Movable Singularities of Nonlinear ODEs

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(Joint work with Galina Filipuk, Warsaw)

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Outline

Fixed versus movable singularities

- Cauchy's theorem guarantees that the initial value problem

$$y' = \frac{1}{2(z+1)}\left(y - y^3\right), \qquad y(0) = c,$$

has a unique solution in a neighbourhood of $z = 0$.

- This solution is

$$y(z) = c\sqrt{\frac{1+z}{1+c^2 z}}.$$

- The singularity at $z = -1$ is said to be *fixed* .
- The singularity at $z = -1/c^2$ is said to be *movable* .

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Examples of movable singularities

- The general solution of

$$y'' + (y')^2 = 0$$

is $y(z) = \log(z - z_0)$, which has a movable logarithmic branch point at z_0.

- The general solution of

$$(yy'' - y'^2)^2 + 4yy'^3 = 0$$

is $y(z) = c\exp\{(z - z_0)^{-1}\}$, which has a movable essential singularity at z_0.

- The general solution of the Chazy equation

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The subject of this talk

For certain classes of equations, one can

1. find a list of some kind of series expansions (or other characterisations) of solutions in the neighbourhood of movable singularities,
2. show that these series have non-zero radii of convergence, and
3. show that the list obtained is complete in the sense that any singularity that can be reached by analytic continuation is of one of the types obtained in 1.

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Outline

The Painlevé property

Definition (the Painlevé property)

An ODE is said to possess the *Painlevé property* if all solutions are single-valued about all movable singularities.

The only equation with this property of the form

$$\frac{dy}{dz} = R(z; y),$$

where R is rational in y, is the Riccati equation

$$\frac{dy}{dz} = p(z)y^2 + q(z)y + r(z),$$

which is linearizable.

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- Kowalevskaya (classical top)
- Painlevé, Gambier, Fuchs (classification): $y'' = F(y, y'; z)$
- There are six Painlevé equations. The first two are

$$P_I \quad y'' = 6y^2 + z \quad \text{and} \quad P_{II} \quad y'' = 2y^3 + zy + \alpha$$

- Ablowitz, Ramani and Segur conjecture:

All ODE reductions of equations solvable by the inverse scattering transform (IST) possess the Painlevé property (possibly after a transformation of variables.)

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The Painlevé equations

- Each of the Painlevé equations is the compatibility condition for an iso-monodromy problem.
- These linear problems play a similar role to that played by the related spectral problems underlying soliton equations such as KdV.
- The Painlevé transcendents are nonlinear special functions.
- They arise in many areas, especially in describing the asymptotics of certain PDEs and in problems in random matrix theory.

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Painlevé analysis

- For which analytic function f does

$$\frac{d^2y}{dz^2} = 6y^2 + f(z)$$

possess the Painlevé property?

- **Leading Order Behavior:**
Look for solutions of the form

$$y \sim \alpha(z - z_0)^p, \quad \Re(p) < 0.$$

$$\begin{aligned} LHS &\sim \alpha p(p-1)(z-z_0)^{p-2}, \\ RHS &\sim 6\alpha^2(z-z_0)^{2p}, \end{aligned}$$

so $p = -2$ and $\alpha = 1$.

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The resonance condition

We look for a series solution of the form

$$y(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n-2}, \qquad a_0 = 1.$$

- We get $a_1 = a_2 = a_3 = 0$ and the recurrence relation

$$(n+1)(n-6)a_n = 6 \sum_{m=1}^{n-1} a_m a_{n-m} + \frac{1}{(n-4)!} f^{(n-4)}(z_0).$$

- There is a resonance at $n = 6$ which gives $f''(z_0) = 0$. This is true for all z_0 so

$$\frac{d^2y}{dz^2} = 6y^2 + Az + B,$$

where A and B are constants.

