Painlevé Equations — Nonlinear Special Functions

Peter A Clarkson

School of Mathematics, Statistics and Actuarial Science
University of Kent, Canterbury, CT2 7NF, UK
P.A.Clarkson@kent.ac.uk

Formal and Analytic Solutions of Differential and Difference Equations
Będlowe, Poland, August 2011

University of Kent

Outline

1. Introduction

2. Classical solutions of the **second** and **fourth Painlevé equations**

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha$$
$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w}$$

and the **second** and **fourth Painlevé σ-equations**

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2$$
$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\theta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\theta_\infty\right) = 0$$

3. Coalescence of Equations

4. Painlevé Challenges
   - Equivalence problem
   - Numerical solution of Painlevé equations

Classical Special Functions

- **Airy, Bessel, Whittaker, Kummer, hypergeometric functions**
- Special solutions in terms of rational and elementary functions (for certain values of the parameters)
- Solutions satisfy **linear** ordinary differential equations and **linear** difference equations
- Solutions related by **linear** recurrence relations

Painlevé Transcendents — Nonlinear Special Functions

- Special solutions such as rational solutions, algebraic solutions and special function solutions (for certain values of the parameters)
- Solutions satisfy **nonlinear** ordinary differential equations and **nonlinear** difference equations
- Solutions related by **nonlinear** recurrence relations

Definition 1

An ODE has the **Painlevé property** if its solutions have **no movable singularities except poles**.

Definition 2

An ODE has the **Painlevé property** if its solutions have **no movable branch points**.

- **Single-valued**

$$w(z) = \frac{1}{z - z_0} \qquad \textbf{pole}$$

$$w(z) = \exp\left(\frac{1}{z - z_0}\right) \qquad \textbf{essential singularity}$$

- **Multi-valued**

$$w(z) = \sqrt{z - z_0} \qquad \textbf{algebraic branch point}$$

$$w(z) = \ln(z - z_0) \qquad \textbf{logarithmic branch point}$$

$$w(z) = \tan[\ln(z - z_0)] \qquad \textbf{essential singularity}$$

Reference

- **Cosgrove**, "Painlevé classification problems featuring essential singularities", *Stud. Appl. Math.*, **98** (1997) 355–433. [See also **Cosgrove**, *Stud. Appl. Math.*, **104** (2000) 1–65; **104** (2000) 171–228; **116** (2006) 321–413.]

Painlevé Equations

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 6w^2 + z \qquad \mathrm{P_I}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w} \qquad \mathrm{P_{III}}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w} \qquad \mathrm{P_{IV}}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1} \qquad \mathrm{P_V}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right\} \qquad \mathrm{P_{VI}}$$

where α, β, γ and δ are arbitrary constants.

Painlevé σ-Equations

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - 2\sigma = 0 \qquad \mathrm{S_I}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad \mathrm{S_{II}}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + \left[4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - 1\right]\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) + \lambda_0\lambda_1\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \tfrac{1}{4}\left(\lambda_0^2 + \lambda_1^2\right) \qquad \mathrm{S_{III}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S_{IV}}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - \left[2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - z\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \sigma\right]^2 + 4\prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \kappa_j\right) = 0 \qquad \mathrm{S_V}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left[z(z-1)\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right]^2 + \left[\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left\{2\sigma - (2z-1)\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right\} + \beta_1\beta_2\beta_3\beta_4\right]^2 = \prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \beta_j^2\right) \qquad \mathrm{S_{VI}}$$

History of the Painlevé Equations

- Derived by **Painlevé, Gambier** and colleagues in the late 19th/early 20th centuries.
- Studied in Minsk, Belarus by **Erugin, Lukashevich, Gromak** *et al.* since 1950's; much of their work is published in the journal *Diff. Eqns.*, translation of *Diff. Urav.*.
- **Barouch, McCoy, Tracy & Wu [1973, 1976]** showed that the correlation function of the two-dimensional Ising model is expressible in terms of solutions of P_{III}.
- **Ablowitz & Segur [1977]** demonstrated a close connection between completely integrable PDEs solvable by inverse scattering, the so-called **soliton equations**, such as the **Korteweg-de Vries equation** and the **nonlinear Schrödinger equation**, and the Painlevé equations.
- **Flaschka & Newell [1980]** introduced the **isomonodromy deformation method** (inverse scattering for ODEs), which expresses the Painlevé equation as the compatibility condition of two **linear** systems of equations and are studied using **Riemann-Hilbert** methods. Subsequent developments by **Deift, Fokas, Its, Zhou, . . .**
- Algebraic and geometric studies of the Painlevé equations by **Okamoto** in 1980's. Subsequent developments by **Noumi, Umemura, Yamada, . . .**
- The Painlevé equations are a chapter in the "**Digital Library of Mathematical Functions**", which is a rewrite/update of **Abramowitz & Stegun's** "**Handbook of Mathematical Functions**" — see `http://dlmf.nist.gov`.

Some Properties of the Painlevé Equations

- $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have **Bäcklund transformations** which relate solutions of a given Painlevé equation to solutions of the same Painlevé equation, though with different values of the parameters with associated **Affine Weyl groups** that act on the parameter space.
- $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have **rational**, **algebraic** and **special function solutions** expressed in terms of the classical special functions [$\mathrm{P_{II}}$: **Airy** $\mathrm{Ai}(z)$, $\mathrm{Bi}(z)$; $\mathrm{P_{III}}$: **Bessel** $J_\nu(z)$, $Y_\nu(z)$, $J_\nu(z)$, $K_\nu(z)$; $\mathrm{P_{IV}}$: **parabolic cylinder** $D_\nu(z)$; $\mathrm{P_V}$: **Whittaker** $M_{\kappa,\mu}(z)$, $W_{\kappa,\mu}(z)$ [equivalently **Kummer** $M(a,b,z)$, $U(a,b,z)$ or **confluent hypergeometric** ${}_1F_1(a;c;z)$]; $\mathrm{P_{VI}}$: **hypergeometric** ${}_2F_1(a,b;c;z)$], for certain values of the parameters.
- These rational, algebraic and special function solutions of $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$, called **classical solutions**, can usually be written in **determinantal form**, frequently as **wronskians**. Often these can be written as **Hankel determinants** or **Toeplitz determinants**.
- $\mathrm{P_I}$–$\mathrm{P_{VI}}$ can be written as a (non-autonomous) **Hamiltonian system** and the Hamiltonians satisfy a second-order, second-degree differential equations ($\mathrm{S_I}$–$\mathrm{S_{VI}}$).
- $\mathrm{P_I}$–$\mathrm{P_{VI}}$ possess **Lax pairs** (**isomonodromy problems**).
- $\mathrm{P_I}$–$\mathrm{P_{VI}}$ and $\mathrm{S_I}$–$\mathrm{S_{VI}}$ form a **coalescence cascade**

$$\begin{array}{ccccc} \mathrm{P_{VI}} & \longrightarrow & \mathrm{P_V} & \longrightarrow & \mathrm{P_{IV}} & & \\ & & \downarrow & & \downarrow & & \\ & & \mathrm{P_{III}} & \longrightarrow & \mathrm{P_{II}} & \longrightarrow & \mathrm{P_I} \end{array} \qquad \begin{array}{ccccc} \mathrm{S_{VI}} & \longrightarrow & \mathrm{S_V} & \longrightarrow & \mathrm{S_{IV}} & & \\ & & \downarrow & & \downarrow & & \\ & & \mathrm{S_{III}} & \longrightarrow & \mathrm{S_{II}} & \longrightarrow & \mathrm{S_I} \end{array}$$

Hamiltonian Representation

$P_{\rm II}$ can be written as the **Hamiltonian system**

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathcal{H}_{\rm II}}{\partial p} = p - q^2 - \tfrac{1}{2}z, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathcal{H}_{\rm II}}{\partial q} = 2qp + \alpha + \tfrac{1}{2} \qquad \mathrm{H}_{\rm II}$$

where $\mathcal{H}_{\rm II}(q,p,z;\alpha)$ is the Hamiltonian defined by

$$\mathcal{H}_{\rm II}(q,p,z;\alpha) = \tfrac{1}{2}p^2 - (q^2 + \tfrac{1}{2}z)p - (\alpha + \tfrac{1}{2})q$$

Eliminating p then $q = w$ satisfies $P_{\rm II}$ whilst eliminating q yields

$$p\frac{\mathrm{d}^2p}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)^2 + 2p^3 - zp^2 - \tfrac{1}{2}(\alpha + \tfrac{1}{2})^2 \qquad P_{34}$$

Theorem (**Okamoto [1986]**)

The function

$$\sigma(z;\alpha) = \mathcal{H}_{\rm II} \equiv \tfrac{1}{2}p^2 - (q^2 + \tfrac{1}{2}z)p - (\alpha + \tfrac{1}{2})q$$

satisfies

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad \mathrm{S}_{\rm II}$$

and conversely

$$q(z;\alpha) = \frac{2\sigma''(z) + \alpha + \frac{1}{2}}{4\sigma'(z)}, \qquad p(z;\alpha) = -2\frac{\mathrm{d}\sigma}{\mathrm{d}z}$$

is a solution of $\mathrm{H}_{\rm II}$.

Hamiltonian Representation

P_{IV} can be written as the **Hamiltonian system**

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathcal{H}_{IV}}{\partial p} = 4qp - q^2 - 2zq - 2\vartheta_0, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathcal{H}_{IV}}{\partial q} = -2p^2 + 2pq + 2zp - \vartheta_\infty \quad (\mathrm{H}_{IV})$$

where $\mathcal{H}_{IV}(q, p, z; \vartheta_0, \vartheta_\infty)$ is the Hamiltonian defined by

$$\mathcal{H}_{IV}(q, p, z; \vartheta_0, \vartheta_\infty) = 2qp^2 - (q^2 + 2zq + 2\vartheta_0)p + \vartheta_\infty q$$

Eliminating p then $w = q$ satisfies P_{IV} with $\alpha = 1 - \vartheta_0 + 2\vartheta_\infty$ and $\beta = -2\vartheta_0^2$, whilst eliminating q then $w = -2p$ satisfies P_{IV} with $\alpha = 2\vartheta_0 - \vartheta_\infty - 1$ and $\beta = -2\vartheta_\infty^2$.

Theorem

(Okamoto [1986])

The function

$$\sigma(z; \vartheta_0, \vartheta_\infty) = \mathcal{H}_{IV} \equiv 2qp^2 - (q^2 + 2zq + 2\vartheta_0)p + \vartheta_\infty q$$

satisfies

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S}_{IV}$$

and conversely

$$q = \frac{\sigma'' - 2z\sigma' + 2\sigma}{2(\sigma' + 2\vartheta_\infty)}, \qquad p = \frac{\sigma'' + 2z\sigma' - 2\sigma}{4(\sigma' + 2\vartheta_0)}$$

are solutions of H_{IV}.

Classical Solutions of the Second Painlevé Equation and the Second Painlevé σ-Equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad \mathrm{S_{II}}$$

Classical Solutions of $\mathrm{P_{II}}$ and $\mathrm{S_{II}}$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad \mathrm{S_{II}}$$

Theorem

- $\mathrm{P_{II}}$ *and* $\mathrm{S_{II}}$ *have* ***rational solutions*** *if and only if* $\alpha = n$, *with* $n \in \mathbb{Z}$.
- $\mathrm{P_{II}}$ *and* $\mathrm{S_{II}}$ *have solutions expressible in terms of the Riccati equation*

$$\varepsilon\frac{\mathrm{d}w}{\mathrm{d}z} = w^2 + \tfrac{1}{2}z, \qquad \varepsilon = \pm 1 \qquad (1)$$

if and only if $\alpha = n + \tfrac{1}{2}$, *with* $n \in \mathbb{Z}$, *which has solution*

$$w(z) = -\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\varphi(z)$$

where

$$\varphi(z) = C_1\,\mathrm{Ai}(\zeta) + C_2\,\mathrm{Bi}(\zeta), \qquad \zeta = -2^{-1/2}z$$

with $\mathrm{Ai}(\zeta)$ *and* $\mathrm{Bi}(\zeta)$ *the* ***Airy functions***.

Rational Solutions of $\mathrm{P_{II}}$ and $\mathrm{S_{II}}$

$$\frac{d^2w}{dz^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^3 + 2\frac{d\sigma}{dz}\left(z\frac{d\sigma}{dz} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad \mathrm{S_{II}}$$

Theorem

Define the polynomial $\varphi_j(z)$ by

$$\sum_{j=0}^{\infty} \varphi_j(z)\lambda^j = \exp\left(z\lambda - \tfrac{4}{3}\lambda^3\right)$$

and the ***Yablonskii–Vorob'ev polynomials $Q_n(z)$*** *given by*

$$Q_n(z) = c_n \mathcal{W}(\varphi_1, \varphi_3, \ldots, \varphi_{2n-1})$$

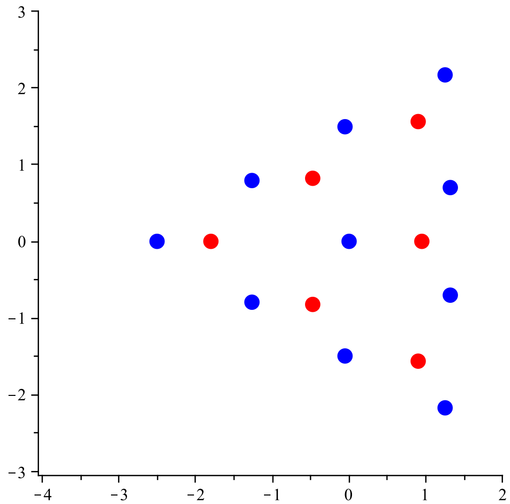
where $\mathcal{W}(\varphi_1, \varphi_3, \ldots, \varphi_{2n-1})$ is the Wronskian and c_n a constant, then

$$w(z;n) = \frac{d}{dz}\ln\frac{Q_{n-1}(z)}{Q_n(z)}, \qquad \sigma(z;n) = -\tfrac{1}{8}z^2 + \frac{d}{dz}\ln Q_n(z)$$

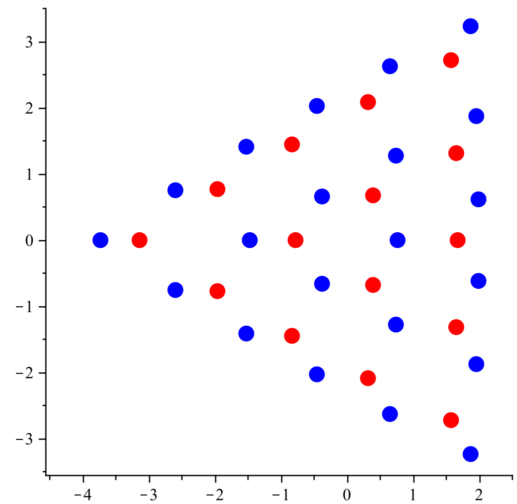
respectively satisfy $\mathrm{P_{II}}$ and $\mathrm{S_{II}}$ with $\alpha = n$, for $n \in \mathbb{Z}$.

