty)\\(b,\infty)}}\right) \tag{92}$$

Problem

Can the IIKS theory be applied? Can we obtain a Lax representation?

Note that by different methods, Tracy and Widom (2004) do obtain PDEs for the gap probabilities, but no Lax representation.

Equivalence of determinants



The following determinants are equal

$$\det\left(\mathrm{Id}_{\mathbb{R}^2} - A(\bullet;\tau_1,\tau_2)_{\substack{(a,\infty)\\(b,\infty)}}\right) = \det(\mathrm{Id}-K) \tag{93}$$

where K acts on $L^2(i\mathbb{R}_1 \cup i\mathbb{R}_2 \cup \gamma_R, \mathbb{C}^2)$ with kernel ($i\mathbb{R}_j := i\mathbb{R}+\tau_j$, $\lambda_j := \lambda - \tau_j,\ \mu_j := \mu - \tau_j$)

$$K(\lambda,\mu) = \frac{f^T(\lambda)g(\mu)}{\lambda-\mu} \tag{94}$$

$$f(\lambda) := \begin{bmatrix} e^{\frac{\lambda_1^3}{6}}\chi_{\gamma_R} & e^{\frac{\lambda_2^3}{6}}\chi_{\gamma_R} \\ e^{a\lambda_1}\chi_{i\mathbb{R}_1} & 0 \\ 0 & e^{b\lambda_2}\chi_{i\mathbb{R}_2} \end{bmatrix}, \quad g(\mu) := \begin{bmatrix} e^{-\frac{\mu_1^3}{3}}\chi_{i\mathbb{R}_1} & e^{-\frac{\mu_2^3}{3}}\chi_{i\mathbb{R}_2} \\ e^{\frac{\mu_1^3}{6}-a\mu_1}\chi_{\gamma_R} & e^{\frac{\mu_1^3-\mu_2^3}{3}-a\mu_1}\chi_{i\mathbb{R}_2} \\ 0 & e^{\frac{\mu_2^3}{6}-b\mu_2}\chi_{\gamma_R} \end{bmatrix} \tag{95}$$

The problem is thus reduced to one with integrable kernel (one has to check $f(\lambda) \cdot g^T(\lambda) \equiv 0$) and hence it is associated in a canonical way to a RHP for a matrix $\Gamma(\lambda)$ of size 3×3 on the union of contours depicted before.
Writing out the jumps one realizes furthermore that the matrix

$$\Psi(\lambda) := \Gamma(\lambda) \mathrm{e}^{\mathrm{T}} \quad (96)$$

$$T(\lambda; \tau_1, \tau_2, a, b) := \mathrm{diag}\left(\frac{\frac{\lambda_1^3 + \lambda_2^3}{3} + a\lambda_1 + b\lambda_2}{3}, \frac{\frac{\lambda_2^3 - 2\lambda_1^3}{3} + b\lambda_2 - 2a\lambda_1}{3}, \cdots \right) \quad (97)$$

solves a RHP with **constant** jumps, and hence solves an ODE in λ (which can be easily written) as well as isomonodromic deformations in a, b, τ_1, τ_2. It can be also shown that

Proposition

The Jimbo-Miwa-Ueno isomonodromic tau function coincides with the Fredholm determinant(s)

$$\partial \ln \tau_{JMU} = -\text{“} \mathop{\mathrm{res}}_{\lambda=\infty} \text{“} \, \mathrm{Tr} \left(\Gamma^{-1} \Gamma'(\lambda) \partial T \right) \mathrm{d}\lambda \quad (98)$$

Some details on the proof

The equivalence of determinants is actually unitary ($a_1 = a, a_2 = b$, $\chi_{I_j} := [a_j, \infty)$)

$$
\begin{aligned}
A_{ij}(x,y)\chi_{I_i}(x) &= \int_{i\mathbb{R}+\tau_i} \frac{d\xi}{2\pi i} e^{\xi_i(a_i - x)} \times \\
& \left[\int_{i\mathbb{R}+\tau_j} \frac{d\lambda}{2\pi i} \int_{\gamma_R} \frac{d\mu}{2\pi i} \frac{\mathrm{e}^{\theta(a_i,\mu_i)-\theta(0,\lambda_j)+y\lambda_j}}{(\xi-\mu)(\mu-\lambda)} + \right. \\
& \left. + \chi_{\tau_i<\tau_j} \int_{i\mathbb{R}+\tau_j} \frac{d\mu}{2\pi i} \frac{\mathrm{e}^{\theta(a_i,\mu_i)-\theta(0,\mu_j)+y\mu_j}}{\xi-\mu} \right]
\end{aligned}
$$