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Painlevé's example

- Painlevé considered the equation

$$\frac{d^2y}{dz^2} = \left(\frac{2y-1}{y^2+1}\right)\left(\frac{dy}{dz}\right)^2.$$

- It is elementary to find a two-parameter family of Laurent series solutions:

$$y(z) = \frac{\beta}{z-z_0} - \frac{1}{2} + O((z-z_0)).$$

- The general solution is $y(z) = \tan\log(A(z-\alpha))$, which has poles at $z = \alpha + A^{-1}\exp\left\{-\left(n+\frac{1}{2}\right)\pi\right\}$, which accumulate at a movable branch point at $z = \alpha$.

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A third-order example

- The general solution of the third-order ODE

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{(4y^3 - g_2 y - g_3)y'}{(12y^2 - g_2)(y')^2 - 2(4y^3 - g_2 y - g_3)y''}\right] = \frac{1}{4}$$

- is $y(z) = \wp\left(\frac{az+b}{cz+d}; g_2, g_3\right)$, where a, b, c and d are arbitrary constants such that $ad - bc = 1$.

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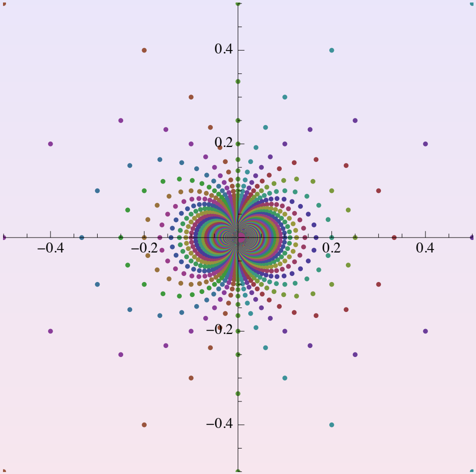
Poles of $y(z) = \wp(-1/z)$; $\omega_1 = 1$, $\omega_2 = \mathrm{i}$



Proofs of the Painlevé property

- Painlevé himself provided a proof that the first Painlevé equation $y'' = 6y^2 + z$ possesses the Painlevé property.
- This proof, which appears in a number of forms in the literature (e.g., Ince and Golubev), had some gaps in it that have been filled by several authors
  - Hukuhara;
  - Hinkkanen and Laine;
  - Shimomura.
- There are other approaches e.g. Miwa, Fokas and Its, Malgrange (using the isomonodromy problem), Steinmetz (differential inequalities), Erugin, and Joshi and Kruskal.
- Shimomura proved that the ODE $y'' = \frac{2(2k+1)}{(2k-1)^2} y^{2k} + z$ possesses the "quasi-Painlevé property."

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Painlevé's Lemma — first-order case

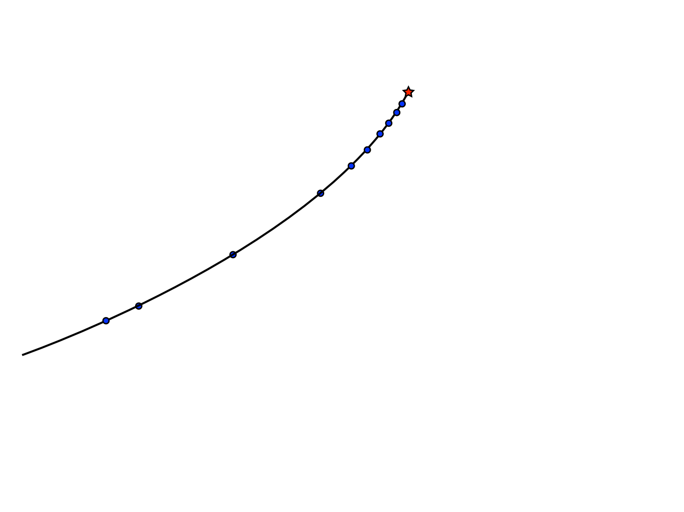
Painlevé's lemma

Let f be an analytic function in a neighbourhood of the point $(\alpha, \eta) \in \mathbb{C}^2$. Let γ be a curve with end point α and suppose that y is analytic on $\gamma \setminus \{\alpha\}$ and satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}z} = f(z, y).$$

Let (z_n) be a sequence of points such that $z_n \in \gamma$, $z_n \to \alpha$ and $y(z_n) \to \eta$ as $n \to \infty$. Then y is analytic at α.