Roots of some Yablonskii–Vorob'ev polynomials

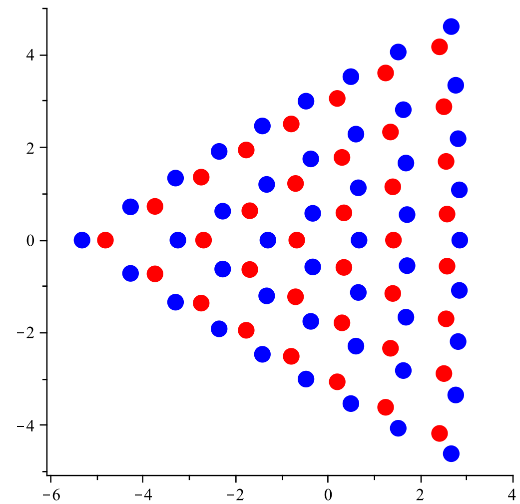
(PAC & Mansfield [2003])



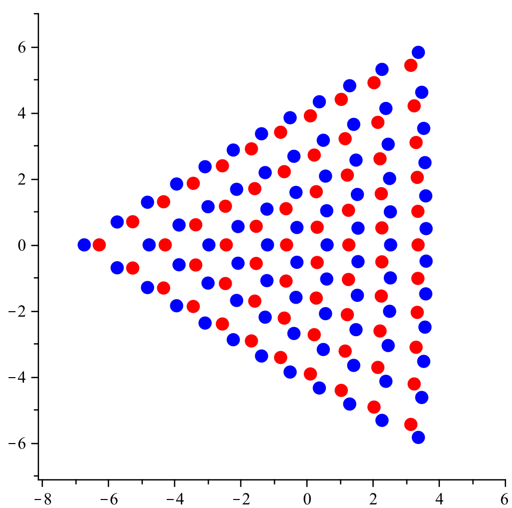
$Q_3(z)$, $Q_4(z)$



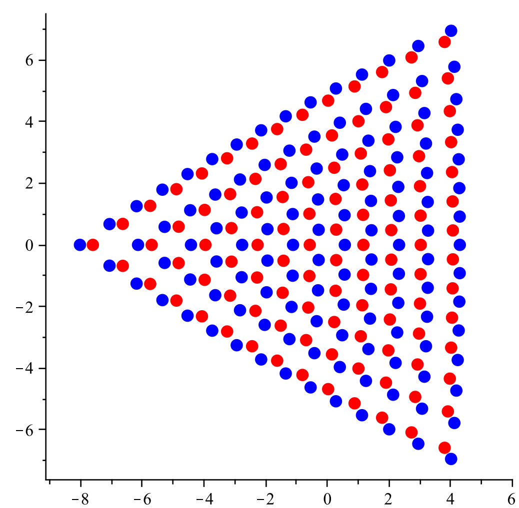
$Q_5(z)$, $Q_6(z)$



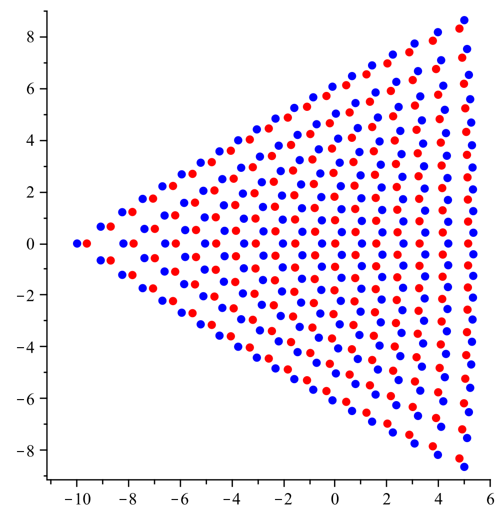
$Q_8(z)$, $Q_9(z)$



$Q_{11}(z)$, $Q_{12}(z)$



$Q_{14}(z)$, $Q_{15}(z)$



$Q_{19}(z)$, $Q_{20}(z)$

Airy Solutions of $\mathrm{P_{II}}$ and $\mathrm{S_{II}}$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad \mathrm{S_{II}}$$

Theorem

Let

$$\varphi(z) = C_1\,\mathrm{Ai}(\zeta) + C_2\,\mathrm{Bi}(\zeta), \qquad \zeta = -2^{-1/2}z$$

with $\mathrm{Ai}(\zeta)$ *and* $\mathrm{Bi}(\zeta)$ ***Airy functions****, and* $\tau_n(z)$ *be the Wronskian*

$$\tau_n(z) = \mathcal{W}\left(\varphi, \frac{\mathrm{d}\varphi}{\mathrm{d}z}, \ldots, \frac{\mathrm{d}^{n-1}\varphi}{\mathrm{d}z^{n-1}}\right)$$

then

$$w(z; n+\tfrac{1}{2}) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{\tau_n(z)}{\tau_{n+1}(z)}\right), \qquad \sigma(z; n+\tfrac{1}{2}) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n(z)$$

respectively satisfy $\mathrm{P_{II}}$ *and* $\mathrm{S_{II}}$ *with* $\alpha = n + \tfrac{1}{2}$*, for* $n \in \mathbb{Z}$*.*

Classical Solutions of $\mathrm{P_{IV}}$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w} \qquad \mathrm{P_{IV}}$$

Theorem

- $\mathrm{P_{IV}}$ *has rational solutions if and only if*

  $$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \left(\boldsymbol{m}, -\mathbf{2}(\mathbf{2}\boldsymbol{n} - \boldsymbol{m} + \mathbf{1})^{\mathbf{2}}\right) \quad \text{or} \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}) = \left(\boldsymbol{m}, -\mathbf{2}(\mathbf{2}\boldsymbol{n} - \boldsymbol{m} + \tfrac{\mathbf{1}}{\mathbf{3}})^{\mathbf{2}}\right)$$

  with $m, n \in \mathbb{Z}$. *Further the rational solutions for these parameter values are unique.*

- $\mathrm{P_{IV}}$ *has solutions expressible in terms of the Riccati equation*

  $$z\frac{\mathrm{d}w}{\mathrm{d}z} = \varepsilon(w^2 + 2zw) - 2(1+\varepsilon\alpha), \qquad \varepsilon = \pm 1$$

  if and only if

  $$\beta = -2(2n+1+\varepsilon\alpha)^2 \qquad \text{or} \qquad \beta = -2n^2$$

  with $n \in \mathbb{Z}$. *The Riccati equation has solution*

  $$w(z) = -\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\varphi(z)$$

  where

  $$\varphi_\nu(z;\varepsilon) = \{C_1 D_\nu(\zeta) + C_2 D_{-\nu}(\zeta)\}\exp(\tfrac{1}{2}\varepsilon z^2), \qquad \nu = -\tfrac{1}{2}(1+2\alpha+\varepsilon), \quad \zeta = \sqrt{2}\,z$$

  with $D_\nu(\zeta)$ *the* ***parabolic cylinder function***.

$\mathrm{P_{IV}}$ — Generalized Hermite Polynomials

Theorem (**Kajiwara & Ohta [1998], Noumi & Yamada [1998]**)

Define the ***generalized Hermite polynomial*** $\boldsymbol{H_{m,n}(z)}$, which has degree mn, by

$$H_{m,n}(z) = a_{m,n}\,\mathcal{W}\left(H_m(z), H_{m+1}(z), \ldots, H_{m+n-1}(z)\right), \qquad m,n \geq 1$$

where $\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is the Wronskian, $H_n(z)$ is the n^{th} Hermite polynomial and $a_{m,n}$ is a constant. Then

$$w_{m,n}^{(\text{i})}(z) = w(z; \alpha_{m,n}^{(\text{i})}, \beta_{m,n}^{(\text{i})}) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{H_{m+1,n}(z)}{H_{m,n}(z)}$$

$$w_{m,n}^{(\text{ii})}(z) = w(z; \alpha_{m,n}^{(\text{ii})}, \beta_{m,n}^{(\text{ii})}) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{H_{m,n}(z)}{H_{m,n+1}(z)}$$

$$w_{m,n}^{(\text{iii})}(z) = w(z; \alpha_{m,n}^{(\text{iii})}, \beta_{m,n}^{(\text{iii})}) = -2z + \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{H_{m,n+1}(z)}{H_{m+1,n}(z)}$$

are respectively solutions of $\mathrm{P_{IV}}$ for

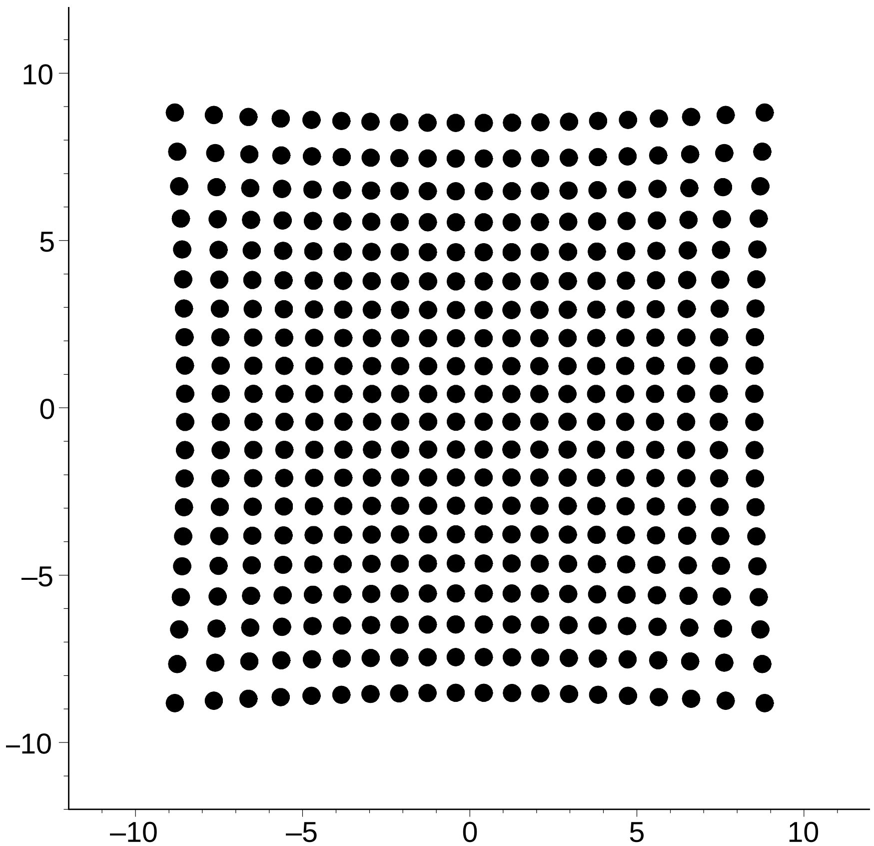
$$(\alpha_{m,n}^{(\text{i})}, \beta_{m,n}^{(\text{i})}) = (2m+n+1, -2n^2)$$

$$(\alpha_{m,n}^{(\text{ii})}, \beta_{m,n}^{(\text{ii})}) = (-m-2n-1, -2m^2)$$

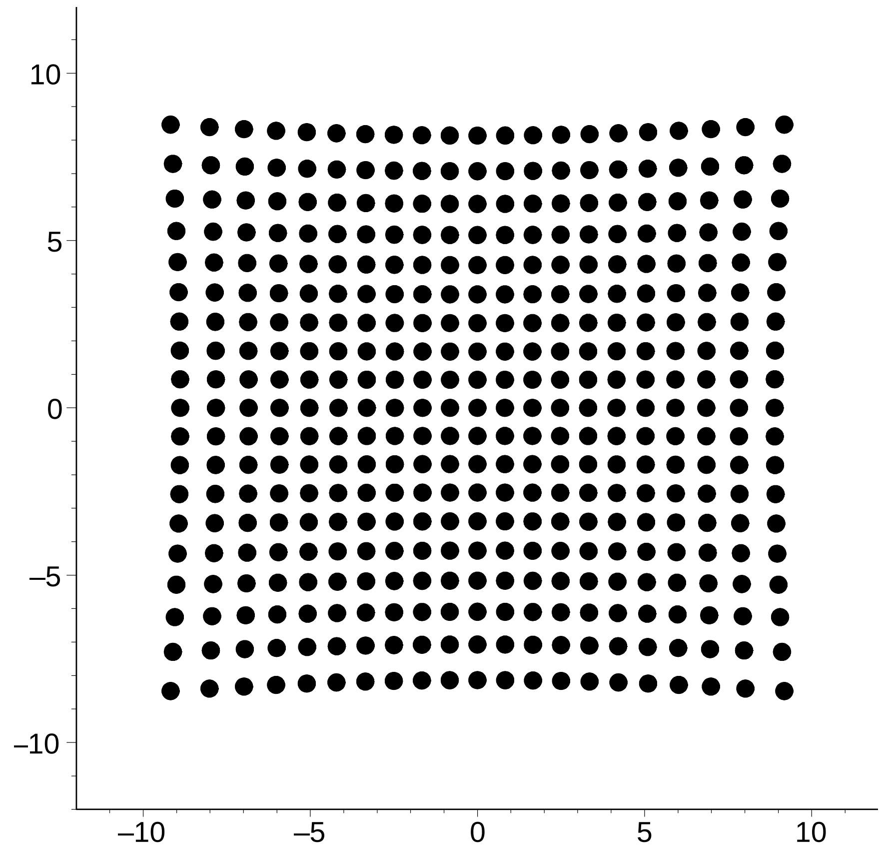
$$(\alpha_{m,n}^{(\text{iii})}, \beta_{m,n}^{(\text{iii})}) = (n-m, -2(m+n+1)^2)$$

Roots of the Generalized Hermite Polynomials $H_{m,n}(z)$

(PAC [2003])



$H_{20,20}(z)$



$H_{21,19}(z)$

$m \times n$ “rectangles"

Properties of the Generalized Hermite Polynomials

- The generalized Hermite polynomial $H_{m,n}(z)$ can be expressed as the multiple integral

$$H_{m,n}(z) = \frac{\pi^{m/2}\prod_{k=1}^{m} k!}{2^{m(m+2n-1)/2}} \int_{-\infty}^{\infty} \overset{n}{\cdots} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \prod_{j=i+1}^{n} (x_i - x_j)^2 \prod_{k=1}^{n} (z - x_k)^m \times \exp\left(-x_1^2 - x_2^2 - \ldots - x_n^2\right) \mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n$$

  which arises in random matrix theory (**Brézin & Hikami [2000], Forrester & Witte [2001], Kanzieper [2002]**).

