On the Malgrange isomonodromic deformations of non-resonant meromorphic connections

Yu.P. Bibilo, R.R. Gontsov

HSE, IITP RAS

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Meromorphic system

Consider a system

$$\frac{dy}{dz} = B(z)\,y, \qquad B(z) = \sum_{i=1}^{n} \sum_{j=1}^{r_i+1} \frac{B_{ij}^0}{(z-a_i^0)^j}, \tag{1}$$

- $y(z) \in \mathbb{C}^p$,
- B_{ij}^0 are $(p \times p)$-matrices;
- $a_1^0, \ldots, a_m^0$ are irregular non-resonant sing. points;

An irregular sing. point a_i^0 is non-resonant if the eigenvalues of B_{i,r_i+1}^0 are pairwise distinct.

- $a_{m+1}^0, \ldots, a_n^0$ are Fuchsian sing. points.

Fuchsian case

A Fuchsian system

$$\frac{dy}{dz} = \sum_{i=1}^{n} \frac{B_i^0}{z - a_i^0}\, y$$

can be included into the Schlesinger isomonodromic family

$$\frac{dy}{dz} = \sum_{i=1}^{n} \frac{B_i(a)}{z - a_i}\, y, \quad B_i(a^0) = B_i^0,$$

$$dB_i(a) = -\sum_{j=1, j\neq i}^{n} \frac{[B_i(a), B_j(a)]}{a_i - a_j}\, d(a_i - a_j).$$

Fuchsian case

According to the Malgrange theorem,

the matrix functions $B_i(a)$ holomorphic in a neighbourhood of the point a^0, can be meromorphically extended to the universal cover $\widetilde{Z}^n$ of the space

$$Z^n = \mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\}.$$

Bolibrukh's theorem on Schlesinger (2×2)-families:

If the monodromy of the (2×2)-family is irreducible, then the pole orders of the matrices $B_i(a)$ do not exceed 2.

Holomorphic bundle and connection on it

There is a correspondence

$$\frac{dy}{dz} = B(z)y \quad \Longrightarrow \quad (E^0, \nabla^0)$$

- E^0 is a holomorphically trivial vector bundle of rank p over $\overline{\mathbb{C}}$,
- ∇^0 is a meromorphic connection on E^0.

As known,

$$B(z)dz \overset{formally}{\sim} \omega_{\Lambda_i^0} = \sum_{j=1}^{r_i+1} \frac{\Lambda_{ij}^0}{(z - a_i^0)^j}\, dz,$$

Λ_{ij}^0 are diagonal matrices.

Normal form and the set of parameters

The deformation parameter $t \in \mathcal{D}$ contains:

- $a_1, \ldots, a_m$ (irregular singularities);
- $\Lambda_1, \ldots, \Lambda_m$, where $\Lambda_i = \{\Lambda_{i2}, \ldots, \Lambda_{i,r_i+1}\}$ coming from the formal normal form

$$\omega_{\Lambda_i} = \sum_{j=2}^{r_i+1} \frac{\Lambda_{ij}}{(z-a_i)^j}\, dz + \frac{\Lambda_{i1}^0}{z-a_i}\, dz;$$

- $a_{m+1}, \ldots, a_n$ (Fuchsian singularities).

The initial parameter value is

$$t^0 = (a_1^0, \ldots, a_n^0, \Lambda_1^0, \ldots, \Lambda_m^0)$$

Malgrange isomonodromic deformations

There is a holomorphic vector bundle E of rank p over $\overline{\mathbb{C}} \times \mathcal{D}$ with a meromorphic connection ∇ such that

- $(E, \nabla)|_{\overline{\mathbb{C}}\times\{t^0\}} \cong (E^0, \nabla^0)$;
- $\nabla|_{\overline{\mathbb{C}}\times\{t\}} \overset{formally}{\sim} \nabla_{\Lambda_i(t)}$ near $z = a_i(t)$;

$$\nabla_{\Lambda_i(t)} = d - \omega_{\Lambda_i(t)} = d - \sum_{j=1}^{r_i+1} \frac{\Lambda_{ij}(t)}{(z - a_i(t))^j}\, dz,$$

- the Stokes' matrices of $\nabla|_{\overline{\mathbb{C}}\times\{t\}}$ do not depend on t.

This pair (E, ∇) is called the Malgrange isomonodromic deformation of the initial pair (E^0, ∇^0).

Malgrange isomonodromic deformations

So, there is an isomonodromic family

$$\frac{dy}{dz} = B(z,t)\,y, \qquad B(z,t) = \sum_{i=1}^{n} \sum_{j=1}^{r_i+1} \frac{B_{ij}(t)}{(z-a_i(t))^j},$$

t varies in some small disk, $t \in D(t^0)$.

Malgrange Θ-divisor

Malgrange Θ-divisor

$$\Theta = \{t \in \mathcal{D} \mid E|_{\overline{\mathbb{C}} \times \{t\}} \text{ is non-trivial } \}.$$

- Θ-divisor can be locally described in terms of auxillary meromorphic system with one additional Fuchsian singularity.
- In particular case Θ-divisor structure can be described in more details. We will consider a system with
  - 2×2 coefficient matrix;
  - two irregular points a_1^0, a_2^0 of Poincare ranks $r_1 = r_2 = 1$;
  - $n - 2$ Fuchsian points.

Meromorphic 2×2-connections

Consider a system

$$\frac{dy}{dz} = \left(\frac{B^0_{12}}{(z-a^0_1)^2} + \frac{B^0_{22}}{(z-a^0_2)^2} + \sum_{i=1}^{n} \frac{B^0_{i1}}{z-a^0_i} \right) y.$$

Let

- (E, ∇) be its Malgrange isomonodromic deformation;
- the monodromy be irreducible;
- $t^* \in \Theta$ such that $E|_{\overline{\mathbb{C}}\times\{t^*\}} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$.

Then

- in a neighbourhood $D(t^*)$ of t^* the Θ-divisor is given as a zero set of an irreducible holomorphic function τ;
- the matrix functions $B_{ij}(t)$ have poles of at most second order along $D(t^*) \cap \Theta$.

Global τ-function: Fuchsian case

Miwa's theorem on Schlesinger deformations

Malgrange Θ-divisor is the zero set of τ-function, holomorphic on

$$\widetilde{Z^n}, \quad Z^n = \mathbb{C}^n \setminus \bigcup_{i \neq j} \{a_i = a_j\},$$

and

$$d \log \tau(a) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{tr(B_i(a) B_j(a))}{a_i - a_j} d(a_i - a_j).$$

Global τ-function: Irregular case

Θ-divisor is the zero set of τ-function, holomorphic on $\mathcal{D}$,

$$d_a \log \tau(t) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \frac{tr(B_{i1}(t)B_{j1}(t))}{a_i - a_j} d(a_i - a_j) +$$

$$\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\sum_{s=1}^{r_i+1}\sum_{k=s+1}^{r_j+1} (k, s-1)\frac{tr(B_{is}(t)B_{jk}(t))}{(a_i - a_j)^{s+k-1}} d(a_i - a_j),$$

$$(\alpha, \beta) = \prod_{m=1}^{\beta} \frac{\alpha - m + 1}{m}.$$