Asymptotic forms and asymptotic expansions of solutions to the Painlevé equations

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We study ordinary differential equations (ODE's) that can be reduced to the form

$$f(x, y, y', y'', \ldots, y^{(n)}) = 0, \tag{1}$$

where x and y are independent and dependent variables correspondingly, $y' = dy/dx$, f is a polynomial in its variables $x, y, y', y'', \ldots, y^{(n)}$.

We can find all asymptotic expansions of solutions to the equation (1) near its singular and nonsingular points of the six types.

Namely, power, power-logarithmic, complicated, exotic, semiexotic and exponential.

Types of asymptotic expansions of solutions

At $x = 0$ we seek all asymptotic expansions of solutions to the equation (1) of the form

$$y = c_r x^r + \sum_s c_s x^s, \qquad (2)$$

where power exponents r and s are complex, $\operatorname{Re} s > \operatorname{Re} r$, $\operatorname{Re} s$ increase. They belong to one of three types: power, power-logarithmic or complicated. Expansions of these types have finite number of power exponents s with the same $\operatorname{Re} s$.

Coefficients c_r and c_s are varied according to types:

Type 1. c_r and c_s are constants (power expansions);

Type 2. c_r is constant, c_s are polynomials in $\log x$ (power-logarithmic expansions);

Type 3. c_r and c_s are series in decreasing powers of $\log x$ (complicated expansions).

At $x = 0$ we also consider expansions of the form

$$y = \sum_\rho c_\rho x^\rho + \sum_s c_s x^s, \qquad (3)$$

where power exponents ρ and s are complex, all $\operatorname{Re}\rho$ are the same, $\operatorname{Re}\rho < \operatorname{Re}s$, $\operatorname{Re}s$ increase, first sum contains more than one term, complex coefficients c_ρ and c_s are polynomials in $\log x$.

We also suppose $\arg x$ bounded.

We differ two types of the expansions (3).

Type 4. The first sum in (3) contains infinite number of terms, but $\operatorname{Im}\rho$ are bounded either below or above. Coefficient c_ρ is constant for extreme value of $\operatorname{Im}\rho$ (exotic expansions).

Type 5. The first sum in

$$y = \sum_\rho c_\rho x^\rho + \sum_s c_s x^s, \tag{3}$$

contains a finite number of terms. Coefficients c_ρ are constants for both extreme values of $\mathrm{Im}\,\rho$. The numbers of power exponents s with the same real parts $\mathrm{Re}\,s$ are finite (semiexotic expansions);

Let $\min(\mathrm{Im}\rho)$ in (3) be reached at $\rho = \rho_1$, and $\max(\mathrm{Im}\rho)$ at $\rho = \rho_2$. A semiexotic expansion (3) has two inverse expansions

$$(y^{-1})_1 = c_{\rho_1}^{-1} x^{-\rho_1}\left(\sum_\sigma b_\sigma x^\sigma + \sum_t b_t x^t\right), \tag{4}$$

$$(y^{-1})_2 = c_{\rho_2}^{-1} x^{-\rho_2}\left(\sum_\sigma \tilde{b}_\sigma x^\sigma + \sum_t \tilde{b}_t x^t\right), \tag{5}$$

where $\mathrm{Re}\,\sigma = 0$, which are exotic ones.

Type 6. In addition, there are exponential expansions

$$y = \sum_{k=0}^{\infty} b_k(x) C^k e^{k\phi(x)}, \qquad (6)$$

where $b_k(x)$ and $\phi(x)$ are power series, but C is arbitrary constant.

All Painlevé equations (P1 – P6 equations) can be reduced to the form

$$f(x, y, y', y'') = 0,$$

where f is a polynomial of its variables. So all expansions of their solutions of the six types can be found.

The sixth Painlevé (P6) equation has form

$$
\begin{aligned}
y'' &= \frac{(y')^2}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) - y' \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) \\
&+ \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[a + b\frac{x}{y^2} + c\frac{x-1}{(y-1)^2} + d\frac{x(x-1)}{(y-x)^2} \right],
\end{aligned} \tag{7}
$$

where a, b, c, d are complex parameters, x and y are complex variables independent and dependent correspondingly. It has three singular points $x = 0, 1, \infty$.