- The orthogonal polynomials on the real line with respect to the weight

$$w(x; z, m) = (x - z)^m \exp(-x^2)$$

  satisfy the three-term recurrence relation

$$x p_n(x) = p_{n+1}(x) + a_n(z; m) p_n(x) + b_n(z; m) p_{n-1}(x)$$

  where

$$a_n(z; m) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{H_{n+1,m}}{H_{n,m}}, \qquad b_n(z; m) = \frac{n H_{n+1,m} H_{n-1,m}}{2 H_{n,m}^2}$$

  (**Chen & Feigen [2006]**).

$\mathrm{P_{IV}}$ — Generalized Okamoto Polynomials

Theorem (**Kajiwara & Ohta [1998], Noumi & Yamada [1998], PAC [2006]**)

Let $\varphi_k(z) = 3^{k/2}\mathrm{e}^{-k\pi\mathrm{i}/2}H_k\left(\tfrac{1}{3}\sqrt{3}\,\mathrm{i}z\right)$*, with* $H_k(\zeta)$ *the* k^{th} *Hermite polynomial, then define the* ***generalized Okamoto polynomial*** $Q_{m,n}(z)$ *by*

$$Q_{m,n}(z) = \mathcal{W}(\varphi_1, \varphi_4, \ldots, \varphi_{3m+3n-5}; \varphi_2, \varphi_5, \ldots, \varphi_{3n-4})$$

with $m, n \geq 1$*, where* $\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ *is the Wronskian. Then*

$$\widetilde{w}^{(\mathrm{i})}_{m,n}(z) = w(z; \widetilde{\alpha}^{(\mathrm{i})}_{m,n}, \widetilde{\beta}^{(\mathrm{i})}_{m,n}) = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{Q_{m+1,n}(z)}{Q_{m,n}(z)}$$

$$\widetilde{w}^{(\mathrm{ii})}_{m,n}(z) = w(z; \widetilde{\alpha}^{(\mathrm{ii})}_{m,n}, \widetilde{\beta}^{(\mathrm{ii})}_{m,n}) = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{Q_{m,n}(z)}{Q_{m,n+1}(z)}$$

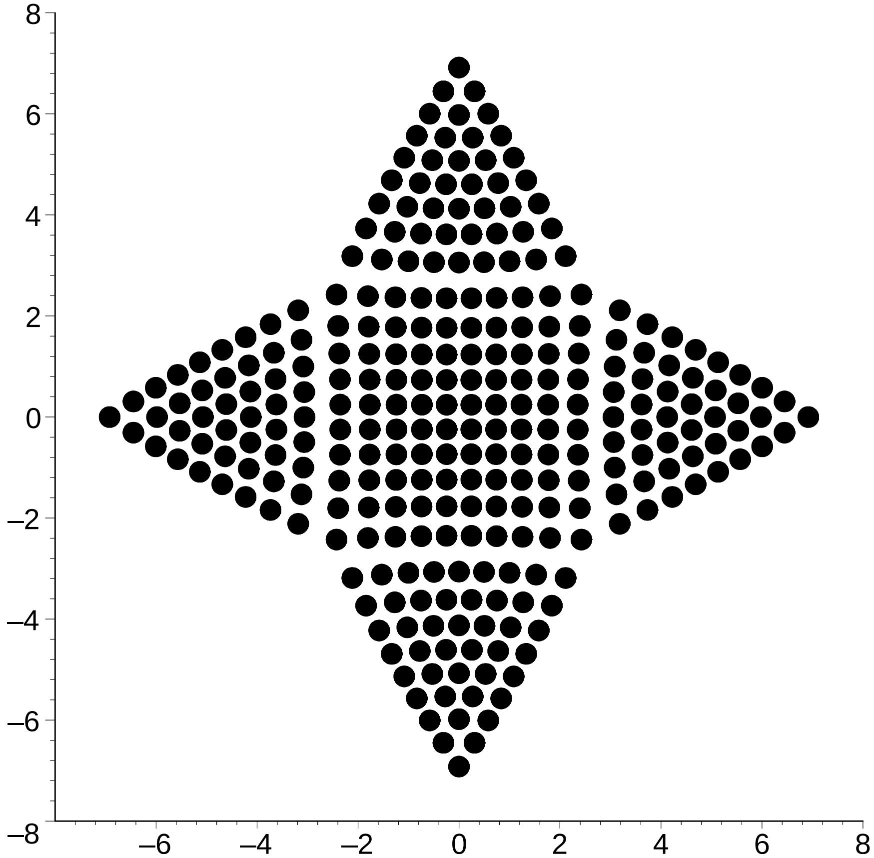
$$\widetilde{w}^{(\mathrm{iii})}_{m,n}(z) = w(z; \widetilde{\alpha}^{(\mathrm{iii})}_{m,n}, \widetilde{\beta}^{(\mathrm{iii})}_{m,n}) = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{Q_{m,n+1}(z)}{Q_{m+1,n}(z)}$$

are respectively solutions of $\mathrm{P_{IV}}$ *for*

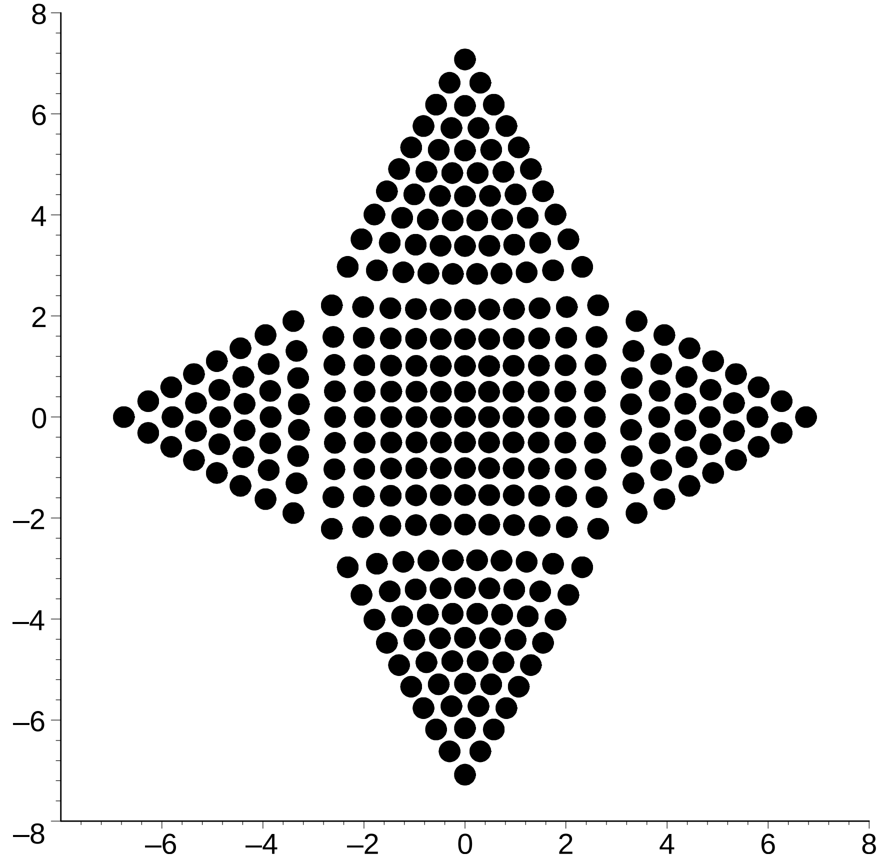
$$(\widetilde{\alpha}^{(\mathrm{i})}_{m,n}, \widetilde{\beta}^{(\mathrm{i})}_{m,n}) = (2m+n, -2(n-\tfrac{1}{3})^2)$$
$$(\widetilde{\alpha}^{(\mathrm{ii})}_{m,n}, \widetilde{\beta}^{(\mathrm{ii})}_{m,n}) = (-m-2n, -2(m-\tfrac{1}{3})^2)$$
$$(\widetilde{\alpha}^{(\mathrm{iii})}_{m,n}, \widetilde{\beta}^{(\mathrm{iii})}_{m,n}) = (n-m, -2(m+n+\tfrac{1}{3})^2)$$

Roots of the Generalized Okamoto Polynomials $Q_{m,n}(z)$, $m,n > 0$

(PAC [2003])



$Q_{10,10}(z)$



$Q_{11,9}(z)$

$m \times n$ "rectangles" and "equilateral triangles" with sides $m-1$ and $n-1$

Parabolic Cylinder Function Solutions of $\mathrm{P_{IV}}$

For $\mathrm{P_{IV}}$ the associated Riccati equation is

$$w' = \varepsilon(w^2 + 2zw) + 2\nu, \qquad \varepsilon^2 = 1$$

Letting $w(z) = -\varepsilon\varphi_\nu'(z;\varepsilon)/\varphi_\nu(z;\varepsilon)$ yields the Weber-Hermite equation

$$\varphi_\nu'' - 2\varepsilon z\varphi_\nu' + 2\varepsilon\nu\varphi_\nu = 0$$

which, provided that $\nu \notin \mathbb{Z}$, has general solution

$$\varphi_\nu(z;\varepsilon) = \left\{C_1 D_\nu(\sqrt{2\varepsilon}\, z) + C_2 D_{-\nu}(\sqrt{2\varepsilon}\, z)\right\} \exp\left(\tfrac{1}{2}\varepsilon z^2\right)$$

where C_1 and C_2 are arbitrary constants and $D_\nu(\zeta)$ is the **parabolic cylinder function** that satisfies

$$\frac{\mathrm{d}^2 D_\nu}{\mathrm{d}\zeta^2} = (\tfrac{1}{4}\zeta^2 - \nu - \tfrac{1}{2})D_\nu$$

with boundary condition

$$D_\nu(\zeta) \sim \zeta^\nu \exp\left(-\tfrac{1}{4}\zeta^2\right), \qquad \text{as} \quad \zeta \to +\infty$$

Equivalently

$$\varphi_\nu(z;\varepsilon) = \left\{\widetilde{C}_1 M_{\frac{1}{2}\nu+\frac{1}{4},\frac{1}{4}}(\varepsilon z^2) + \widetilde{C}_2 W_{\frac{1}{2}\nu+\frac{1}{4},\frac{1}{4}}(\varepsilon z^2)\right\} z^{-1/2} \exp\left(\tfrac{1}{2}\varepsilon z^2\right)$$

where $M_{\kappa,\mu}(\xi)$ and $W_{\kappa,\mu}(\xi)$ are **Whittaker functions**.

Parabolic Cylinder Function Solutions of $\mathrm{P_{IV}}$

Theorem (**Okamoto [1986], Forrester & Witte [2001]**)

Suppose $\tau_{\nu,n}(z;\varepsilon)$, for $\nu \notin \mathbb{Z}$ and with $\varepsilon = \pm 1$, is given by

$$\tau_{\nu,n}(z;\varepsilon) = \mathcal{W}\left(\varphi_{\nu}(z;\varepsilon), \varphi_{\nu+1}(z;\varepsilon), \ldots, \varphi_{\nu+n-1}(z;\varepsilon)\right)$$

where $\varphi_{\nu}(z;\varepsilon)$ is given by

$$\varphi_{\nu}(z;\varepsilon) = \left\{C_1 D_{\nu}(\sqrt{2\varepsilon}\, z) + C_2 D_{-\nu}(\sqrt{2\varepsilon}\, z)\right\} \exp\left(\tfrac{1}{2}\varepsilon z^2\right)$$

with $D_{\nu}(\zeta)$ the **parabolic cylinder function** *and C_1 and C_2 are arbitrary constants, and $\mathcal{W}(\varphi_{\nu}, \varphi_{\nu+1}, \ldots, \varphi_{\nu+n-1})$ is the usual Wronskian. Then solutions of $\mathrm{P_{IV}}$ are given by*

$$w\left(z; \varepsilon(2\nu+n+1), -2n^2\right) = \varepsilon \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left(\frac{\tau_{\nu+1,n}(z;\varepsilon)}{\tau_{\nu,n}(z;\varepsilon)}\right)\right\}$$

$$w\left(z; -\varepsilon(\nu+2n+1), -2\nu^2\right) = \varepsilon \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left(\frac{\tau_{\nu,n}(z;\varepsilon)}{\tau_{\nu,n+1}(z;\varepsilon)}\right)\right\}$$

$$w\left(z; \varepsilon(n-\nu), -2(\nu+n+1)^2\right) = -2z + \varepsilon \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left(\frac{\tau_{\nu,n+1}(z;\varepsilon)}{\tau_{\nu+1,n}(z;\varepsilon)}\right)\right\}$$

If $\nu = n \in \mathbb{Z}^+$ then

$$D_n(\zeta) = 2^{-n/2} H_n\left(\tfrac{1}{2}\sqrt{2}\,\zeta\right) \exp(-\tfrac{1}{4}\zeta^2)$$

with $H_n(z)$ the **Hermite polynomial**.