Painlevé's Lemma



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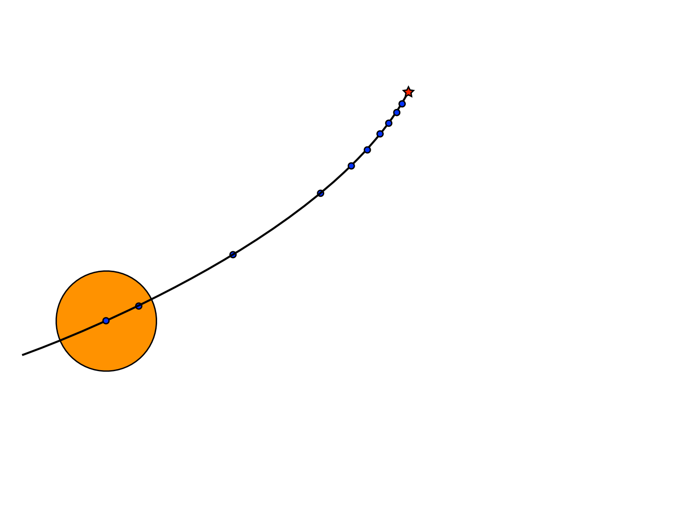

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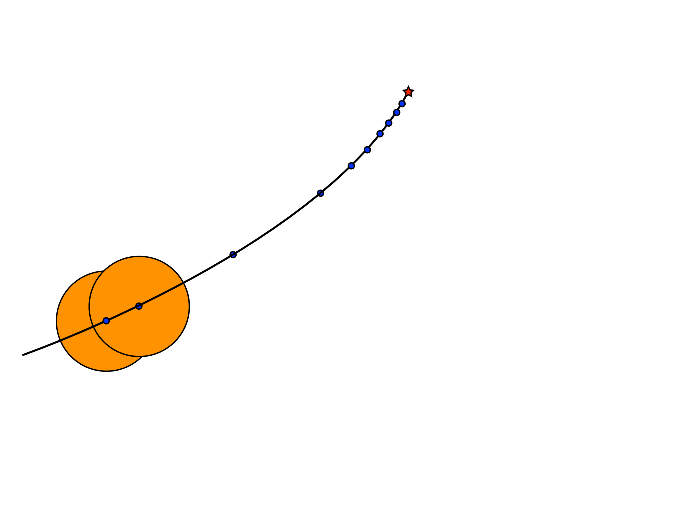

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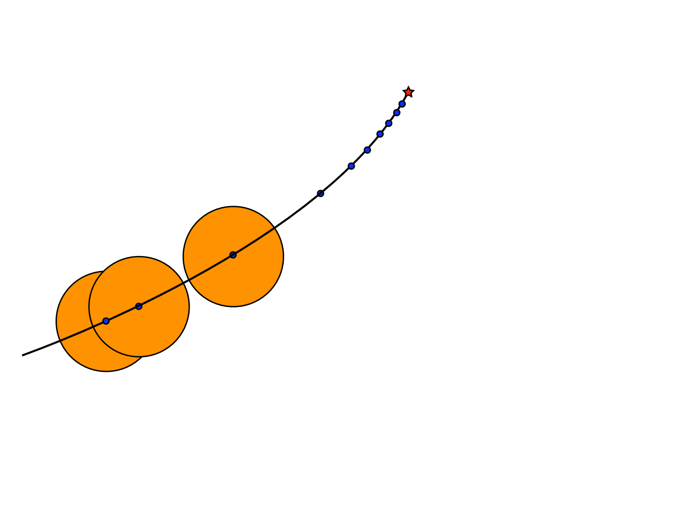

