Regular coordinates and reduction of deformation equations for Fuchsian systems

Yoshishige Haraoka

Department of Mathematics
Kumamoto University

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Problem

"Construct Fuchsian systems for prescribed Riemann schemes"

Fuchsian system:

$$(F) \qquad \frac{dY}{dx} = \left(\sum_{j=1}^{p} \frac{A_j}{x - t_j} \right) Y$$

$$A_j : \ n \times n\text{-}\mathit{constant\ matrix} \quad (1 \leq j \leq p)$$

$$A_0 := -\sum_{j=1}^{p} A_j, \quad t_0 = \infty$$

Assume: for each j,

$$A_j \ \mathit{is} \begin{cases} \mathit{diagonalizable} \\ \lambda, \mu : \mathit{eigenvalues\ of}\ A_j, \lambda \neq \mu \Rightarrow \lambda - \mu \notin \mathbb{Z} \end{cases}$$

Riemann scheme: the table which describes the characteristic exponents at each singular point

$$
(RS) \qquad \left\{
\begin{array}{l}
x = t_0 : \overbrace{\lambda_{01}, \ldots, \lambda_{01}}^{m_{01}}, \ldots, \overbrace{\lambda_{0n_0}, \ldots, \lambda_{0n_0}}^{m_{0n_0}} \\
\quad \vdots \\
x = t_j : \overbrace{\lambda_{j1}, \ldots, \lambda_{j1}}^{m_{j1}}, \ldots, \overbrace{\lambda_{jn_j}, \ldots, \lambda_{jn_j}}^{m_{jn_j}} \\
\quad \vdots \\
x = t_p : \overbrace{\lambda_{p1}, \ldots, \lambda_{p1}}^{m_{p1}}, \ldots, \overbrace{\lambda_{pn_p}, \ldots, \lambda_{pn_p}}^{m_{pn_p}}
\end{array}
\right\}
$$

$m_j := (m_{j1}, \ldots, m_{jn_j})$: the spectral type of A_j

$$(F) \quad \longrightarrow \quad (RS)$$

$$\underset{Problem}{\longleftarrow}$$

Problem: Construct tuples $(A_0, A_1, \ldots, A_p)$ with sum zero and with prescribed eigenvalues $\{\lambda_{01}(m_{01}), \ldots, \lambda_{pn_p}(m_{pn_p})\}$

The Problem

- seems fundamental
- is open (far from the perfect solution)
- is deeply related to the deformation theory

Precise formulation of the problem

$$A_j \sim \begin{pmatrix} \lambda_{j1} I_{m_{j1}} & & \\ & \ddots & \\ & & \lambda_{jn_j} I_{m_{jn_j}} \end{pmatrix} =: C_j$$

$$\mathcal{O}_j := \{A \in \mathrm{M}(n \times n, \mathbb{C}) \mid A \sim C_j\}$$

We set

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_{\mathcal{O}_0,\dots,\mathcal{O}_p} \\ &:= \{(A_0, \dots, A_p) \in \mathcal{O}_0 \times \cdots \times \mathcal{O}_p \mid \sum_{j=0}^{p} A_j = O\} / \sim, \end{aligned}$$

where

$$\begin{aligned} & (A_0, \dots, A_p) \sim (B_0, \dots, B_p) \\ \overset{def}{\Leftrightarrow} \quad & \exists P \in \mathrm{GL}(n, \mathbb{C}),\ A_j = P B_j P^{-1} \quad (\forall j) \end{aligned}$$

We have a map

$$[(A_0, \ldots, A_p)] \overset{\varphi}{\mapsto} (\mathcal{O}_0, \ldots, \mathcal{O}_p)$$

Our problem is to describe

$$\varphi^{-1}((\mathcal{O}_0, \ldots, \mathcal{O}_p)) = \mathcal{M}_{\mathcal{O}_0, \ldots, \mathcal{O}_p}$$

Related results

1. φ is not surjective.
We have an obvious necessary condition $\sum_{j=0}^{p} \mathrm{tr}\mathcal{O}_j = 0$, which is not sufficient.

$$\vec{m} := (m_0, m_1, \ldots, m_p) : \ \textit{the spectral type of } (F)$$

For which $\vec{m}$, does an irreducible $[(A_0, \ldots, A_p)]$ exist?
(for generic values of $\{\lambda_{jk}\}$)

Deligne-Simpson Problem

- V.P. Kostov
- W. Crawley-Boevey — in terms of Kac-Moody root systems

2. For an irreducibly realizable $\vec{m}$,

$$\dim \mathcal{M} = (p-1)n^2 - \sum_{j=0}^{p} \dim Z(\mathcal{O}_j) + 2 =: \alpha$$

A coordinate system of $\mathcal{M}$ is called *accessory parameters*.