Classical Solutions of $\mathrm{S_{IV}}$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0$$

- **Rational solutions**

$$\sigma_{m,n}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln H_{m,n}(z), \qquad \vartheta_0 = -n, \qquad \vartheta_\infty = m$$

$$\widetilde{\sigma}_{m,n}(z) = \tfrac{4}{27}z^3 - \tfrac{2}{3}(m-n)z + \frac{\mathrm{d}}{\mathrm{d}z}\ln Q_{m,n}(z), \qquad \vartheta_0 = -n + \tfrac{1}{3}, \quad \vartheta_\infty = m - \tfrac{1}{3}$$

where $H_{m,n}(z)$ is the **generalized Hermite polynomial** and $Q_{m,n}(z)$ is the **generalized Okamoto polynomial**.

- **Parabolic cylinder function solutions**

$$\sigma_{\nu,n}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln \tau_{\nu,n}(z;\varepsilon), \qquad \vartheta_0 = -n, \quad \vartheta_\infty = \varepsilon\nu$$

where

$$\tau_{\nu,n}(z;\varepsilon) = \mathcal{W}\left(\varphi_\nu(z;\varepsilon), \varphi_{\nu+1}(z;\varepsilon), \ldots, \varphi_{\nu+n-1}(z;\varepsilon)\right)$$

with

$$\varphi_\nu(z;\varepsilon) = \left\{C_1 D_\nu(\sqrt{2\varepsilon}\,z) + C_2 D_{-\nu}(\sqrt{2\varepsilon}\,z)\right\}\exp\left(\tfrac{1}{2}\varepsilon z^2\right)$$

and $D_\nu(\zeta)$ the **parabolic cylinder function**.

Application of $\mathrm{P_{IV}}$ to Orthogonal Polynomials

(**Filipuk, van Assche & Zhang [2011]; Forrester & Witte [2005]**)

Consider the orthogonal polynomials with respect to the semi-classical Laguerre weight

$$w(x;z) = x^\lambda \exp(-x^2 + zx), \qquad x \in \mathbb{R}^+, \qquad \lambda > -1$$

and seek polynomials $P_n(x;z)$ which satisfy

$$\int_0^\infty P_m(x;z)P_n(x;z)w(x;z)\,\mathrm{d}x = h_n(z)\delta_{m,n}$$

Consequently they satisfy the three term recurrence relation

$$xP_n(x;z) = P_{n+1}(x;z) + a_n(z)P_n(x;z) + b_n(z)P_{n-1}(x;z)$$

where $a_n(z)$ and $b_n(z)$ are expressible in terms of solutions of $\mathrm{P_{IV}}$ with

$$(\alpha, \beta) = \left(1 + 2n + \lambda, -2\lambda^2\right)$$

which is the condition for $\mathrm{P_{IV}}$ to have **parabolic cylinder function solutions**.

We note that $D_\nu(\zeta)$, the **parabolic cylinder function**, has the integral representation

$$D_\nu(z) = \frac{\exp(-\frac{1}{4}z^2)}{\Gamma(-\nu)} \int_0^\infty x^{\nu-1} \exp(-\tfrac{1}{2}x^2 - zx)\,\mathrm{d}x$$

and $H_n(z)$, the **Hermite polynomial**, has the integral representation

$$H_n(z) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^\infty (z + \mathrm{i}x)^n \exp(-x^2)\,\mathrm{d}x$$

Further Examples

- The weight
$$w(x;z) = \exp\left(\tfrac{1}{3}x^3 + zx\right), \qquad x^3 < 0$$
is associated with solutions of $P_{\rm II}$ (**Magnus [1995]**).
- The weight
$$w(x;z) = x^\alpha \mathrm{e}^{-x-z/x}$$
is associated with solutions of $P_{\rm III}$ (**Chen & Its [2010]**).
- The weight
$$w(x;z) = |x|^{2\alpha+1} \exp\left(-x^4 + zx^2\right), \quad \alpha > -1$$
is associated with solutions of $P_{\rm IV}$ (**Filipuk, van Assche & Zhang [2011]; Forrester & Witte [2005]**).
- The weight
$$w(x;z) = x^\alpha (1-x)^\beta \mathrm{e}^{-z/x}, \qquad x \in [0,1], \qquad \alpha > 0, \;\; \beta > 0$$
is associated with solutions of $P_{\rm V}$ (**Basor, Chen & Ehrhardt [2010]; Chen & Dai [2010]**).
- The weight
$$w(x;z) = x^\alpha (1-x)^\beta (z-x)^\gamma, \quad x \in [0,1]$$
is associated with solutions of $P_{\rm VI}$ (**Dai & Zhang [2010]; Forrester & Witte [2006]**).

Coalescence of Equations

$$
\begin{array}{ccccc}
P_{\mathrm{VI}} & \longrightarrow & P_{\mathrm{V}} & \longrightarrow & P_{\mathrm{IV}} & & \\
 & & \downarrow & & \downarrow & & \\
 & & P_{\mathrm{III}} & \longrightarrow & P_{\mathrm{II}} & \longrightarrow & P_{\mathrm{I}}
\end{array}
\qquad
\begin{array}{ccccc}
S_{\mathrm{VI}} & \longrightarrow & S_{\mathrm{V}} & \longrightarrow & S_{\mathrm{IV}} & & \\
 & & \downarrow & & \downarrow & & \\
 & & S_{\mathrm{III}} & \longrightarrow & S_{\mathrm{II}} & \longrightarrow & S_{\mathrm{I}}
\end{array}
$$

Coalescence of $\mathrm{P_{IV}}$ to $\mathrm{P_{II}}$

Making the transformation

$$w(z;\alpha,\beta)=\frac{y(x;a)}{\varepsilon}+\frac{1}{4\varepsilon^3},\qquad z=\varepsilon x-\frac{1}{4\varepsilon^3},\qquad \alpha=-2a-\frac{1}{32\varepsilon^6},\qquad \beta=-\frac{1}{512\varepsilon^{12}}$$

with a an arbitrary constant in $\mathrm{P_{IV}}$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}=\frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2+\frac{3}{2}w^3+4zw^2+2(z^2-\alpha)w+\frac{\beta}{w}$$

yields

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2y^3+xy+a+\left\{2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-2y^4+4xy^2+4ay+\tfrac{1}{2}x^2\right\}\varepsilon^2+\mathcal{O}(\varepsilon^4)$$

and so in the limit as $\varepsilon\to 0$ we obtain $\mathrm{P_{II}}$.

We remark that if we make the transformation

$$w(z;\alpha,\beta)=-2\varepsilon u(x;b),\qquad z=\varepsilon x-\frac{1}{4\varepsilon^3},\qquad \alpha=\frac{1}{16\varepsilon^6},\qquad \beta=-2b^2$$

in $\mathrm{P_{IV}}$, then in the limit as $\varepsilon\to 0$ we obtain $\mathrm{P_{34}}$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}=\frac{1}{2u}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2+2u^2-xu-\frac{b^2}{2u}$$

Making the transformation

$$w(z) = 2\varepsilon y^2(x), \qquad z = \varepsilon x + \frac{1}{2\varepsilon^3}, \qquad \alpha = \frac{1}{4\varepsilon^6}$$

in $\mathrm{P_{IV}}$ with $\beta = 0$, i.e.

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w$$

yields

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y^3 + xy + \left(3y^5 + 4xy^3 + yx^2\right)\varepsilon^4$$

and so in the limit as $\varepsilon \to 0$ we obtain $\mathrm{P_{II}}$ with $\alpha = 0$.

Coalescence of Hamiltonian Systems

If we let

$$q(z;\theta_0,\theta_\infty) = \frac{Q(x;a)}{\varepsilon} + \frac{1}{4\varepsilon^3}, \qquad p(z;\theta_0,\theta_\infty) = \varepsilon P(x;a)$$
$$z = \varepsilon x - \frac{1}{4\varepsilon^3}, \qquad \theta_0 = \frac{1}{32\varepsilon^6}, \qquad \theta_\infty = -\kappa$$

with κ an arbitrary constant, in the Hamiltonian system for $\mathrm{P_{IV}}$

$$\frac{\mathrm{d}q}{\mathrm{d}z} = 4qp - q^2 - 2zq - 2\theta_0, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -2p^2 + 2qp + 2zp - \theta_\infty$$

then $Q(x;b)$ and $P(x;b)$ satisfy

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = P - Q^2 - \tfrac{1}{2}x + 2Q(2P - x)\varepsilon^2 + \mathcal{O}(\varepsilon^4)$$
$$\frac{\mathrm{d}P}{\mathrm{d}x} = 2QP + \kappa + 2P(x - P)\varepsilon^2 + \mathcal{O}(\varepsilon^4)$$

and so in the limit as $\varepsilon \to 0$

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = P - Q^2 - \tfrac{1}{2}x, \qquad \frac{\mathrm{d}P}{\mathrm{d}x} = 2QP + \kappa$$

which is the Hamiltonian system for $\mathrm{P_{II}}$.

Coalescence of σ Equations

If we let

$$\sigma(z;\theta_0,\theta_\infty)=\frac{h(x;b)}{\varepsilon}-\frac{b}{2\varepsilon^3},\qquad z=\varepsilon x-\frac{1}{4\varepsilon^3},\qquad \theta_0=\frac{1}{32\varepsilon^6},\qquad \theta_\infty=-2b$$

with b an arbitrary constant, in S_{IV}, the "σ-equation" for P_{IV}

$$\left(\frac{d^2\sigma}{dz^2}\right)^2-4\left(z\frac{d\sigma}{dz}-\sigma\right)^2+4\frac{d\sigma}{dz}\left(\frac{d\sigma}{dz}+2\theta_0\right)\left(\frac{d\sigma}{dz}+2\theta_\infty\right)=0$$

then $h(x;b)$ satisfies

$$\begin{aligned}\left(\frac{d^2h}{dx^2}\right)^2&+4\left(\frac{dh}{dx}\right)^3+2\frac{dh}{dx}\left(x\frac{dh}{dx}-h\right)\\&=b^2+4b\left\{4\left(\frac{dh}{dx}\right)^2+x\frac{dh}{dx}-h\right\}\varepsilon^2+\mathcal{O}(\varepsilon^4)\end{aligned}$$

and so in the limit as $\varepsilon\to 0$

$$\left(\frac{d^2h}{dx^2}\right)^2+4\left(\frac{dh}{dx}\right)^3+2\frac{dh}{dx}\left(x\frac{dh}{dx}-h\right)=b^2$$

which is S_{II}, the "σ-equation" for P_{II}.

Coalescence of Special Function Solutions

Making the transformation

$$w(z;\alpha,\beta) = \frac{y(x;a)}{\varepsilon} + \frac{1}{4\varepsilon^3}, \qquad z = \varepsilon x - \frac{1}{4\varepsilon^3}, \qquad \nu = \frac{1}{32\varepsilon^6}$$

with a an arbitrary constant in the Riccati equation associated with $\mathrm{P_{IV}}$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = w^2 + 2zw + 2\nu$$

yields

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 2\varepsilon^2 xy + \tfrac{1}{2}x$$

and so in the limit as $\varepsilon \to 0$ we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + \tfrac{1}{2}x$$

which is the Riccati equation associated with $\mathrm{P_{II}}$.

Consequently the special function solutions of $\mathrm{P_{IV}}$, which are expressed in terms of **parabolic cylinder functions** (or equivalently **Whittaker functions**), coalesce to the special function solutions of $\mathrm{P_{II}}$, which are expressed in terms of **Airy functions**.

Remark

Making the transformation

$$\varphi(z) = \psi(x) \exp\left(-\frac{x}{4\varepsilon^2}\right), \qquad z = \varepsilon x - \frac{1}{4\varepsilon^3}, \qquad \nu = \frac{1}{32\varepsilon^6}$$

in

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}\varphi}{\mathrm{d}z} + 2\nu\varphi = 0$$

which is the linearization of the P_{IV} Riccati equation, yields

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - 2\varepsilon^2 x\frac{\mathrm{d}\psi}{\mathrm{d}x} + \tfrac{1}{2}x\psi = 0$$

and so in the limit as $\varepsilon \to 0$ we obtain

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \tfrac{1}{2}x\psi = 0$$

which is the linearization of the P_{II} Riccati equation.