Near its three singular points $x = 0$, 1, ∞ and near its nonsingular points $x = x_0 \neq 0, 1, \infty$ for all values its four complex parameters a, b, c, d we found all asymptotic expansions of its solutions of the five types:

power, power-logarithmic, complicated, exotic, semiexotic.
Exponential expansions are absent.

The equation (7) has three main symmetries that allow to transfer solutions near of one its singular point to solutions near the same or another its singular point.

These symmetries are associated with three changes of variables

$$
\begin{aligned}
&1.\ x = z,\ y = z/w; \\
&2.\ x = 1/z,\ y = 1/w; \\
&3.\ x = 1 - z,\ y = 1 - w.
\end{aligned}
\tag{8}
$$

The first symmetry takes the expansions of solutions in a neighborhood of zero to the expansions of solutions in a neighborhood of infinity (and vice versa).

The second symmetry preserves the position of singular points.

The third symmetry takes the expansions of solutions in a neighborhood of zero to the expansions of solutions in a neighborhood of unity.

On the sixth Painlevé equation

First we obtain asymptotic expansions of such solutions to the equation (7) near the singular point $x = 0$, for which the order of the first term is less than unity. We refer to these expansions as *basic expansions*. They form 21 families.

We obtain all other asymptotic expansions of solutions to the equation (7) near its three singular points $x = 0$, $x = 1$, and $x = \infty$ from the basic expansions using symmetries (8) of the equation. Altogether they form 117 families [1].

Studies of asymptotic forms and asymptotic expansions of solutions to P6 were performed by authors:
S. Shimomura, M. Jimbo, H. Kimura, K. Okamoto, I. V. Gromak, B. A. Dubrovin, M. Mazzocco, D. Guzzetti, and many others.

On results obtained for P6

We obtain all asymptotic expansions of solutions to the sixth Painlevé equation near all its three singular points $x = 0$, $x = 1$, and $x = \infty$ for all values of four complex parameters a, b, c, d of this equation [1]. They belong to expansions of five types: power, power-logarithmic, complicated, semiexotic, and exotic. They form 117 families. Among them 33 families of expansions are power, 12 families are power-logarithmic, 18 families are complicated, 48 families are exotic, 6 families are semiexotic. The most of them are new.

In neighborhood of a regular point $x = x_0 \neq 0, 1, \infty$ of the sixth Painlevé equation (7) for all values its four parameters there exist 17 families of power expansions of its solutions [2]. They are Laurent or Taylor series. Among them 1 family of expansions has the pole of the second order, 2 families of expansions have poles of the first order, other ones are families of Taylor expansions. 8 families of these expansions are new.

For $x \to 0$ there exists family of power expansions of solutions to the equation (7) with two parameters c_r and r, and constant coefficients, which has form

$$\mathscr{A}_0 : y = c_r x^r + \sum_s c_s x^s, \qquad (9)$$

where complex power exponent r is arbitrary with $\operatorname{Re} r \in (0, 1)$, complex power exponents
$s \in \{r + lr + m(1 - r), l, m \geq 0; l + m > 0; l, m \in \mathbb{Z}\}$; complex coefficient c_r is arbitrary nonzero constant, other complex coefficients c_s are uniquely determined constants.

The family $\mathscr{A}_0$ was known. Family $\mathscr{A}_0$ exists for all values of parameters of equation (7).

Altogether there are 33 families of power expansions of solutions to P6.

For $x \to 0$ and $a = c \neq 0$, $2\sqrt{2a} \in \mathbb{Z} \backslash \{0\}$ there exists family of power-logarithmic expansions of solutions to the equation (7) of the form

$$\mathscr{B}_2 : \quad y = 2 + \sum_{s=1}^{\infty} c_s(\log x) x^s, \qquad (10)$$

where coefficients $c_s(\log x)$ are polynomials in $\log x$.

Altogether there are 12 families of power-logarithmic expansions of solutions to P6.