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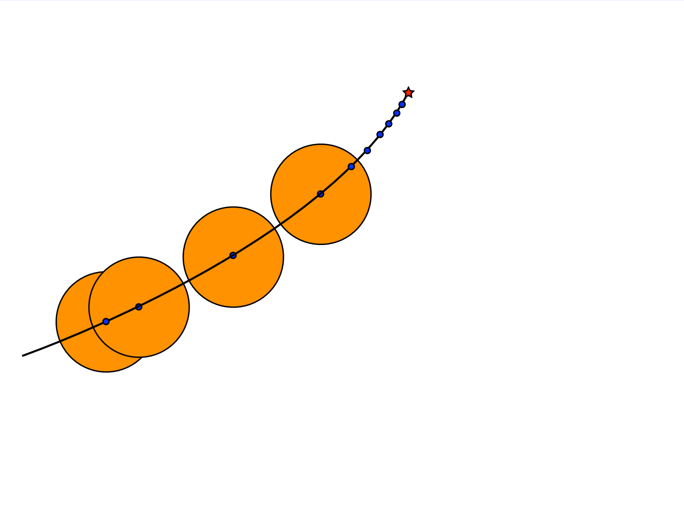

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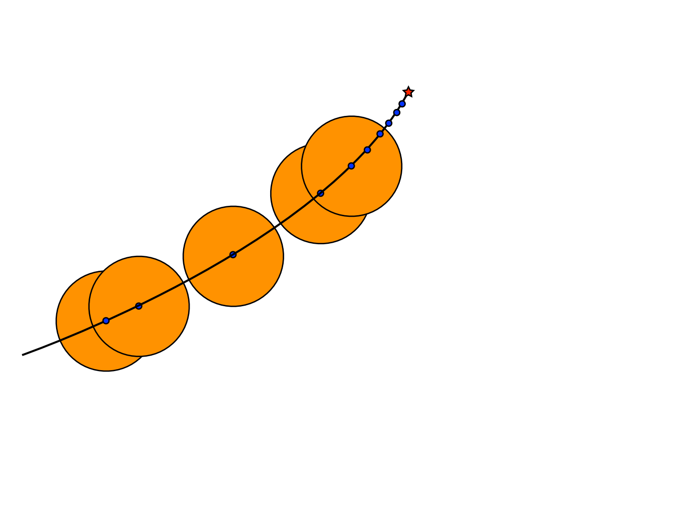

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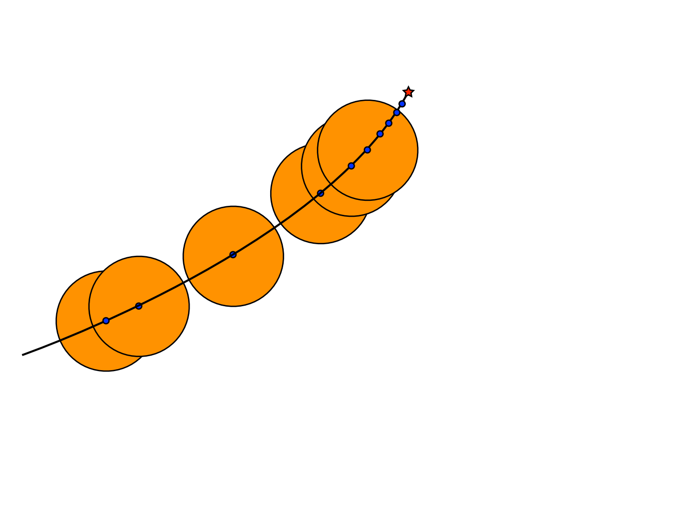

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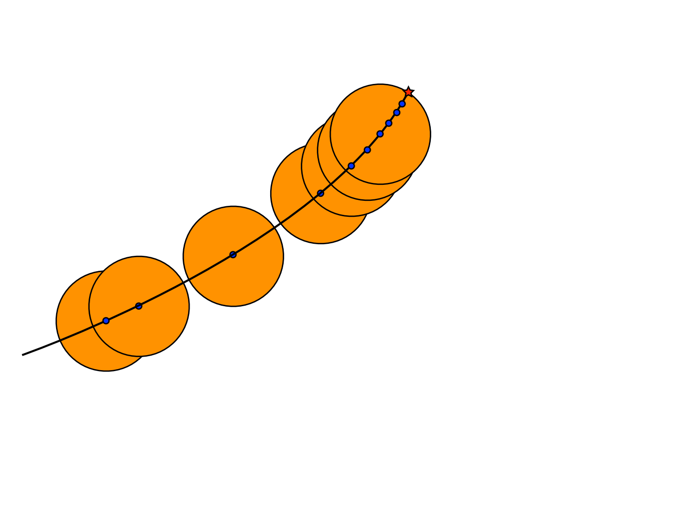