Coalescence of Rational Solutions ($\mathrm{P_{III}}$ to $\mathrm{P_{II}}$)

Making the transformation

$$w(z;\alpha,\beta,1,-1)=1-\varepsilon y(x;a),\qquad z=\frac{x}{\varepsilon}+\frac{4}{\varepsilon^3},\qquad \alpha=-\frac{8}{\varepsilon^3}-2a,\qquad \beta=\frac{8}{\varepsilon^3}-2a$$

with a an arbitrary constant, in $\mathrm{P_{III}}$ (with $\gamma=-\delta=1$)

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}=\frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2-\frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z}+\frac{\alpha w^2+\beta}{z}+w^3-\frac{1}{w}$$

yields

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2y^3+xy+a+\left\{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-y^4+\frac{1}{2}xy^2\right\}\varepsilon+\mathcal{O}(\varepsilon^2)$$

and so in the limit as $\varepsilon\to 0$ we obtain $\mathrm{P_{II}}$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2y^3+xy+a$$

Thus

$$y(x;a)=\lim_{\varepsilon\to 0}\frac{1-w\left(\dfrac{x}{\varepsilon}+\dfrac{4}{\varepsilon^3};-\dfrac{8}{\varepsilon^3}-2a,\dfrac{8}{\varepsilon^3}-2a,1,-1\right)}{\varepsilon}$$

$$y(x;a) = \lim_{\varepsilon\to 0} \frac{1 - w\left(\dfrac{x}{\varepsilon} + \dfrac{4}{\varepsilon^3}; -\dfrac{8}{\varepsilon^3} - 2a, \dfrac{8}{\varepsilon^3} - 2a, 1, -1\right)}{\varepsilon}$$

$$w(z;\kappa+2, 2-\kappa) = \frac{2z+\kappa-1}{2z+\kappa+1} = \frac{2x-\varepsilon}{2x+\varepsilon} \qquad \left(z = \frac{x}{\varepsilon} + \frac{4}{\varepsilon^3}, \quad \kappa = -\frac{8}{\varepsilon^3}\right)$$

$$\Longrightarrow \frac{1-w}{\varepsilon} = \frac{2}{2x+\varepsilon} \xrightarrow[\varepsilon\to 0]{} \frac{1}{x} = y(x;-1)$$

$$w(z;\kappa+4, 4-\kappa) = \frac{2z+\kappa+1}{2z+\kappa-1}\,\frac{8z^3 + 12(\kappa-1)z^2 + 6z(\kappa-1)^2 z + (\kappa^2-1)(\kappa-3)}{8z^3 + 12(\kappa+1)z^2 + 6z(\kappa+1)^2 z + (\kappa^2-1)(\kappa+3)}$$

$$= \frac{(2x+\varepsilon)(8x^3 + 32 - 12\varepsilon x^2 + 6\varepsilon^2 x + 3\varepsilon^3)}{(2x-\varepsilon)(8x^3 + 32 + 12\varepsilon x^2 + 6\varepsilon^2 x - 3\varepsilon^3)}$$

$$\Longrightarrow \frac{1-w}{\varepsilon} = \frac{8(4x^3 - 8 - 3\varepsilon^2 x)}{(2x-\varepsilon)(8x^3 + 32 + 12\varepsilon x^2 + 6\varepsilon^2 x - 3\varepsilon^3)}$$

$$\xrightarrow[\varepsilon\to 0]{} \frac{2(x^3-2)}{x(x^3+4)} = y(x;-2)$$

Some Rational solutions of $\mathrm{P_{IV}}$

$$-\tfrac{2}{3}z \pm \frac{1}{z}, \qquad -\tfrac{2}{3}z \pm \frac{2z^2 \pm 3}{z(2z^2 \mp 3)}, \qquad -\tfrac{2}{3}z \pm \frac{4z}{2z^2 \pm 3}$$

$$-\tfrac{2}{3}z + \frac{24z}{(2z^2-3)(2z^2+3)} \qquad -\tfrac{2}{3}z + \frac{48z(4z^4+9)}{(4z^4-12z^2-9)(4z^4+12z^2-9)}$$

Some Rational solutions of $\mathrm{P_{II}}$

$$\pm\frac{1}{x}, \qquad \pm\frac{2(x^3-2)}{x(x^3+4)}, \qquad \pm\frac{3x^2(x^6+8x^3+160}{(x^3+4)(x^6+20x^3-80)}$$

Painlevé Challenges

1. **Equivalence problem**
   - Given an equation with the Painlevé property, how do we know which Painlevé equation, or Painlevé σ-equation, it is related to?
2. **Numerical solution of Painlevé equations**
   - How do we use the special properties of the Painlevé equations, e.g. that they are solvable by the isomonodromy method through an associated Riemann-Hilbert problem, in the development of numerical software?

Painlevé Equivalence Problem

- Given an equation with the Painlevé property, how do we know which equation, in particular a Painlevé equation or Painlevé σ-equation, it is solvable in terms of?

For linear ODEs, if we can solve the equation in terms of the classical special functions then we regard that the equation is solved.

Example

The linear ODEs

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + z^2 v = 0, \qquad \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \mathrm{e}^{2z} w = 0,$$

respectively have the solutions

$$v(z) = \sqrt{z}\left\{C_1 J_{1/4}\left(\tfrac{1}{2}z^2\right) + C_2 J_{-1/4}\left(\tfrac{1}{2}z^2\right)\right\}$$
$$w(z) = C_1 J_0(\mathrm{e}^z) + C_2 Y_0(\mathrm{e}^z),$$

with C_1 and C_2 arbitrary constants, $J_\nu(\zeta)$ and $Y_\nu(\zeta)$ **Bessel functions**.

MAPLE can easily find such solutions of linear ODEs.

However MAPLE is not as clever for nonlinear ODEs.

MAPLE's `odeadvisor` command will tell you that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6y^2 + x$$

is the first Painlevé equation, but gives “none", i.e. “don't know", as the answer for

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6y^2 - x$$

which is obtained by making the simple transformation $x \to -x$.

Example

Consider the equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + w^3 - 1 \tag{1}$$

This can be shown to possess the Painlevé property, but which equation is it equivalent to? It's not in the list of 50 equations given by **Ince [1956]**.

Equation (1) arises from the symmetry reduction

$$u(x,t) = \ln w(z), \qquad z = 2\sqrt{xt}$$

of the **Tzitzeica equation** (**Tzitzeica [1910]**)

$$u_{xt} = \exp(2u) - \exp(-u)$$

which is also known as the **Bullough-Dodd-Mikhailov-Shabat-Zhiber equation**.

The Painlevé classification is up to a Möbius (bilinear rational) transformation

$$W(\zeta) = \frac{a(z)w(z) + b(z)}{c(z)w(z) + d(z)}, \qquad \zeta = \phi(z)$$

where $a(z)$, $b(z)$, $c(z)$, $d(z)$ and $\phi(z)$ are locally analytic functions.

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + w^3 - 1 \qquad (1)$$

Canonical Equations of Type II

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \qquad \text{XI}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \alpha w^3 + \beta w^2 + \gamma + \frac{\delta}{w} \qquad \text{XII}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w} \qquad \text{XIII}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + q(z)w + \frac{r(z)}{w} + q'(z)w^3 - r'(z) \qquad \text{XIV}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}z} + r(z)w^2 - w\frac{\mathrm{d}}{\mathrm{d}z}\frac{r'(z)}{r(z)} \qquad \text{XV}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - q'(z)\frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}z} + w^3 - q(z)w^2 + q''(z) \qquad \text{XVI}$$

Example

Consider the equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + w^3 - 1 \tag{1}$$

This can be shown to possess the Painlevé property, but which equation is it equivalent to? It's not in the list of 50 equations given by **Ince [1956]**

Answer

Making the transformation

$$w(z) = x^{1/3}y(x), \qquad z = \tfrac{3}{2}x^{2/3} \tag{2}$$

yields

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + y^3 - \frac{1}{x} \tag{3}$$

which is the special case of $\mathrm{P_{III}}$ with $\alpha = 0$, $\beta = -1$, $\gamma = 1$ and $\delta = 0$.

Remark

The transformation (2) is suggested by the asymptotic expansions of (1) and (3)

$$w(z) \sim 1 + \lambda z^{-1/2}\exp\left(-\sqrt{3}\,z\right), \qquad \text{as} \quad z \to \infty$$

$$y(x) \sim x^{-1/3}\left\{1 + \kappa x^{-1/3}\exp\left(-\tfrac{3}{2}\sqrt{3}\,x^{2/3}\right)\right\}, \qquad \text{as} \quad x \to \infty$$

with λ and κ constants.

Example Consider the equation

$$w'''' + ww''' + (w')^2 - z^2 w'' - 3zw' = 0$$

which arises as a scaling reduction of the **Boussinesq equation**

$$u_{tt} + uu_{xx} + u_x^2 + u_{xxxx} = 0$$

Let $w = v'$ to give

$$\begin{aligned} v''''' + v'v''' + (v'')^2 - z^2 v''' - 3zv' &= 0 \\ \Rightarrow \qquad v'''' + v'v'' - z^2 v'' - zv' + v &= 0 \\ \Rightarrow \qquad v''' + \tfrac{1}{2}(v')^2 - z^2 v' + zv &= 6C \end{aligned}$$

with C an arbitrary constant of integration. Multiply by v'' and integrate again to give

$$(v'')^2 - (zv' - v)^2 + \tfrac{1}{3} v' \left\{ (v')^2 - 36C \right\} = 0$$

which has the same functional form as the S_{IV}, the P_{IV} σ-equation. Specifically, letting $v(z) = 6\sqrt{2}\,\sigma(\zeta)$, with $z = \sqrt{2}\,\zeta$, yields

$$\left(\frac{d^2\sigma}{d\zeta^2}\right)^2 - 4\left(\zeta\frac{d\sigma}{d\zeta} - \sigma\right)^2 + 4\frac{d\sigma}{d\zeta}\left\{\left(\frac{d\sigma}{d\zeta}\right)^2 - C\right\} = 0$$

Asymptotics for $\mathrm{P_I}$

(**Bender & Orszag [1969]; Holmes & Spence [1984]; Joshi & Kruskal [1992]**)

There are four families of solutions of the initial value problem for $\mathrm{P_I}$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 6w^2 + x, \qquad w(0) = \kappa, \quad \frac{\mathrm{d}w}{\mathrm{d}x}(0) = \mu$$

where κ and μ are arbitrary constants.

- Solutions which oscillate infinitely often, remain bounded for all finite $x < 0$, with

  $$w(x) = -\left(-\tfrac{1}{6}x\right)^{1/2} + d|x|^{-1/8}\sin\{\varphi(x)\} + o(|x|^{-1/8}), \qquad \text{as} \quad x \to -\infty$$

  where

  $$\varphi(x) = \sqrt[4]{24}\left(\tfrac{4}{5}|x|^{5/4} - \tfrac{5}{8}d^2 \ln|x| - \theta_0\right)$$

  with d and θ_0 parameters (**Qin & Lu [2008]**).
- A unique, monotone increasing, solution, which is bounded for all finite $x < 0$ (known as the **tri-tronquée solution**).
- Solutions with $w(x) \sim +\left(-\tfrac{1}{6}x\right)^{1/2}$, as $x \to -\infty$ (a **tronquée solution**).
- Solutions, each of which has a pole at a finite, real x_0, with $-\infty < x_0 < 0$.

Open Question:

- **How are these solutions related to κ and μ, e.g. how do d and θ_0 depend on κ and μ?**

Numerical Studies of $\mathrm{P_I}$

Consider the initial value problem

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 6w^2 + x, \qquad w(0) = 0, \quad \frac{\mathrm{d}w}{\mathrm{d}x}(0) = \mu$$

where μ is an arbitrary constant. Numerical studies show that:

- $w(x)$ has at least one pole on the real axis;
- there are two special values of μ, namely μ_1 and μ_2, with the properties

  $$-0.451428 < \mu_1 < -0.451427, \qquad 1.851853 < \mu_2 < 1.851855$$

  such that:
  - if $\mu < \mu_1$, then $w(x) > 0$ for $x_0 < x < 0$, where x_0 is the first pole on negative real axis;
  - if $\mu_1 < \mu < \mu_2$, then $w(x)$ oscillates about and is asymptotic to $-\sqrt{\frac{1}{6}|x|}$;
  - if $\mu_2 < \mu$, then $w(x)$ changes sign once, from positive to negative as x passes from x_0 to 0.
- **Fornberg & Weideman [2011]** have recently shown that

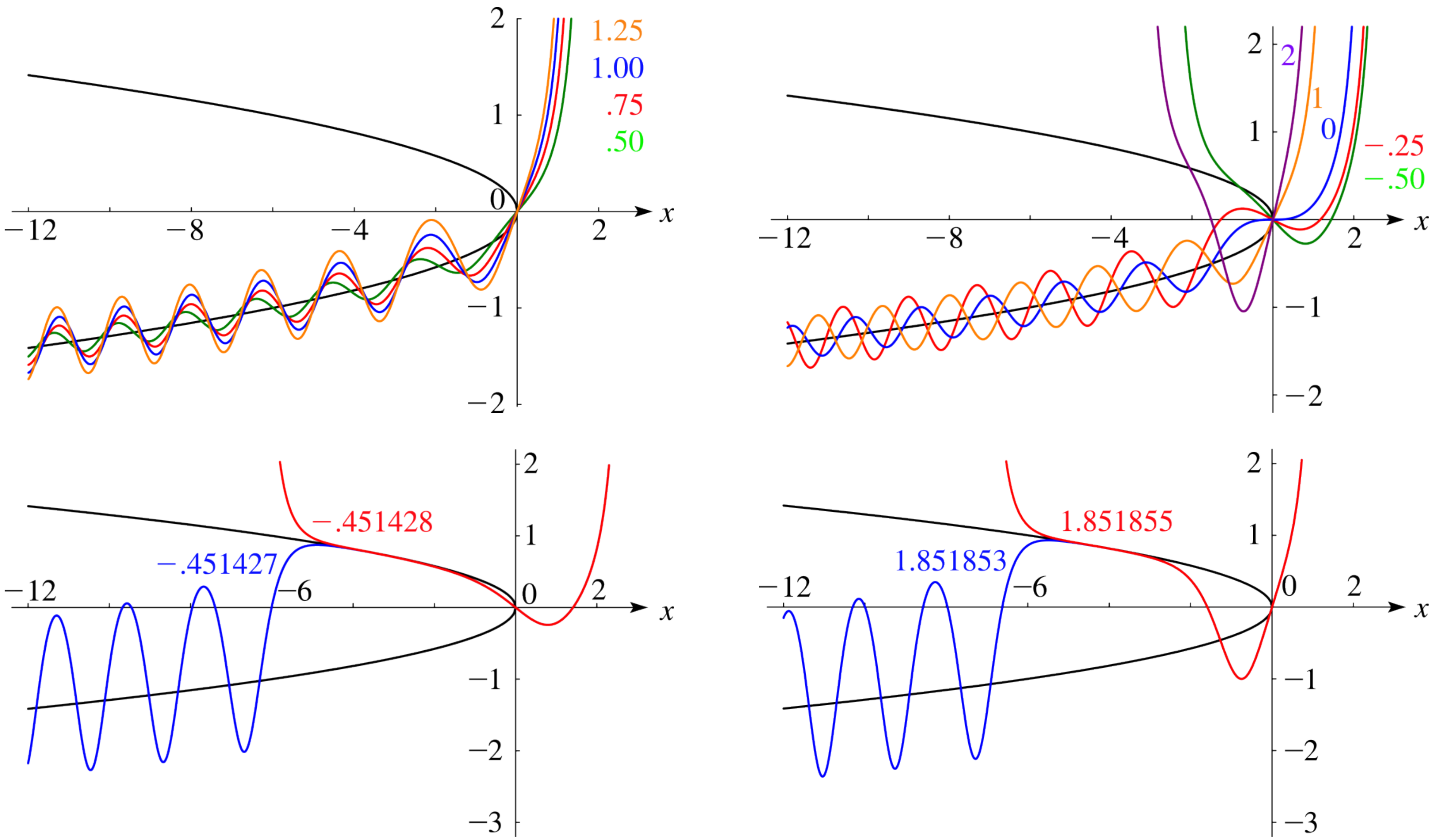
  $$\mu_1 \approx -0.451427404741774, \qquad \mu_2 \approx 1.851854033760367$$