3. Scalar equation case.
Toshio Oshima solved the Problem for scalar equations

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0.$$

The moduli space is a smooth manifold.

However,

$$\# \textit{ of a.p. for scalar equation} = \frac{\alpha}{2}.$$

α parameters are necessary for the deformation, because α is equal to the dimension of the conjugacy classes of the monodromy representations.

4. Case $\vec{m} = (11, 11, 11, 11)$.
$\overline{\mathcal{M}}$ is constructed by Saito-Inaba-Iwasaki.
$\Rightarrow$ Painlevé property for Painlevé VI.

Our Approach

- $\diamond$ Do not go into the compactification (too serious)
- $\diamond$ Consider only generic points of $\mathcal{M}$
- $\diamond$ Find *good* representatives $(A_0, A_1, \ldots, A_p)$

It would be good if there is a set of a.p. $z = (z_1, z_2, \ldots, z_\alpha)$ s.t. $\forall$entries of $\forall A_j$ are rational functions in z.

We call such set of a.p. a *regular coordinate*.

A regular coordinate may be different from the canonical coordinate.

How to find regular coordinates

Lemma 1. For a generic pair A, B of $n \times n$-matrices, $\exists P \in \mathrm{GL}(n, \mathbb{C})$ s.t.

$$P^{-1}AP = \textit{lower triangular}$$
$$P^{-1}BP = \textit{upper triangular}$$

Lemma 2. Let C be a diagonalizable $n \times n$-matrix with spectral type $(n_1, n_2, \ldots, n_q)$.
(i) C can be parametrized by

$$n^2 - \sum_{i=1}^{q} {n_i}^2$$

parameters besides the eigenvalues.

(ii) Let γ_i be the eigenvalue of multiplicity n_i. Then C can be (generically) parametrized as follows.

$$
\begin{aligned}
C &= \gamma_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_{n-n_1} & P_1 \end{pmatrix} \qquad C_1 : (n-n_1) \times (n-n_1) \\
C_1 + P_1 U_1 &= \gamma_2 - \gamma_1 + \begin{pmatrix} C_2 \\ U_2 \end{pmatrix} \begin{pmatrix} I_{n-n_1-n_2} & P_2 \end{pmatrix} \\
C_2 + P_2 U_2 &= \gamma_3 - \gamma_2 + \begin{pmatrix} C_3 \\ U_3 \end{pmatrix} \begin{pmatrix} I_{n-n_1-n_2-n_3} P_3 \end{pmatrix} \\
&\vdots \\
C_{q-1} + P_{q-1} U_{q-1} &= \gamma_q - \gamma_{q-1}
\end{aligned}
$$

parameters: P_i, U_i $(1 \leq i \leq q-1)$

Note that $\sum_{i=1}^{q} {n_i}^2 = \dim Z(C)$

$\vec{m} = (m_0, m_1, \ldots, m_p)$: given
First we assume two m_i are 1^n.

$$m_0 = m_p = 1^n$$

By Lemma 1, we can take a representative $(A_0, A_1, \ldots, A_p)$ s.t.

$$A_0 = \begin{pmatrix} a_{01} & & O \\ & \ddots & \\ * & & a_{0n} \end{pmatrix}, \quad A_p = \begin{pmatrix} a_{p1} & & * \\ & \ddots & \\ O & & a_{pn} \end{pmatrix}$$

Parametrize $A_1, \ldots, A_{p-1}$ by Lemma 2.

The number of parameters we use is

$$\sum_{j=1}^{p-1} \left(n^2 - \dim Z(A_j) \right).$$

We can normalize the tuple $(A_0, \ldots, A_p)$ by $\mathrm{GL}(1)^n$ (with center $\mathbb{C}^\times$).
Since $\sum_{j=0}^{p} A_j = O$, we have

$$(*) \qquad a_{oi} + \sum_{j=1}^{p-1} \big((i,i)\text{-}entry\ of\ A_j\big) + a_{pi} = 0$$

for $i = 1, \ldots, n-1$, which are $n-1$ relations for the parameters.
Thus

$$\begin{aligned}
&\sum_{j=1}^{p-1} \Big(n^2 - \dim Z(A_j)\Big) - (n-1) - (n-1) \\
&= (p-1)n^2 - \sum_{j=1}^{p-1} \dim Z(A_j) - n - n + 2 \\
&= \alpha.
\end{aligned}$$

If we can take α parameters $(z_1, z_2, \ldots, z_\alpha)$ s.t. the solution of $(*)$ can be written as rational functions of $(z_1, z_2, \ldots, z_\alpha)$, this set of the parameters is a regular coordinate.

