Asymptotics for the Second Painlevé equation

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Setup and outline

This talk discusses joint work with Alexander Its on the large x asymptotical behavior of solutions u of the second Painlevé equation (PII)

$$u_{xx} = xu + 2u^3, \tag{1}$$

assuming that u is real-valued for real x.

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$$\{s_k\}_{k=1}^{6} \subset \mathbb{C}: \; s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0, \; s_{k+3} = -s_k, \; s_1 = \bar{s}_3, s_2 = \bar{s}_2,$$

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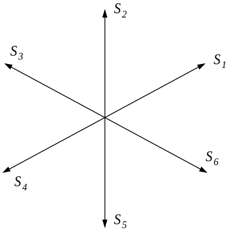
put $S_k = \left(\begin{smallmatrix} 1 & 0 \\ s_k & 1 \end{smallmatrix}\right)$ if k odd, $S_k = \left(\begin{smallmatrix} 1 & s_k \\ 0 & 1 \end{smallmatrix}\right)$ if k even and consider the following Riemann-Hilbert problem (RHP) which consists in finding the piecewise analytic 2×2 function $\Psi(\lambda)$:

$\Psi(\lambda)$ is analytic on $\mathbb{C} \setminus \cup_1^6 \Gamma_k$



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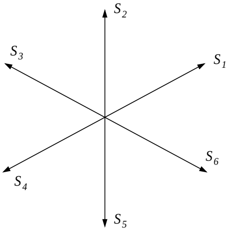

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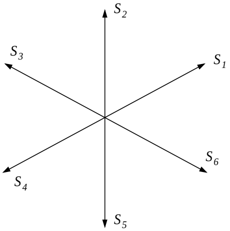

$$\begin{aligned} \Psi_+(\lambda) &= \Psi_-(\lambda) S_k, \ \lambda \in \Gamma_k \\ \Psi(\lambda) &= \big(I + O(\lambda^{-1})\big) e^{-i(\frac{4}{3}\lambda^3 + x\lambda)\sigma_3}, \ \lambda \to \infty \end{aligned}$$

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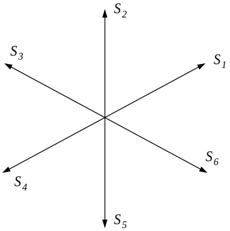

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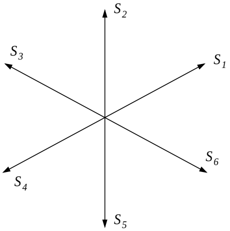

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This RHP is for any $s \equiv (s_1, s_2, s_3)$ meromorphically w.r.t. x solvable [Bolibruch, Its, Kapaev, 2004] and its solution determines the solution of (1) via

$$u(x) \equiv u(x; s) = 2 \lim_{\lambda \to \infty} \big(\lambda (\Psi(\lambda) e^{i(\frac{4}{3}\lambda^3 + x\lambda)\sigma_3})_{12}\big), \quad \bar{u}(x) = u(\bar{x}).$$

Theorem (Kapaev, 1992; Bolts, 2010)

As $x \to -\infty$ and $|s_1| > 1$, the following asymptotical behavior holds for real-valued solutions of PII equation (1)

$$u(x) = \frac{\sqrt{-x}}{\sin\left(\frac{2}{3}(-x)^{3/2} + \beta \ln\left(8(-x)^{3/2}\right) + \varphi\right) + O\left((-x)^{-3/2}\right)} + O\left((-x)^{-1}\right)$$

where

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However as $x \to +\infty$ and $s_2 \neq 0$, we have

$$u(x) = \sigma\sqrt{\frac{x}{2}}\cot\left(\frac{\sqrt{2}}{3}x^{3/2} + \frac{\gamma}{2}\ln\left(8\sqrt{2}x^{3/2}\right) + \phi\right) + O(x^{-1}),$$

where

$$\sigma = \mathrm{sgn}\ s_2,\ \gamma = \frac{1}{\pi}\ln|s_2|,\ \phi = -\frac{1}{2}\arg\Gamma\left(\frac{1}{2} + i\gamma\right) - \frac{1}{2}\arg\left(1 + s_2 s_3\right) + \frac{\pi}{2}.$$

Sketch of the proof as $x \to -\infty$

The solution of the given RHP

$$\begin{aligned} \Psi_+(\lambda) &= \Psi_-(\lambda) \underbrace{e^{-\theta(\lambda)\sigma_3} S_k e^{\theta(\lambda)\sigma_3}}_{\equiv G(\lambda)}, \ \lambda \in \Gamma_k; \quad \theta(\lambda) = i\Big(\frac{4}{3}\lambda^3 + x\lambda\Big) \\ \Psi(\lambda) &\to I, \ \lambda \to \infty \end{aligned}$$

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so as $|x| \to \infty$ a "steepest descent evaluation" for the oscillatory integral is needed.