- **Tronquée solutions** with these special values **both** satisfy the BVP

  $$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 6w^2 + x, \qquad w(0) = 0, \quad w(x) \sim \sqrt{-\tfrac{1}{6}x} \quad \text{as} \quad x \to -\infty$$

Painlevé I

$$w'' = 6w^2 + x, \quad w(0) = 0, \quad w'(0) = \mu$$



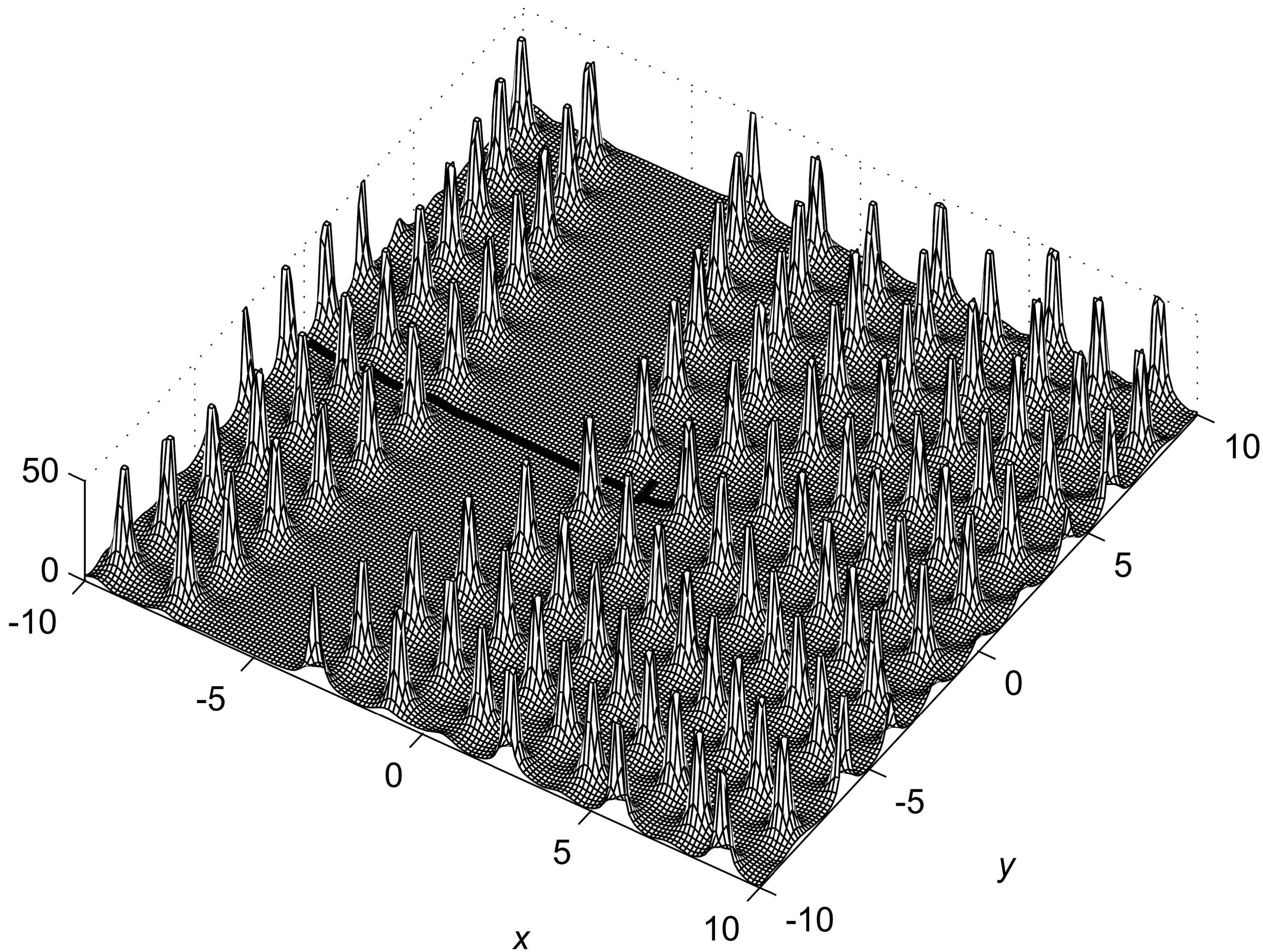
$\mu_1 \approx -0.451427404741774$

$\mu_2 \approx 1.851854033760367$

Painlevé I

$$w'' = 6w^2 + x, \quad w(0) = 0, \quad w'(0) = 1.8518$$

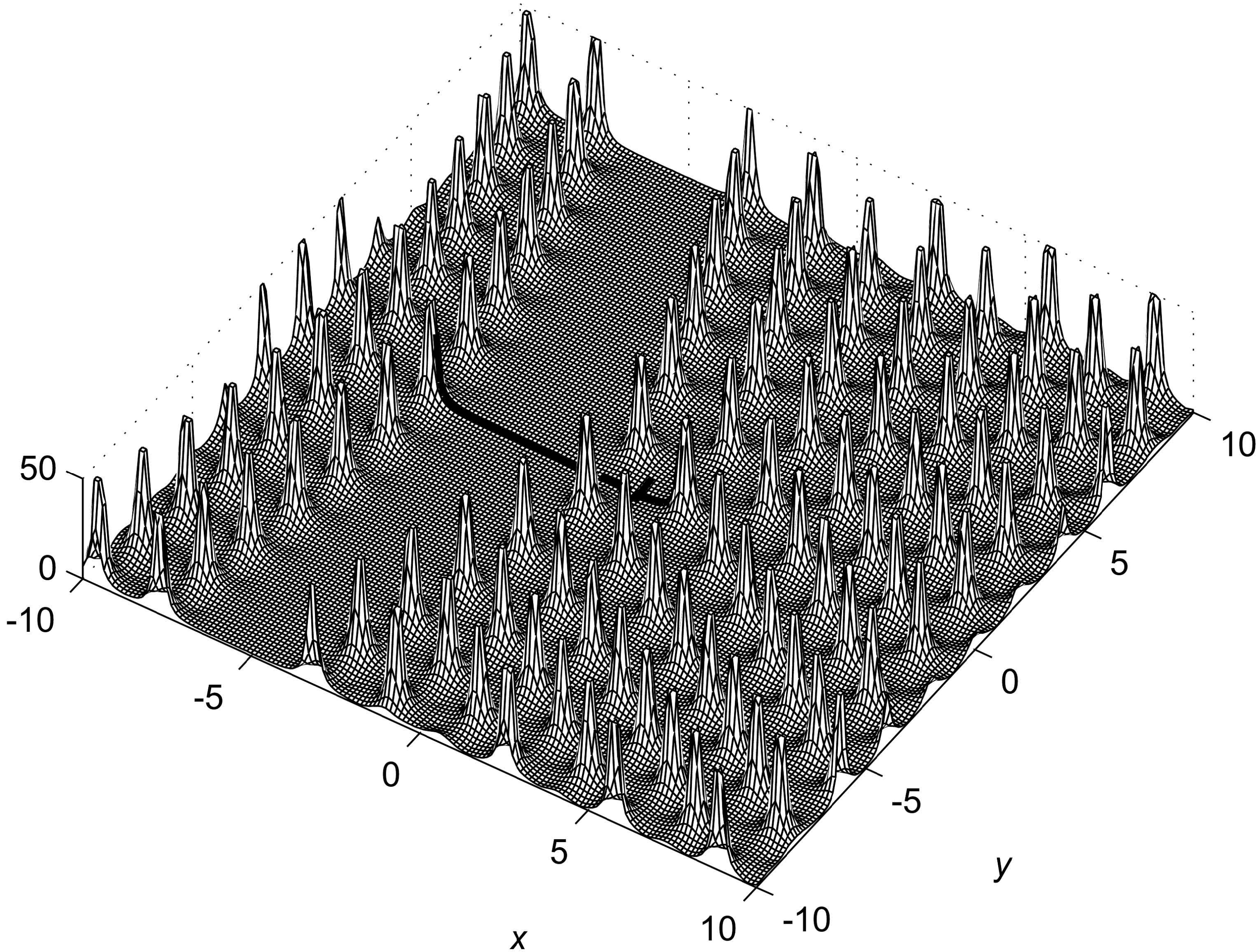
(**Fornberg & Weideman [2011]**)



Painlevé I

$$w'' = 6w^2 + x, \quad w(0) = 0, \quad w'(0) = 1.8519$$

(**Fornberg & Weideman [2011]**)

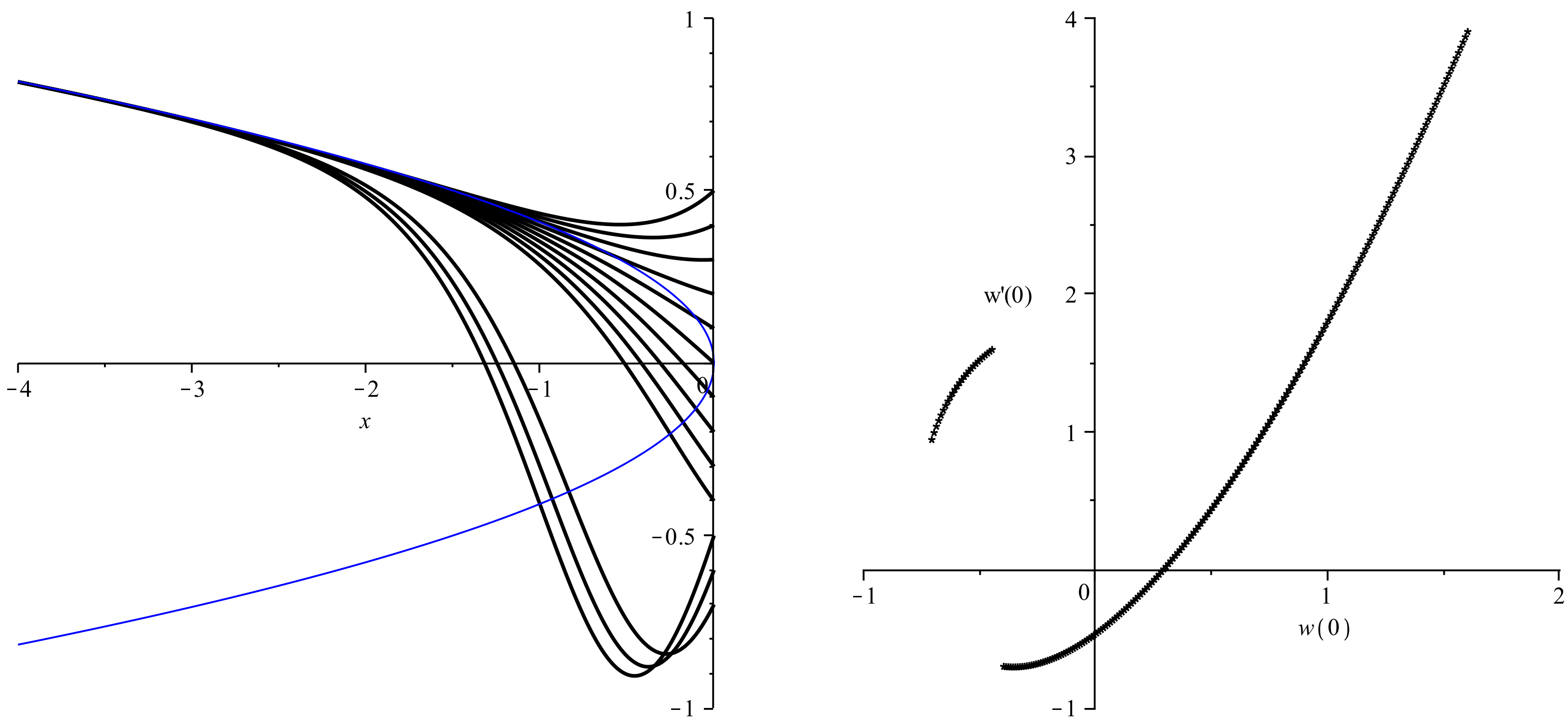


Boundary-Value Problem for $\mathrm{P_I}$

Consider

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 6w^2 + x \qquad \begin{cases} w(0) = \kappa, \\ w(x) \sim \sqrt{-\tfrac{1}{6}x}, \qquad \text{as} \quad x \to -\infty \end{cases}$$