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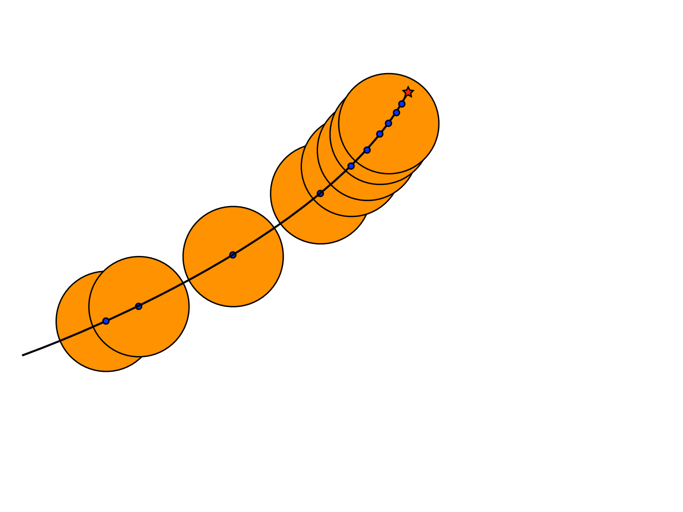

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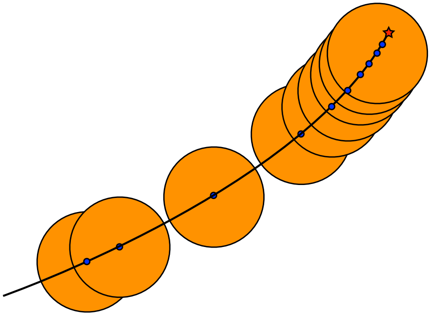

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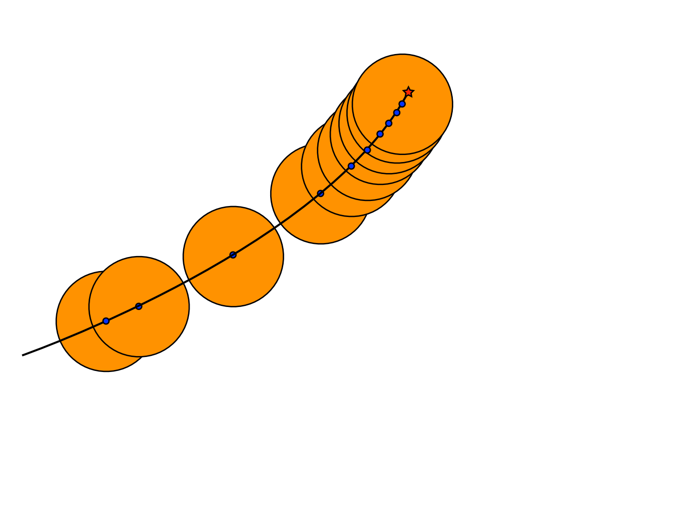

First-order ODEs

- Let y be a solution of $\dfrac{\mathrm{d}y}{\mathrm{d}z} = \displaystyle\sum_{n=0}^{N} a_n(z)y^n$, $N > 1$, on a curve $\gamma \setminus \{\alpha\}$, where y is singular at the endpoint $z = \alpha$ of γ.
- Furthermore, suppose that the a_n's are analytic in a neighbourhood of $z = \alpha$ and that $a_N(\alpha) \neq 0$.
- Then Painlevé's Lemma says that $\lim\limits_{\gamma \ni z \to \alpha} y(z) = \infty$.
- Let $u = 1/y$. Then the ODE becomes

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{u^{N-2}}{a_N(z) + a_{N-1}(z)u + \cdots + a_0(z)u^N},$$

and $z \to \alpha$ along γ as $u \to 0$.