We start with a scaling

$$\lambda \to z(\lambda) = \frac{\lambda}{\sqrt{-x}}, \qquad \Psi(z) \equiv \Psi\big(\lambda(z)\big),$$

$$\theta(\lambda) \to t\vartheta(z), \qquad \vartheta(z) = i\left(\frac{4}{3}z^3 - z\right), \; t = (-x)^{3/2}.$$

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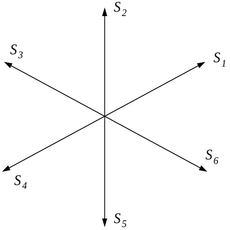
Then use analytic structure of $G(\lambda)$ and deform jump contour $\cup_1^6 \Gamma_k$ to steepest descent contours of the exponent $\vartheta(z)$:

From



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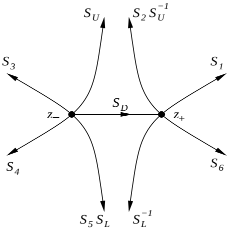

applying

a Gauss LU-decomposition (since $1 - s_1 s_3 = 1 - |s_1|^2 \neq 0$)

$$
\begin{aligned}
(S_3 S_4 S_5)^{-1} &= \begin{pmatrix} 1 - s_1 s_3 & s_1 \\ s_1 & 1 - s_1 s_3 \end{pmatrix} = S_L S_D S_U \\
&= \begin{pmatrix} 1 & 0 \\ \frac{s_1}{1 - s_1 s_3} & 1 \end{pmatrix} (1 - s_1 s_3)^{\sigma_3} \begin{pmatrix} 1 & \frac{s_1}{1 - s_1 s_3} \\ 0 & 1 \end{pmatrix}
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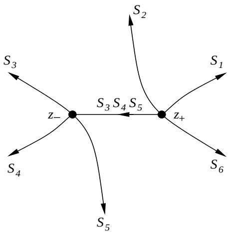

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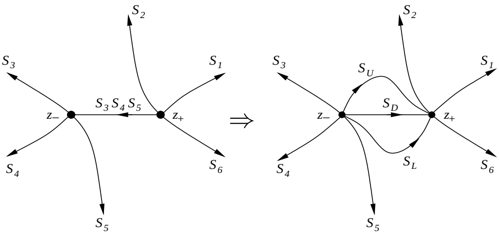

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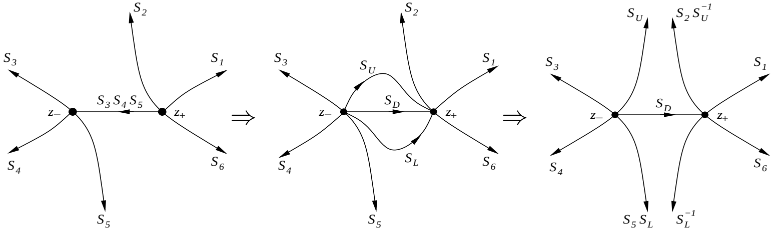

Now observe that triangularity of $G(z)$ and the sign of $\Re\vartheta(z)$ imply for $|z - z_\pm| > r, 0 < r < \frac{1}{2}$

$$\|G - I\|_{L^2\cap L^\infty(\text{infinite branches})} = O\big(e^{-ct|z-z_\pm|}\big), c > 0 \quad t \to \infty,$$

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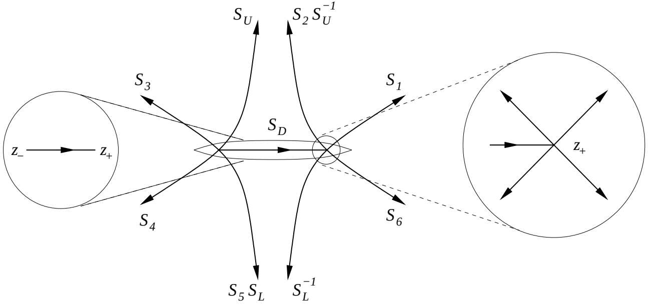
Construct model solutions/parametrices:



Along $[z_-, z_+]$ we solve the diagonal RHP applying the Plemelj -Sokhotskii formulae: For $z \in \mathbb{C} \backslash [z_-, z_+]$

$$\Psi^D(z) = \left(\frac{z - z_-}{z - z_+} \right)^{\nu \sigma_3} e^{-t\vartheta(z)\sigma_3}, \;\; \nu = -\frac{1}{2\pi i} \ln(1 - s_1 s_3)$$

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In the neighborhood of z_+ we notice the quadratic local behavior of $\vartheta(z)$:

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and make use of the parabolic cylinder function $D_\nu(\zeta)$, a solution of

$$\frac{d^2 D_\nu}{d\zeta^2} + \left(\nu + \frac{1}{2} - \frac{\zeta^2}{4}\right) D_\nu = 0, \quad D_\nu(\zeta) \sim \zeta^\nu e^{-\frac{\zeta^2}{4}}, \; \zeta \to +\infty. \qquad (3)$$