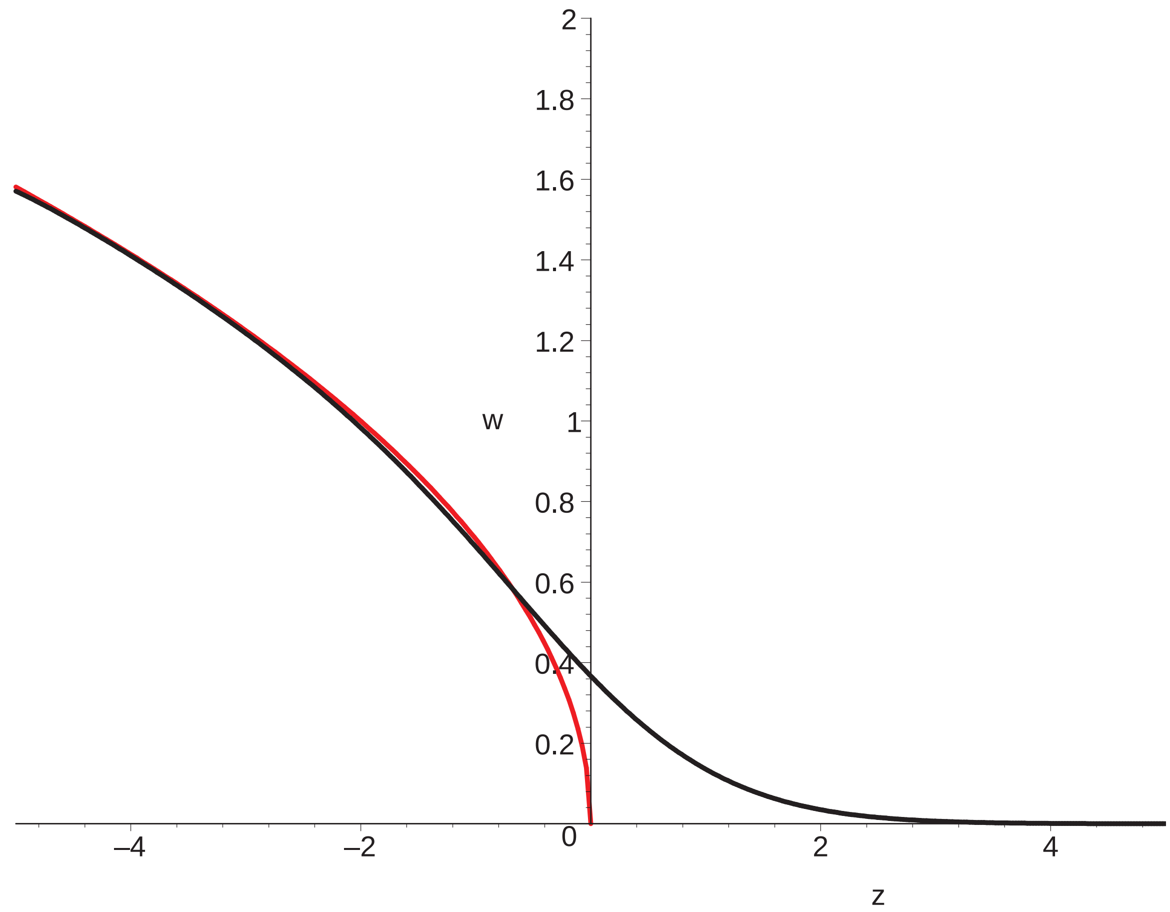
with κ an arbitrary parameter. There are **two** solutions of this BVP for several values of κ, though (naively) using MAPLE's numerical BVP solver only gives one solution.



The Hastings-McLeod Solution

Hastings & McLeod [1980] showed that there is a **unique**, **monotonically decreasing**, solution of the boundary value problem for P_{II} with $\alpha = 0$, for $x \in \mathbb{R}$

$$\frac{\mathrm{d}^2 w_{\mathrm{HM}}}{\mathrm{d}x^2} = 2w_{\mathrm{HM}}^3 + x w_{\mathrm{HM}}, \qquad w_{\mathrm{HM}}(x) \sim \begin{cases} \mathrm{Ai}(x), & \text{as} \quad x \to \infty \\ \left(-\tfrac{1}{2}x\right)^{1/2}, & \text{as} \quad x \to -\infty \end{cases}$$



The Tracy-Widom Distribution

The **Tracy-Widom distribution** is given by

$$F_2(s) = \exp\left\{ -\int_s^\infty (x-s) w_{\mathrm{HM}}^2(x)\,\mathrm{d}x \right\}$$

where $w_{\mathrm{HM}}(x)$ is the **Hastings-McLeod solution**, which can be expressed in terms of solutions of S_{II}, the P_{II} σ-equation.

Theorem

(**Tracy & Widom [1994]**)

In Random Matrix Theory, the limiting distribution for the normalized largest eigenvalue in the Gaussian Unitary Ensemble of $N \times N$ complex Hermitian matrices in the edge scaling limit, is

$$\lim_{N\to\infty} \textbf{Prob}\left(\left(\lambda_{\max} - 2\sqrt{N}\right)\sqrt{2}\,N^{1/6} \le s\right) = F_2(s)$$

$$F_1(s) = \sqrt{F_2(s)}\exp\left\{-\tfrac{1}{2}\int_s^\infty w_{\mathrm{HM}}(x)\,\mathrm{d}x\right\}$$

Gaussian Orthogonal Ensemble
$N \times N$ real symmetric matrices

$$F_4(s/2^{2/3}) = \sqrt{F_2(s)}\cosh\left\{-\tfrac{1}{2}\int_s^\infty w_{\mathrm{HM}}(x)\,\mathrm{d}x\right\}$$

Gaussian Sympletic Ensemble
$N \times N$ self-dual Hermitian matrices

Deift, "*Universality for mathematical and physical systems*", arXiv:math-ph/0603038

Theorem (**Its & Kapaev [2003]**)

There is a unique solution of $\mathrm{P_{II}}$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 2w^3 + xw + \alpha \tag{1}$$

which satisfies the boundary conditions

$$w(x) \sim \begin{cases} -\dfrac{\alpha}{x}, & \textit{as} \quad x \to \infty \\ -\operatorname{sgn}(\alpha)\sqrt{-\tfrac{1}{2}x}, & \textit{as} \quad x \to -\infty \end{cases} \tag{2}$$

Theorem (**Claeys, Kuijlaars & Vanlessen [2008]**)

The solution $w(x)$ of (1) satisfying the boundary conditions (2), which is the analog of the ***Hastings-McLeod solution****, is a meromorphic function with no poles on the real line.*

The asymptotics of (1) as $|x| \to \infty$ are given by

$$w(x) = w_1(x) + kx^{-1/4}\exp\left(-\tfrac{2}{3}x^{3/2}\right)\left\{1 + \mathcal{O}\left(x^{-3/4}\right)\right\} + \mathcal{O}\left(x^{-7/4}\exp\left(-\tfrac{4}{3}x^{3/2}\right)\right)$$

where $w_1(x) \sim -\alpha/x$ and k is a parameter which takes different values in different sectors (**Its & Kapaev [2003]**).

- If $\alpha = n \in \mathbb{Z}$ then $w_1(x)$ is the associated rational solution.
- If $\alpha = n \notin \mathbb{Z}$ then $w_1(x)$ is a divergent series.

Conjecture (PAC [2008])

Let k be an arbitrary, non-zero real number and $w_k(x;n)$ be the solution of $\mathrm{P_{II}}$ *with* $\alpha = n \in \mathbb{Z}$, *for* $x \in \mathbb{R}$

$$\frac{\mathrm{d}^2 w_k}{\mathrm{d}x^2} = 2w_k^3 + xw_k + n, \qquad n \in \mathbb{Z}, \quad x \in \mathbb{R}$$

with boundary condition

$$w_k(x;n) \sim q_n(x) + k\,\mathrm{Ai}(x), \qquad \textit{as} \quad x \to \infty$$

where $q_n(x)$ is the rational solution of $\mathrm{P_{II}}$ *for $\alpha = n$.*

- *There exists a unique k_n^* such that for $k < k_n^*$, then $w_k(x;n)$ blows up at a finite x_1, with*

$$w_k(x;n) \sim -\frac{\mathrm{sgn}(n)}{x - x_1}, \qquad \textit{as} \quad x \downarrow x_1$$

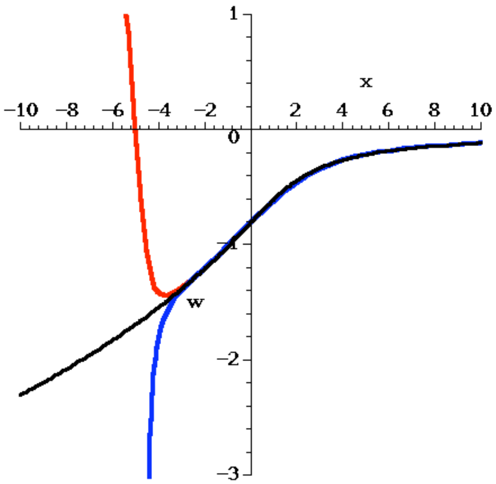
  and for $k > k_n^*$, then $w_k(x;n)$ blows up at a finite x_2, with

$$w_k(x;n) \sim \frac{\mathrm{sgn}(n)}{x - x_2}, \qquad \textit{as} \quad x \downarrow x_2$$

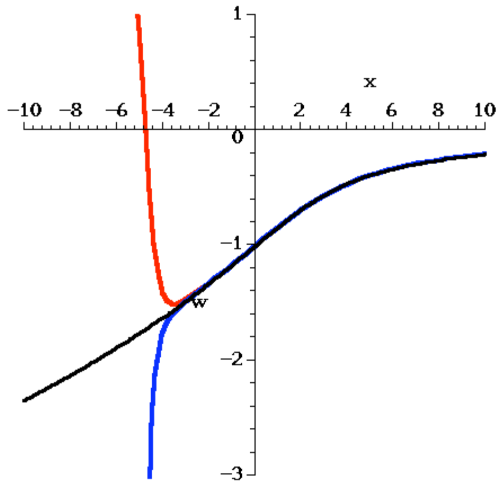
- *For $n > 0$, $w_{k_n^*}(x;n)$ is a* ***negative, monotonically increasing*** *solution, and for $n < 0$, $w_{k_n^*}(x;n)$ is a* ***positive, monotonically decreasing*** *solution. Further*

$$w_{k_n^*}(x;n) \sim \begin{cases} -n/x, & \textit{as} \quad x \to +\infty \\ -\,\mathrm{sgn}(n)\sqrt{-\tfrac{1}{2}x}, & \textit{as} \quad x \to -\infty \end{cases}$$

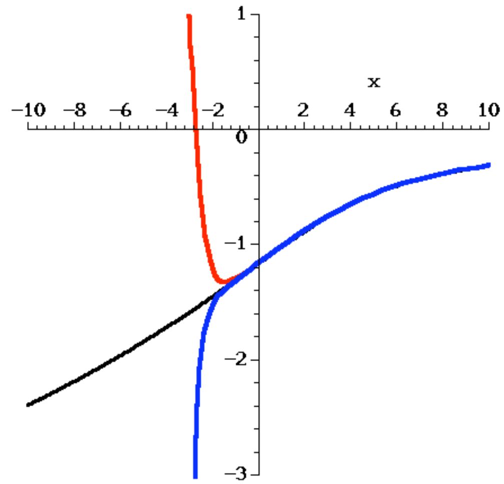
$$w_k'' = 2w_k^3 + xw_k + n, \qquad w_k(x;n) \sim q_n(x) + k\,\mathrm{Ai}(x), \quad \text{as } \ x \to \infty$$



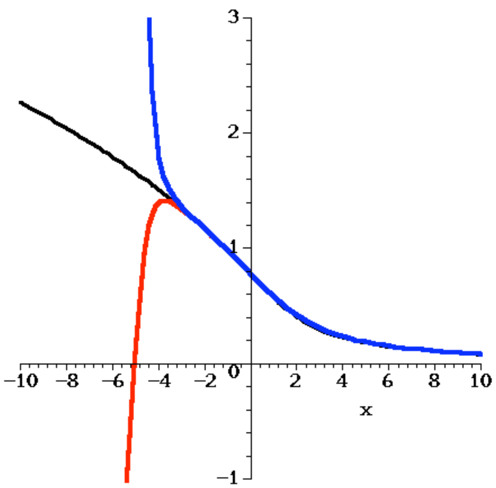
$n = 1$
$k = 1.06493, 1.06494$



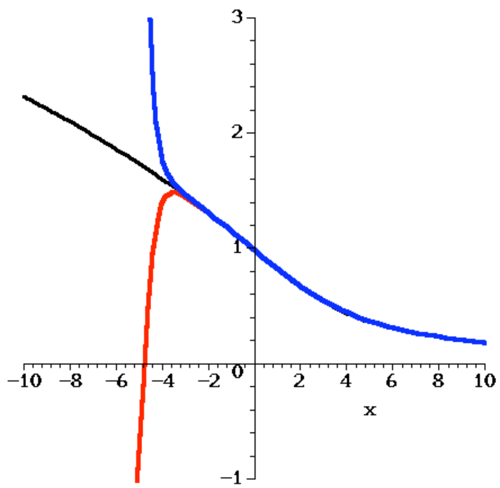
$n = 2$
$k = -1.29624, -1.28623$



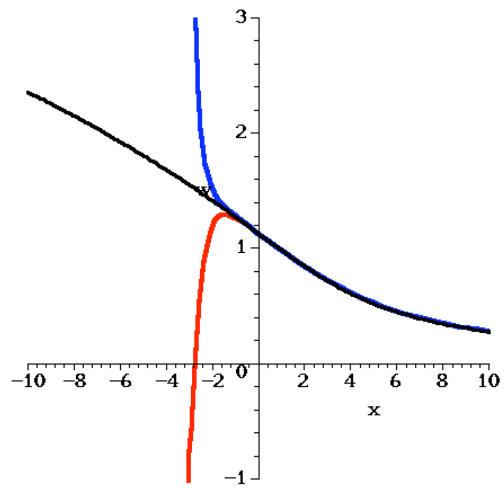
$n = 3$
$k = 1.75176, 1.75177$



$n = -1$
$k = -1.06494, -1.06493$



$n = -2$
$k = 1.29623, 1.28624$



$n = -3$
$k = -1.75177, -1.75176$

- It is interesting to compare these numerical results with the results of **McCoy & Tang [1986]** (see also **Kapaev [1992]**, **Fokas *et al.* [2006]**) which infer that $k_n^* = \pm 1$.
- Recently **Bornemann** (private communication), using Mathematica and a large number of digits, **S. Olver** (private communication), using a method based on the Riemann-Hilbert problem, and **Fornberg & Weideman** (private communication), using a "pole field solver", have also numerically studied the problem and their results suggest that $k_n^* = \pm 1$.
- To date there is no numerical evidence for the existence of bounded solutions of $\mathrm{P_{II}}$ for $\alpha = n \in \mathbb{Z}\backslash\{0\}$, with the boundary condition

$$w_k(x;n) \sim q_n(x) + k\,\mathrm{Ai}(x), \qquad \text{as} \quad x \to \infty$$

  with $q_n(x)$ the rational solution, which have oscillatory behaviour as $x \to -\infty$.
- **Ablowitz & Segur [1981]** suggest that all real solutions of $\mathrm{P_{II}}$ with $\alpha = n \neq 0$ and $|k| < 1$ oscillate as $x \to -\infty$ and that these solutions are not bounded since they all have pole singularities at some finite x. Hence the oscillating behavior can not be observed by a direct numeric continuation from the right to the left side.
- **Kashevarov [1998, 2004]** suggests numerically that there are such solutions of $\mathrm{P_{II}}$ for some non-integer values of α.