Note that the off-diagonal entries of A_0 and A_p are determined by $\sum_{j=1}^{p} A_j = O$:

$$\begin{pmatrix} a_{01} & & O \\ & \ddots & \\ * & & a_{0n} \end{pmatrix} + A_1 + \cdots + A_{p-1} + \begin{pmatrix} a_{p1} & & * \\ & \ddots & \\ O & & a_{pn} \end{pmatrix} = O$$

Next we relax the assumption by a coalescence of eigenvalues.

$$m_0 = 1^n \quad \to \quad 2, 1^{n-2}$$

$$A_0 = \begin{pmatrix} a_{01} & & & & \\ * & a_{01} & & O & \\ & & a_{03} & & \\ & * & & \ddots & \\ & & & & a_{0n} \end{pmatrix} = \begin{pmatrix} a_{01} & & & & \\ 0 & a_{01} & & O & \\ & & a_{03} & & \\ & * & & \ddots & \\ & & & & a_{0n} \end{pmatrix}$$

Then by $\mathrm{GL}(2) \times \mathrm{GL}(1)^{n-2}$ action, we have

$$A_p = \begin{pmatrix} a_{p1} & 0 & & & \\ & a_{p2} & & * & \\ & & a_{p3} & & \\ & O & & \ddots & \\ & & & & a_{pn} \end{pmatrix}$$

$$m_0 = 1^n \ \to\ 3, 1^{n-3}; \quad m_p = 1^n \ \to\ 2, 1^{n-2}$$

$$A_0 = \begin{pmatrix} a_{01} & 0 & 0 & & & \\ 0 & a_{01} & 0 & & & \\ 0 & 0 & a_{01} & & O & \\ & & & a_{04} & & \\ & * & & & \ddots & \\ & & & & & a_{0n} \end{pmatrix}$$

$$A_p = \begin{pmatrix} a_{p1} & 0 & 0 & & & \\ 0 & a_{p1} & 0 & & & \\ 0 & 0 & a_{p3} & & * & \\ & & & a_{p4} & & \\ & O & & & \ddots & \\ & & & & & a_{pn} \end{pmatrix}$$

$\mathrm{GL}(2) \times \mathrm{GL}(1)^{n-2}$ action keeps these forms of A_0 and A_p.

Reductions

1. Katz operations

addition: $Y(x) \mapsto \prod_{j=1}^{p}(x - t_j)^{a_j} \cdot Y(x)$
middle convolution: $Y(x) \mapsto \displaystyle\int_{\Delta}(u - x)^{\lambda} Y(u)\, du$

These operations are realized as operations on $(A_0, A_1, \ldots, A_p)$.

Katz operations keep the number of accessory parameters, irreducibility and the deformation equation invariant.

Theorem. If $(A_0, A_1, \ldots, A_p)$ has a regular coordinate, the result of a Katz operation also has a regular coordinate.

Thus it is enough to find regular coordinates for *basic* $\vec{m}$.

Basic spectral types.

$\alpha = 2$:

$$(11, 11, 11, 11);\ (111, 111, 111),\ (22, 1^4, 1^4),\ (33, 2^3, 1^6)$$

$\alpha = 4$:

$$(11, 11, 11, 11, 11);$$
$$(21, 21, 1^3, 1^3),\ (31, 22, 22, 1^4),\ (22, 22, 22, 211);$$
$$(211, 1^4, 1^4),\ (221, 221, 1^5),\ (32, 1^5, 1^5),\ (2^3, 2^3, 2211),$$
$$(33, 2211, 1^6),\ (44, 2^4, 22211),\ (44, 332, 1^8),\ (55, 3331, 2^5),$$
$$(66, 4^3, 2^5 11)$$

Example. $(33, 222, 1^6) \quad (\alpha = 2)$

$$A = \begin{pmatrix} a_1 & 0 & & & & \\ 0 & a_1 & & & O & \\ & & a_2 & 0 & & \\ & & 0 & a_2 & & \\ & * & & & a_3 & 0 \\ & & & & 0 & a_3 \end{pmatrix}, B = \begin{pmatrix} b_1 & 0 & & & & \\ 0 & b_2 & & & * & \\ & & b_3 & 0 & & \\ & & 0 & b_4 & & \\ & O & & & b_5 & 0 \\ & & & & 0 & b_6 \end{pmatrix}$$

$$C = c_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_3 & P_1 \end{pmatrix}, \quad C_1 + P_1 U_1 = c_2 - c_1$$

Normalization by $\mathrm{GL}(1)^6$ gives

$$U_1 = \begin{pmatrix} 1 & * & * \\ 1 & * & * \\ 1 & 1 & 1 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

Parameters we use: $4 + 9 = 13$
Relations: $4 + 4 + 4 - 1 = 11$
Thus we have

$$13 - 11 = 2 = \alpha.$$

We find we can take a regular coordinate (p_{11}, p_{21}).