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First-order ODEs — algebraic singularities

- Recall that $u = 1/y$ and

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where $z(0) = \alpha$.

- So z is analytic in u near $u = 0$.
- Hence $z = \alpha + u^{N-1}\sum_{n=0}^{\infty} c_n u^n$, where $c_0 \neq 0$.
- So $y(z) = 1/u(z) = \sum_{n=-1}^{\infty} b_n (z-\alpha)^{n/(N-1)}$.

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- Hence $z = \alpha + u^{N-1} \sum_{n=0}^{\infty} c_n u^n$, where $c_0 \neq 0$.
- So $y(z) = 1/u(z) = \sum_{n=-1}^{\infty} b_n (z-\alpha)^{n/(N-1)}$.

First-order ODEs — algebraic singularities

- Recall that $u = 1/y$ and

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{u^{N-2}}{a_N(z) + a_{N-1}(z)u + \cdots + a_0(z)u^N},$$

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Outline

Algebraic singularities

- Let $F(z; u, v)$ be a polynomial in u and v with coefficients that are analytic in z in some common domain. Painlevé showed that the only movable singularities of solutions of the ODE $F(z; y, y') = 0$ are algebraic.
- This is not true in general for higher-order equations such as
$$y'' = \sum_{n=0}^{N} a_n(z) y^n.$$
- Leading order behaviour: $y \sim c_0(z - z_0)^{-2/(N-1)}$, where $c_0^{N-1} = \frac{2}{a_N(z_0)} \frac{N+1}{(N-1)^2}$. Nature depends on the parity of N.

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Theorem (assumptions)

For $N \geq 2$, suppose that there is a domain $\Omega \subset \mathbb{C}$ such that $a_0, \ldots, a_N$ are analytic and that a_N is nowhere 0 on Ω. Suppose further that for each $z_0 \in \Omega$ and for each c_0 such that

$$c_0^{N-1} = \frac{2}{a_N(z_0)} \frac{N+1}{(N-1)^2}, \tag{1}$$

the equation $y'' = \sum_{n=0}^{N} a_n(z) y^n$ admits a formal series solution of the form

$$y(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^{\frac{j-2}{N-1}}. \tag{2}$$

Theorem cont'd (conclusions part 1)

i For each c_0 satisfying (1) and for each $\beta \in \mathbb{C}$, there is a unique formal series solution of the form (2) such that $c_{2(N+1)} = \beta$.

ii Given c_0 and $c_{2(N+1)}$ as above, the series (2) converges in a neighbourhood of z_0.

$$c_0^{N-1} = \frac{2}{a_N(z_0)} \frac{N+1}{(N-1)^2} \tag{1}$$

$$y(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^{\frac{j-2}{N-1}} \tag{2}$$

Theorem cont'd (conclusions part 2)

iii Now let y be a solution of equation $y'' = \sum_{n=0}^{N} a_n(z)y^n$ that can be continued analytically along a curve γ up to but not including the endpoint z_0, where the coefficients a_j are analytic in a neighbourhood of z_0 and $a_N(z_0) \neq 0$. If γ is of finite length, then y has a convergent series expansion about z_0 of the form (2).

iv If y cannot be represented by a series expansion about z_0 of the form (2) then γ is of infinite length but z_0 is an accumulation point of such algebraic singularities.

$$y(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^{\frac{j-2}{N-1}} \tag{2}$$

Main tool

A main tool used in the proof is the following.

Painlevé's lemma

Let $f_1, \ldots, f_m$ be analytic functions in a neighbourhood of the point $(\alpha, \eta_1, \ldots, \eta_m)$ in $\mathbb{C}^{m+1}$. Let γ be a curve with end point α and suppose that y_i is analytic on $\gamma \setminus \{\alpha\}$ for $i = 1, \ldots, m$ and satisfies

$$y_i' = f_i(z; y_1, \ldots, y_m).$$

Let (z_n) be a sequence of points such that $z_n \in \gamma$, $z_n \to \alpha$ and $y_i(z_n) \to \eta_i$ as $n \to \infty$, for all $i = 1, \ldots, n$. Then each y_i is analytic at α.