Asymptotics of P_{IV} — Nonlinear Harmonic Oscillator

Consider the special case of P_{IV} where $w(z) = 2\sqrt{2}\, q^2(\zeta)$ and $\zeta = \sqrt{2}\, z$, with $\alpha = 2\nu+1$ and $\beta = 0$, i.e.

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\zeta^2} = 3q^5 + 2\zeta q^3 + (\tfrac{1}{4}\zeta^2 - \nu - \tfrac{1}{2})q \tag{1}$$

and the boundary condition

$$q(\zeta) \to 0, \qquad \text{as} \quad \zeta \to +\infty \tag{2}$$

This equation has solutions have exponential decay as $\zeta \to \pm\infty$ and so are nonlinear analogues of **bound state solutions** for the **linear harmonic oscillator**.

Let $q_k(\zeta)$ denote the unique solution of (1) which is asymptotic to $kD_\nu(\zeta)$, i.e.

$$\frac{\mathrm{d}^2 q_k}{\mathrm{d}\zeta^2} = 3q_k^5 + 2\zeta q_k^3 + (\tfrac{1}{4}\zeta^2 - \nu - \tfrac{1}{2})q_k$$

with boundary condition

$$q_k(\zeta) \sim kD_\nu(\zeta), \qquad \text{as} \quad \zeta \to +\infty$$

where $D_\nu(\zeta)$ is the **parabolic cylinder function** which satisfies

$$\frac{\mathrm{d}^2 D_\nu}{\mathrm{d}\zeta^2} = (\tfrac{1}{4}\zeta^2 - \nu - \tfrac{1}{2})D_\nu$$

with boundary condition

$$D_\nu(\zeta) \sim \zeta^\nu \exp\left(-\tfrac{1}{4}\zeta^2\right), \qquad \text{as} \quad \zeta \to +\infty$$

Theorem

$$\frac{\mathrm{d}^2 q_k}{\mathrm{d}\zeta^2} = 3q_k^5 + 2\zeta q_k^3 + (\tfrac{1}{4}\zeta^2 - \nu - \tfrac{1}{2})q_k, \qquad q_k(\zeta) \sim k D_\nu(\zeta), \quad \text{as} \quad \zeta \to +\infty$$

- *If* $0 \le k < k_*$, *where*

$$k_*^2 = \frac{1}{2\sqrt{2\pi}\,\Gamma(\nu+1)}$$

then this solution exists for all real ζ *as* $\zeta \to -\infty$.

  - *If* $\nu = n \in \mathbb{N}$

$$q_k(\zeta) \sim \frac{k D_n(\zeta)}{\sqrt{1 - 2\sqrt{2\pi}\, n!\, k^2}}, \qquad \text{as} \quad \zeta \to -\infty$$

  - *If* $\nu \notin \mathbb{N}$, *then for some* d *and* $\theta_0 \in \mathbb{R}$,

$$q_k(\zeta) = (-1)^{[\nu+1]}\left(-\tfrac{1}{6}\zeta\right)^{1/2} + d|\zeta|^{-1/2}\sin\left(\frac{\zeta^2}{2\sqrt{3}} - \frac{4d^2}{\sqrt{3}}\ln|\zeta| - \theta_0\right) + \mathcal{O}\left(|\zeta|^{-3/2}\right), \qquad \text{as} \quad \zeta \to -\infty$$

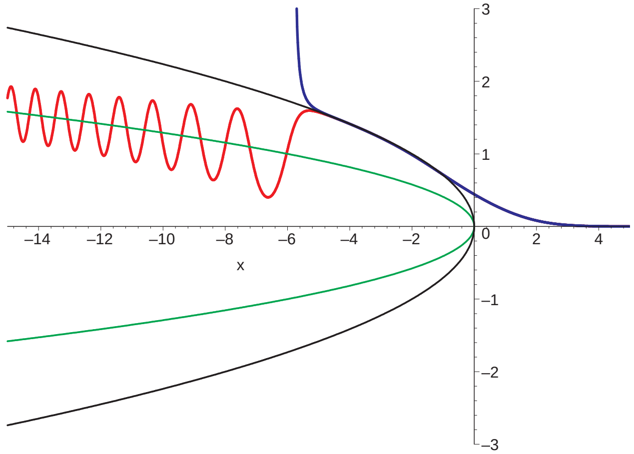
- *If* $k = k_*$, *then*

$$q_k(\zeta) \sim \left(-\tfrac{1}{2}\zeta\right)^{1/2}, \qquad \text{as} \quad \zeta \to -\infty$$

- *If* $k > k_*$ *then* $q_k(\zeta)$ *has a pole at a finite* ζ_0 *depending on* k, *so*

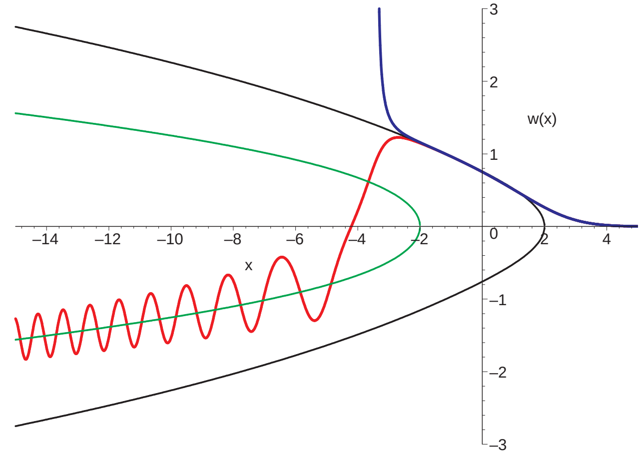
$$q_k(\zeta) \sim (\zeta - \zeta_0)^{-1/2}, \qquad \text{as} \quad \zeta \downarrow \zeta_0$$

$$\frac{\mathrm{d}^2 q_k}{\mathrm{d}\zeta^2} = 3q_k^5 + 2\zeta q_k^3 + (\tfrac{1}{4}\zeta^2 - \nu - \tfrac{1}{2})q_k, \qquad q_k(\zeta) \sim kD_\nu(\zeta), \quad \text{as} \quad \zeta \to +\infty$$



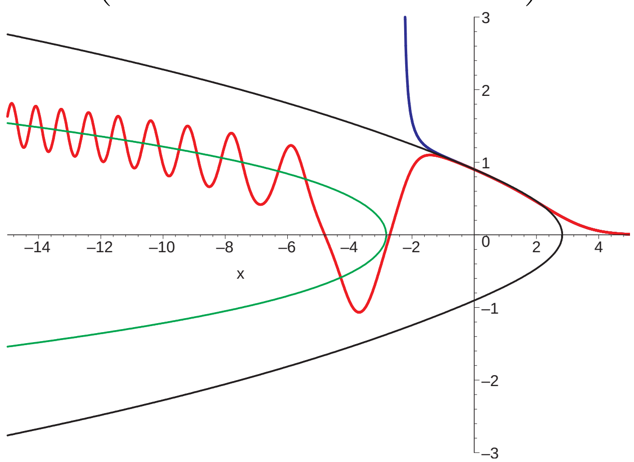
$\nu = -\frac{1}{2}$, $k = 0.33554691, 0.33554692$

$(k_* = 0.3354691334)$



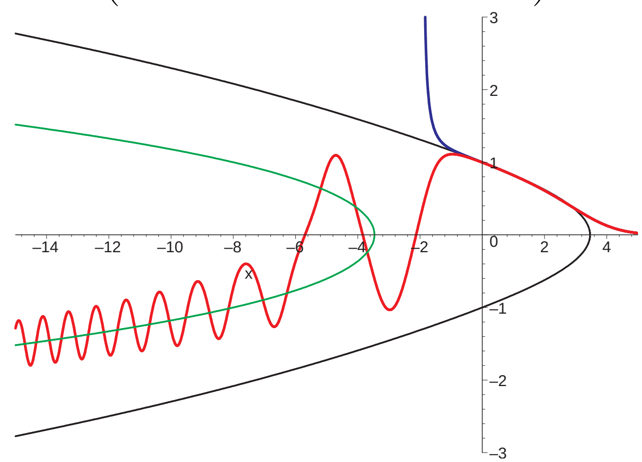
$\nu = \frac{1}{2}$, $k = 0.47442, 0.47443$

$(k_* = 0.4744249982)$



$\nu = \frac{3}{2}$, $k = 0.38736, 0.38737$

$(k_* = 0.3873663890)$



$\nu = \frac{5}{2}$, $k = 0.244992, 0.244993$

$(k_* = 0.2449920156)$

Remark

The solution of the boundary value problem

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\zeta^2} = 3q^5 + 2\zeta q^3 + (\tfrac{1}{4}\zeta^2 - \nu - \tfrac{1}{2})q, \qquad q(\zeta) \sim \begin{cases} k_* D_\nu(\zeta), & \text{as} \quad \zeta \to +\infty \\ \left(-\tfrac{1}{2}\zeta\right)^{1/2}, & \text{as} \quad \zeta \to -\infty \end{cases} \tag{1}$$

where

$$k_*^2 = \frac{1}{2\sqrt{2\pi}\,\Gamma(\nu+1)}$$

is the analog of the **Hastings-McLeod solution** of P_{II} with $\alpha = 0$ which satisfies the boundary value problem

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 2w^3 + xw, \qquad w(x) \sim \begin{cases} \mathrm{Ai}(x), & \text{as} \quad x \to \infty \\ \left(-\tfrac{1}{2}x\right)^{1/2}, & \text{as} \quad x \to -\infty \end{cases} \tag{2}$$

We note that making the transformation

$$q(\zeta) = \varepsilon w(x), \qquad \zeta = 2\varepsilon^2 x + \frac{1}{4\varepsilon^4} \tag{$*$}$$

in

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\zeta^2} = 3q^5 + 2\zeta q^3 + (\tfrac{1}{4}\zeta^2 - \nu - \tfrac{1}{2})q$$

and taking the limit as $\varepsilon \to 0$ yields

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 2w^3 + xw$$

Further that making the transformation $(*)$ in the parabolic cylinder equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\zeta^2} = (\tfrac{1}{4}\zeta^2 - \nu - \tfrac{1}{2})q$$

and taking the limit as $\varepsilon \to 0$ yields the Airy equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = xw$$

Numerical Studies of Painlevé Equations

- My numerical simulations were obtained using MAPLE using the `DEplot` command with option `method=dverk78`, which finds a numerical solution using a seventh-eighth order continuous Runge-Kutta method. This is easy to use, gives plots of solutions quickly with accuracy better than the human eye can detect.
- There have been several numerical studies of the **Hastings-McLeod solution** of $\mathrm{P_{II}}$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 2w^3 + xw, \qquad w(x) \sim \begin{cases} \mathrm{Ai}(x), & \text{as} \quad x \to \infty \\ \left(-\frac{1}{2}x\right)^{1/2}, & \text{as} \quad x \to -\infty \end{cases}$$

  some of which have obtained the solution to high precision [e.g. **Driscoll, Bornemann & Trefethen (2008); Edelman & Raj Rao (2005); Grava & Klein (2008); Prähofer & Spohn (2004)**].
- The Runge-Kutta method, including its variants, is a standard ODE solver. Can we do better for integrable ODEs such as the Painlevé equations?
- Painlevé equations are solvable by the isomonodromy method through an associated Riemann-Hilbert problem (inverse scattering for ODEs). How can we use this in the development of software for studying the Painlevé equations numerically?
- Should we use a “integrable discretization" of the Painlevé equations? It is well known that there **discrete Painlevé equations**, which are integrable discrete equations that tend to the associated Painlevé equations in an appropriate continuum limit.

Objectives

- To provide a complete classification and unified structure of the special properties which the Painlevé equations and Painlevé σ-equations possess — the presently known results are rather fragmentary and non-systematic.
- Develop algorithmic procedures for the classification of equations with the Painlevé property.
- Develop software for numerically studying the Painlevé equations which utilizes the fact that they are integrable equations solvable using isomonodromy methods.
- To produce a general theorem on uniform asymptotics for linear systems to cover all those systems which arise as isomonodromy problems of the Painlevé equations.

Reference

P A Clarkson, Painlevé equations — nonlinear special functions, in "*Orthogonal Polynomials and Special Functions: Computation and Application*" [Editors F Marcellàn and W van Assche], *Lect. Notes Math.*, **1883**, Springer, Berlin (2006) pp 331–411