For $x \to 0$ and $a \neq c \neq 0$ there exists family of complicated expansions of solutions to the equation (7) of the form

$$\mathscr{B}_3: \quad y = \psi_0 + \sum_{\sigma=1}^{\infty} \psi_\sigma x^\sigma, \tag{11}$$

where

$$\psi_0 = \frac{2}{c-a} \frac{1}{\log^2 x} + \frac{c_{-3}}{\log^3 x} + \sum_{s=4}^{\infty} \frac{c_{-s}}{\log^s x} = \frac{2(c-a)}{(c-a)^2 (\log x + C_0)^2 - 2a},$$

coefficients c_{-3} and C_0 are arbitrary constants, other coefficients c_{-s} are uniquely determined constants; ψ_σ are series in decreasing powers of $\log x$.

Altogether there are 18 families of complicated expansions of solutions to P6.

Example of exotic expansions of solutions to P6

For $x \to 0$ there exists family of exotic expansions of solutions to the equation (7) with two parameters C_1 and ρ, and constant coefficients, which has form

$$\mathscr{B}_0^\tau : y = \frac{\rho^2}{\beta \cos^2[\log(C_1 x)^\gamma] + \alpha \sin^2[\log(C_1 x)^\gamma]} + \sum_{\text{Re}\, s \geq 1} c_s x^s =$$

$$= x^\rho \left(c_\rho + \sum_{k=1}^{\infty} \tilde{c}_k x^{k\rho} \right) + \sum_{\text{Re}\, s \geq 1} c_s x^s,$$

(12)

where complex power exponent ρ is pure imaginary nonzero arbitrary constant,
$s \in \{\rho + l\rho + m(1-\rho); l, m \geq 0; l+m > 0; l, m \in \mathbb{Z}\}$, $\tau = \text{sgn}(\text{Im}\,\rho)$,
$\alpha + \beta = (\rho^2 - 2c + 2a)/(2a)$, $\alpha\beta = \rho^2/(2a)$, $2\gamma = i\rho$, complex coefficients c_ρ and C_1 are nonzero arbitrary constants and interrelated, other complex coefficients $\tilde{c}_k$ and c_s are uniquely determined constants. Altogether there are 48 families of exotic expansions of solutions to P6.

Example of semiexotic expansions of solutions to P6

For $x \to 0$ there exists a family of semiexotic expansions to solutions of the equation (7) with two parameters c_ρ and ρ, and constant coefficients, which has form

$$\mathscr{H}_0: \quad y = c_\rho x^\rho + c_1 x + c_{2-\rho} x^{2-\rho} + \sum_s c_s x^s + \dots, \qquad (13)$$

where power exponent ρ is complex, $\rho - 1$ is pure imaginary arbitrary constant, s runs the set $\{l + k(\rho - 1); \, l, k \in \mathbb{Z}; \, l \geq 2, |k| \leq l\}$, complex coefficient c_ρ is arbitrary constant, other complex coefficients c_1, $c_{2-\rho}$ and c_s are uniquely determined constants.

Altogether there are 6 families of semiexotic expansions of solutions to P6.

On convergent formal solutions

We consider an ordinary differential equation of the form

$$f(x, y, y', y'', \ldots, y^{(n)}) = 0, \tag{1}$$

where x and y are independent and dependent variables correspondingly, $y' = dy/dx$, f is a polynomial of its variables $x, y, y', y'', \ldots, y^{(n)}$.

Let for $|x| \to 0$ and $\arg(x) \in (-\pi, \pi)$, equation (1) have the formal solution

$$y = \sum c_s x^s, \ s \in \mathbf{K} \subset \mathbb{C}, \tag{8}$$

where power exponents s are complex, $\operatorname{Re} s$ increase, number of power exponents s with the same real parts $\operatorname{Re} s$ is finite, coefficients c_s are complex constants.