After Fourier transform (some care to be paid) one has an unitarily equivalent operator on $L^2(i\mathbb{R}_1 \cup i\mathbb{R}_2, \mathbb{C}^2)$ with kernel

$$
\begin{aligned}
&(\mathfrak{K})_{ij}(\xi,\lambda) = \\
&= \chi_{i\mathbb{R}_i}(\xi)\chi_{i\mathbb{R}_j}(\lambda) \left(\underbrace{\int_{\gamma_R} \frac{d\mu}{2\pi i} \frac{\mathrm{e}^{\theta(a_i,\mu_i)-\theta(0,\lambda_j)+a_i\xi_i}}{(\xi-\mu)(\mu-\lambda)}}_{\mathcal{G}\circ\mathcal{F}} + \underbrace{\chi_{\tau_i<\tau_j} \frac{\mathrm{e}^{\theta(a_i,\lambda_i)-\theta(0,\lambda_j)+a_i\xi_i}}{\xi-\lambda}}_{\mathcal{H}} \right).
\end{aligned}
$$

$$
\overset{\circlearrowleft \mathcal{H}}{L^2(i\mathbb{R}_1 \cup i\mathbb{R}_2, \mathbb{C}^2)} \underset{\xleftarrow[\mathcal{G}]{}}{\xrightarrow{\mathcal{F}}} L^2(\gamma_R, \mathbb{C}^2) \tag{99}
$$

So we have the determinant of

$$\det(Id - \mathcal{G} \circ \mathcal{F} - \mathcal{H}) \tag{100}$$

$$\overset{\circlearrowleft \mathcal{H}}{L^2(i\mathbb{R}_1) \oplus L^2(i\mathbb{R}_2)} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} L^2(\gamma_R, \mathbb{C}^2) \tag{101}$$

Note that all three operators are **Hilbert-Schmidt** so that $\mathcal{G} \circ \mathcal{F}$ is trace-class but $\mathcal{H}$ is not (at least we cannot prove it directly).
However the matrix kernel of $\mathcal{H}$ is upper-triangular so that it is "traceless" (it is not, technically)
But then the series of $\det_2$ for HS operators (well-defined) coincides with the series of $\det$ for trace-class (ill-defined here); thus, the correct definition is

$$\text{“}\det\text{”}(\mathrm{Id} - \mathcal{G} \circ \mathcal{F} - \mathcal{H}) := \det{}_2(\mathrm{Id} - \mathcal{G} \circ \mathcal{F} - \mathcal{H})\mathrm{e}^{-\mathrm{Tr}\mathcal{G}\circ\mathcal{F}} \tag{102}$$

Finally one uses the identity

$$\det(Id - \mathcal{G} \circ \mathcal{F} - \mathcal{H}) = \det\left(Id - \begin{bmatrix} 0 & \mathcal{F} \\ \mathcal{G} & \mathcal{H} \end{bmatrix}\right) \tag{103}$$

and then recognize that the last operator on $L^2(i\mathbb{R}_1 \cup i\mathbb{R}_2 \cup \gamma_R, \mathbb{C}^2)$ has the postulated kernel.

Nonlinear PDE

Using the Lax pair one can verify the nonlinear PDE for $G(a, b, \tau)$ $(\tau := \tau_2 - \tau_1)$ that was found by Adler-VanMoerbeke using vertex operators:

$$\left(\frac{\tau^2}{2}\partial_W - W\partial_E\right)\left(\partial_E^2 - \partial_W^2\right)G + 2\tau\partial^3_{\tau EW}G = \{\partial^2_{EW}G, \partial^2_E G\}_E \tag{104}$$

where $E = \frac{a+b}{2}$, $W = \frac{a-b}{2}$ and $\{f, g\}_E := \partial_E f\, g - f\partial_E g$.
This confirms that the RHP provides the desired Lax formulation.

Conclusions

- Fredholm determinants of scalar operators are intimately related to Painlevé equations (property);
- Fredholm determinant of **matrix** operators lead to noncommutative versions and/or PDEs with Painlevé property.
- Special solutions of Painlevé type equations come from Fredholm determinants (e.g. Hastings-McLeod, Ablowitz-Segur for P2). Numerical evaluation of Fredholm determinants is **more stable** than numerical integration of nonlinear PDE/ODEs $\Rightarrow$ tools for numerical study (if appropriate det. representation can be found) (see Bornemann, and cf. Prof. Clarkson's talk).

Some bibliography

S. P. Balandin and V. V. Sokolov.
On the Painlevé test for non-abelian equations.
Phys. Lett. A, 246(3-4):267–272, 1998.

M. Bertola.
The dependence on the monodromy data of the isomonodromic tau function.
Comm. Math. Phys., 294(2):539–579, 2010.

Peter A. Clarkson, Nalini Joshi, and Andrew Pickering.
Bäcklund transformations for the second Painlevé hierarchy: a modified truncation approach.
Inverse Problems, 15(1):175–187, 1999.

Vladimir Retakh and Vladimir Rubtsov.
Noncommutative Toda chains, Hankel quasideterminants and Painlevé II equation.
J. Phys. A: Math. Theor. 43 (2010) 505204 (13pp).

Patrik L. Ferrari and Herbert Spohn.
A determinantal formula for the GOE Tracy-Widom distribution.
J. Phys. A, 38(33):L557–L561, 2005.

A.R. Its, A. G. Izergin, V. E. Korepin, N. A. Slavnov,
Differential equations for quantum correlations functions
in *Proc. Conf. on Yang-Baxter Eqs., Conf. Invariance and Int. Systems* 1990

M. Jimbo, T. Miwa, K. Ueno
Monodromy preserving deformation of linear ordinary differential equations [...] I, II, III
Physica D, 1981-1982

Craig A. Tracy and Harold Widom.
Level-spacing distributions and the Airy kernel.
Comm. Math. Phys., 159(1):151–174, 1994.