Applying this to part iii of the theorem shows that y is unbounded on γ.

lim vs limsup

- We write the equation $y'' = \sum_{n=0}^{N} a_n(z) y^n$ as the first-order system

$$y_1' = y_2, \qquad y_2' = \sum_{n=0}^{N} a_n(z) y_1^n.$$

- Then Painlevé's Lemma gives

$$\lim_{\gamma \ni z \to z_0} \max\{|y(z)|, |y'(z)|\} = \infty$$

and

$$\limsup_{\gamma \ni z \to z_0} |y(z)| = \infty.$$

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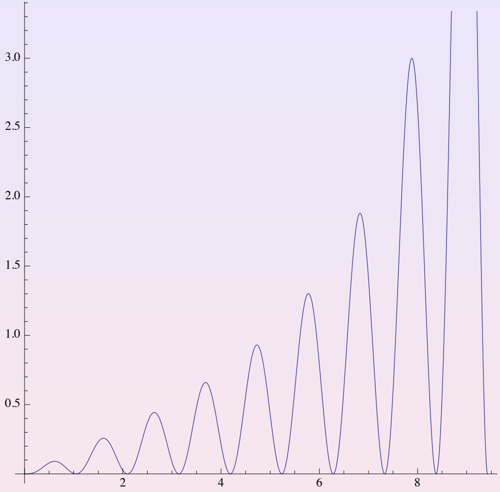

Outline of proof of iii.

- Show that, WLOG, $A := \liminf_{\gamma \ni z \to z_0} |y(z)| > 0$ (Shimomura, Hukuhara)
- Show that there is a bounded function on γ of the form

$$W(z) := y'(z)^2 + \left(\sum_{k=1}^{N-1} \frac{b_k(z)}{y^k(z)} \right) y'(z) - 2 \sum_{k=1}^{N+1} \frac{a_{k-1}(z)}{k} y^k(z).$$

- If $A < \infty$ then y and y' are both bounded on a sequence with limit z_0. Now apply Painlevé's lemma.
- If $A = \infty$ then solve for y': $\quad y' = \sum_{n=0}^{\infty} Y_n(z, y) W^n.$
- Define v by $y' = Y_0(z, y) + Y_1(z, y)v$, set $y = 1/u$ or $y = 1/u^2$ and write down a pair of ODEs for z and v as functions of u that are regular for $z(0) = z_0$ and any $v(0)$.

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and we will use the normalised form of the equation

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- If the resonance condition is not satisfied for solutions of

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of the form $y(z) \sim c_0(z - z_0)^{-2/(N-1)}$, then the Laurent series expansion in fractional powers of $z - z_0$ must be modified to a series of the form

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Accumulation of “finite type” branch points

- By “accumulation point” in part iv of the theorem we mean that given any $\epsilon > 0$ there exists a straight line segment l in the disk of radius ϵ centred at z_0 with endpoints $z_1 \in \gamma$ and z_2 such that analytic continuation of y along γ up to z_1 and then along l ends in an algebraic singularity at z_2.
- This accumulation is much more complicated than the accumulation of poles in Painlevé's example.
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Outline

Equations of Liénard type

- We have proved the analogous result for

$$y'' = P(z, y)y' + Q(z, y),$$

where P and Q are polynomials in y and $\deg_y P \geq \deg_y Q - 1$

- The constant coefficient case with $\deg P \geq \deg Q + 1$ was done by Smith in 1953.
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has a solution with algebraic branch points that accumulate along a curve of infinite length in the finite plane.

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Future directions

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- How do we know when we have a complete list of possible kinds of singularities for a given equation?
- The main goal of this research is to make some first steps towards a "general theory" of movable singularities of solutions of ODEs.
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