2. Good reductions

We consider a coalescence of eigenvalues which sends $\vec{m}$ to $\vec{m}'$.
For example, for $\vec{m} = (m_0, m_1, \ldots, m_p)$ with $m_0 = m_p = 1^n$, we consider the coalescence

$$m_0 = 1^n \mapsto 21^{n-2} =: m_0'.$$

$$A_0 = \begin{pmatrix} a_{01} & 0 & & & \\ f & a_{02} & & O & \\ & & a_{03} & & \\ & * & & \ddots & \\ & & & & a_{0n} \end{pmatrix}, A_p = \begin{pmatrix} a_{p1} & g & & & \\ 0 & a_{p2} & & * & \\ & & a_{p3} & & \\ & O & & \ddots & \\ & & & & a_{pn} \end{pmatrix}$$

Assume that the tuple $(A_0, A_1, \ldots, A_p)$, in particular f and g, are written rationally by a regular coordinate $z = (z_1, \ldots, z_\alpha)$.

The coalescence $a_{02} \to a_{01}$ yields two equations

$$f = 0, \quad g = 0.$$

If this system is linear in two entries z_i, z_j of the regular coordinate z, we can solve the system to get a regular coordinate $z' := (z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_\alpha)$ for $\vec{m}'$.

We call such reduction $\vec{m} \to \vec{m}'$ a *good reduction*.

Example.

$(11, 11, 11, 11, 11) \to (11, 11, 11, 11, 2) = (11, 11, 11, 11)$

$$A_0 = \begin{pmatrix} a_{01} & 0 \\ f & a_{02} \end{pmatrix}, A_4 = \begin{pmatrix} a_{41} & g \\ 0 & a_{42} \end{pmatrix}$$

$$A_j = \begin{pmatrix} a_{j2} - u_j p_j & (a_{j2} - a_{j1} - u_j p_j) p_j \\ u_j & a_{j1} + u_j p_j \end{pmatrix} \quad (j = 1, 2, 3)$$

Normalization: $p_1 = 1$

Relation: $a_{01} + \sum_{j=1}^{3} (a_{j2} - u_j p_j) + a_{41} = 0$

We have a regular coordinate (u_2, p_2, u_3, p_3).

$$\begin{cases} f = -(u_1 + u_2 + u_3) \\ g = -\displaystyle\sum_{j=1}^{3} (a_{j2} - a_{j1} - u_j p_j) p_j \end{cases}$$

Coalescence: $a_{41}, a_{42} \to (a_{41} + a_{42})/2$

$$\Rightarrow \begin{cases} u_1 + u_2 + u_3 = 0 \\ \displaystyle\sum_{j=1}^{3} (a_{j2} - a_{j1} - u_j p_j) p_j = 0 \end{cases}$$

This system is linear in u_2, u_3, and then they can be written rationally in p_2, p_3.
Thus we obtain a regular coordinate (p_2, p_3) after the coalescence.

This is a good reduction, and gives a reduction from Garnier system to Painlevé VI.

Isomonodromic deformation

The isomonodromic deformation of the Fuchsian system

$$(F) \qquad \frac{dY}{dx} = \left(\sum_{j=1}^{p} \frac{A_j}{x - t_j} \right) Y$$

is described by the Schlesinger system

$$(S) \qquad \frac{\partial A_i}{\partial t_i} = -\sum_{k \neq i} \frac{[A_i, A_k]}{t_i - t_k}, \quad \frac{\partial A_j}{\partial A_i} = \frac{[A_i, A_j]}{t_i - t_j} \quad (i \neq j)$$

under the condition

$$(J) \qquad A_j \sim C_j \quad (0 \leq j \leq p).$$

The unknowns of (S) are the entries of $A_1, \ldots, A_p$: pn^2 unknowns, while the rank of (S)+(J) is α.
Thus we must reduce the unknowns of (S) to get a slim deformation equation.

If we have a regular coordinate for $(A_0, A_1, \ldots, A_p)$, we obtain, as isomonodromic deformation equations, algebraic differential equations for the regular coordinate.

If, moreover, we have a good reduction, we get an explicit reduction formula for the deformation equations such as Garnier to Painlevé.

Questions

Q1. Does a regular coordinate exist for any basic spectral type $\vec{m}$? If it does not so, describe the condition.

Q2. Are there any general procedures to find a regular coordinate?

Q3. Can we obtain a regular coordinate for any basic spectral type from a regular coordinate for $(1^n, 1^n, \ldots, 1^n)$ by a finite iteration of good reductions?

Q4. For which pair of spectral types does a good reduction exist? Give the condition in terms of Kac-Moody root systems.

Q5. Irregular singular case?