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$$\Psi^r(z) = \begin{pmatrix} 1 & -\frac{\nu s_3}{h_1} e^{\frac{2it}{3}} \beta(z) \frac{1}{\zeta(z)} \\ -\frac{h_1}{s_3} e^{-\frac{2it}{3}} \beta^{-1}(z) \frac{1}{\zeta(z)} & 1 \end{pmatrix} \Big(I + O(t^{-1}) \Big) \Psi^D(z),$$

as long as $0 < r_1 \leq |z - z_+| \leq r_2 < 1$, with $h_1 = \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi\nu}$,

$$\zeta(z) = 2\sqrt{-t\vartheta(z) + t\vartheta(z_+)} = e^{i\frac{3\pi}{4}} 2\sqrt{2t}(z - z_+)\big(1 + O(z - z_+)\big)$$

and

$$\beta(z) = \left(\zeta(z) \frac{z - z_-}{z - z_+} \right)^{2\nu}, \text{ i.e. } \beta^{\pm 1} \frac{1}{\zeta} = O\big(t^{-\frac{1}{2} \pm \Re\nu}\big), \ \ t \to \infty.$$

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however, here $|s_1| > 1$, so $\nu = -\frac{1}{2\pi i}\ln(|s_1|^2 - 1) - \frac{1}{2} \equiv \nu_0 - \frac{1}{2}$, i.e.

$$\beta\zeta = \frac{\beta_0(z)}{\alpha(z)} = O(1), t \to \infty \text{ i.e. } \Psi^r(z) = \left(E_r(z) + O(t^{-1})\right)\Psi^D(z)$$

with

$$E_r(z) = \begin{pmatrix} 1 & 0 \\ -\frac{h_1}{s_3} e^{-\frac{2it}{3}} \frac{\alpha(z)}{\beta_0(z)} & 1 \end{pmatrix}, \quad \beta_0(z) = \left(\zeta(z)\frac{z - z_-}{z - z_+}\right)^{2\nu_0}$$

and $\alpha(z) = \frac{z - z_-}{z - z_+}$.

Now assemble the model functions

$$\hat{\Psi}(z) = \begin{cases} \Psi^D(z), & |z - z_\pm| > r; \\ \Psi^r(z), & |z - z_+| < r; \\ \Psi^l(z) = \sigma_2 \Psi^r(-z) \sigma_2, & |z - z_-| < r, \end{cases}$$

consider the ratio $\Psi(z) = \chi(z)\hat{\Psi}(z)$ and obtain the following ratio-RHP:

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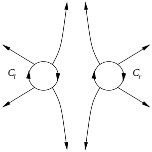

$$\begin{aligned} \chi_+(z) &= \chi_-(z)\Psi^{r,l}(z)\big(\Psi^D(z)\big)^{-1}, z \in C_{r,l} \\ \chi(z) &\to I, \ z \to \infty \end{aligned}$$

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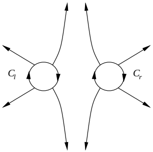

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but $\|\Psi^{r,l}\left(\Psi^D\right)^{-1} - I\| \nrightarrow 0$ as $t \to \infty$. To this end employ an undressing transformation and pass from $\chi(z)$ to $\Phi(z)$:

$$\Phi(z) = \begin{cases} \chi(z)E_r(z), & |z - z_+| < r; \\ \chi(z)\sigma_2 E_r(-z)\sigma_2, & |z - z_-| < r; \\ \chi(z), & |z - z_\pm| > r, \end{cases}$$

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which satisfies a RHP with pole singularities at $z_\pm$, since after all

$$E_r(z) = \frac{R}{z - z_+} + O(1), \ z \to z_+, \tag{4}$$

$$\Phi(z) = \begin{cases} \chi(z)E_r(z), & |z - z_+| < r; \\ \chi(z)\sigma_2 E_r(-z)\sigma_2, & |z - z_-| < r; \\ \chi(z), & |z - z_\pm| > r, \end{cases}$$

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and determine B uniquely from the residue-relations (4).

Finally solve the Y-RHP asymptotically: Since

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$$\|Y_- - I\|_{L^2(\text{jump contour})} \leq \frac{c}{t}, \quad t \to \infty.$$

In the end use the latter estimate together with the integral representation for $Y(z)$

$$\begin{aligned} Y(z) &= I + \frac{1}{2\pi i} \int\limits_{\text{contour}} Y_-(w)\big(G_Y(w) - I\big)\frac{dw}{w-z} \\ &= I + \frac{i}{2\pi z} \int\limits_{\text{contour}} Y_-(w)\big(G_Y(w) - I\big)dw + O\big(z^{-2}\big) \end{aligned}$$

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and extract the required asymptotics via

$$u(x) = 2\sqrt{-x}\lim_{z\to\infty}\big(z(\Psi(z)e^{t\vartheta(z)\sigma_3})_{12}\big)$$

by tracing back: $\Psi(z) \to \chi(z) \to \Phi(z) \to Y(z)$.

Reference

Thomas Bothner and Alexander Its, The Nonlinear Steepest Descent Approach to the Singular Asymptotics of the Second Painlevé Transcendent.