On convergent formal solutions

By the substitution

$$y = \sum_{s=s_0}^{s_m} c_s x^s + u, \qquad (14)$$

where $m \in \mathbb{Z}$, $m \geq 0$, $\operatorname{Re} s_m \geq n$, we reduce equation (1) to the form

$$f_1(x, u) \overset{def}{=} \mathscr{L}(x)u + g(x, u, u', \ldots, u^{(n)}) = 0, \qquad (15)$$

where the linear differential operator

$$\mathscr{L}(x) = x^v \sum_{l=1}^{n} a_l x^l \frac{d^l}{dx^l},$$

$\mathscr{L}(x) \not\equiv 0$, $v \in \mathbb{C}$, a_l are complex constants.

The function g contains terms independent of $u, u', \ldots, u^{(n)}$, linear terms in $u, u', \ldots, u^{(n)}$ of the form $c x^{v_1+l} u^{(l)}$ with $\operatorname{Re} v_1 > \operatorname{Re} v$, $l \leq n$, $c = \operatorname{const} \in \mathbb{C}$, and nonlinear terms in $u, u', \ldots, u^{(n)}$.

On convergent formal solutions

The linear differential operator $\mathscr{L}(x)$ has eigenvalues $\lambda_1, \ldots, \lambda_n$. We order these eigenvalues by increasing real parts: $\operatorname{Re}\lambda_1 \leq \cdots \leq \operatorname{Re}\lambda_n$.

We suppose that $\operatorname{Re} s_m > \operatorname{Re}\lambda_n$ in substitution (2). Then for $|x| \to 0$, $\arg(x) \in (-\pi, \pi)$ equation (15) has a unique solution of the form

$$u = \sum_{s=s_{m+1}}^{\infty} c_s x^s, \tag{16}$$

where power exponents $s \in \mathbb{C}$, $\operatorname{Re} s$ increase, complex coefficients c_s are uniquely determined constants.

In 2004 Prof. Bruno had formulated the theorem on convergence of a power series solution to an ordinary differential equation.

If in the equation

$$f_1(x, u) \overset{def}{=} \mathscr{L}(x)u + g(x, u, u', \ldots, u^{(n)}) = 0,$$

which we obtain after substitution

$$y = \sum_{s=s_0}^{s_m} c_s x^s + u,$$

to the equation

$$f(x, y, y', y'', \ldots, y^{(n)}) = 0,$$

the order of the highest derivative in $\mathscr{L}(x)u$ is equal to the order of the highest derivative in the sum f_1, then the series

$$u = \sum_{s=s_{m+1}}^{\infty} c_s x^s,$$

converges for sufficiently small $|x|$ and $\arg x \in (-\pi, \pi)$.

On convergent formal solutions

In 2010 we have proved Theorem (1) in the case of rational power exponents s of power expansion $y = \sum c_s x^s$. It is published in the article

Bruno, A.D., Goryuchkina, I.V., *Asymptotic expansions of solutions to the sixth Painlevé equation.* Transactions Moscow Math. Soc. 2010. **71**, 1–106 (in English) .

In 2011 we have proved Theorem (1) in the case of complex but not rational power exponents s of power expansion $y = \sum c_s x^s$. It is published in preprint

Goryuchkina, I.V., *Theorem on convergence of a formal solution to ODE.* Preprint no. 25, Keldysh Inst. Appl. Math., Moscow, 2011, 16 p. (in Russian).

According to Theorem (1) all power expansions of solutions to the sixth Painlevé equation near its three singular points are convergent.

Near regular point all expansions converge according to Theorem (1), and in some cases according to the Cauchy Theorem.

Existence Boutroux type asymptotic forms to P6

For $|x| \to \infty$ we also can find asymptotic forms of solutions to the equation (1) of the form

$$y = x^\alpha \varphi(x^\beta),$$

where $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, $\varphi(x^\beta)$ is elliptic or periodic function (Boutroux type asymptotic forms).

Theorem (2)

Near singular points $x = 0, 1, \infty$ of the sixth Painlevé equation there is not Boutroux type asymptotic forms of solutions contrary to other Painlevé equations [6], [7], [8].

In our article [6] we found Boutroux asymptotic forms of solutions to P1 and P2. First they were found by Boutroux. In other two articles [7] and [8] we found Boutroux type asymptotic forms of solutions to P3 and P4 correspondingly.

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