JACOBI POLYNOMIALS IN SIE REPRESENTATIONS OF QUIVERS

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Formal and Analytic Solutions of Differential and Difference Equations
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Quivers and SIE Representations

Heisenberg–Weyl Algebra

h–deformed Heisenberg–Weyl Algebra

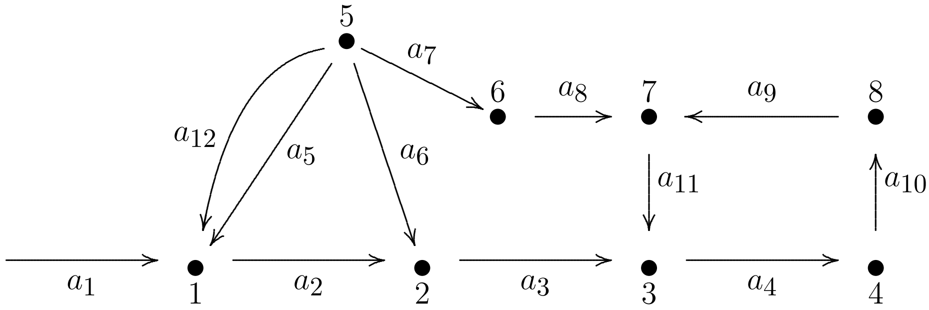
q–deformed Heisenberg–Weyl Algebra

Dimension 1: Ladders (General Theory — Examples)

Dimension 2: Grids (General Theory — Examples)

Dimension 3: The Jacobi Cube

Quivers



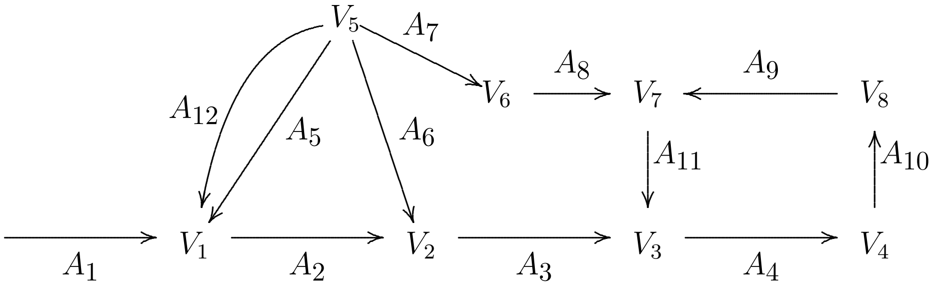
Directed Connected Graph with Vertices $\bullet_i$ / Arrows a_j

Multiple Arrows allowed

Loops allowed

Finite or Infinite

Quiver Representations

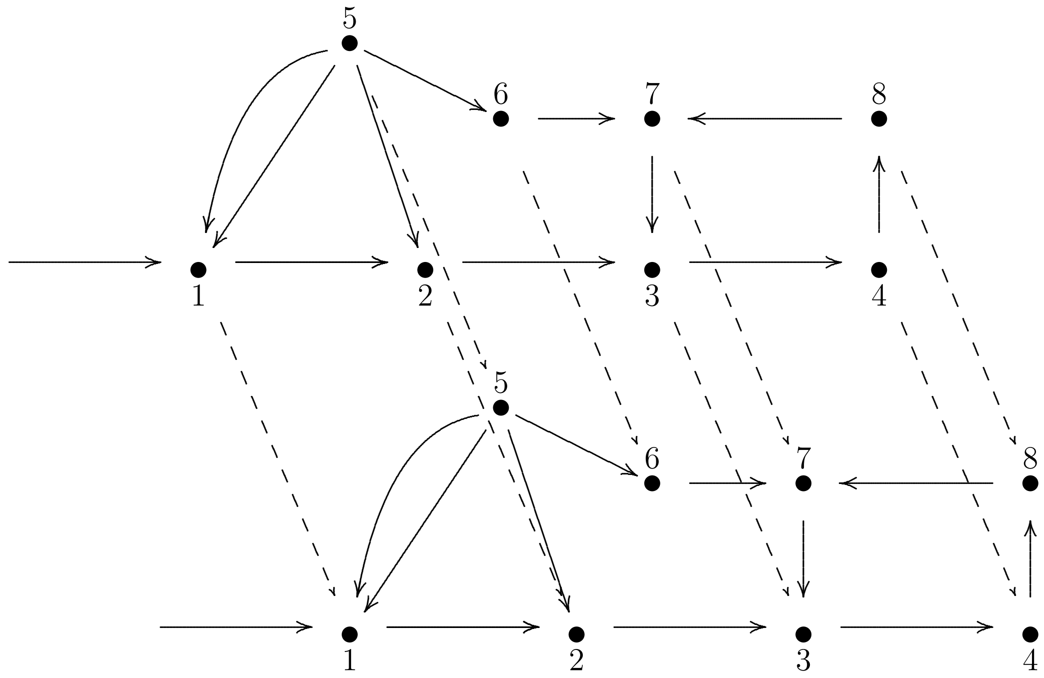


Functor from Category of Directed Graphs to Category of Vector spaces

Vertex $\mapsto$ Vector space
Arrow $\mapsto$ Linear Operator

i.e. Diagram of vector spaces and operators

Homomorphism of two Quiver Representations



All rectangles connecting the two quivers commute

= Natural transformation of two Functors

Quiver Representations

…form an Abelian Category

Subrepresentation

Quotient Representation

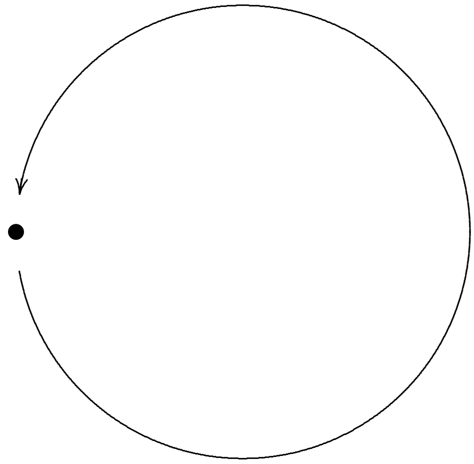
Direct Sum of Representations

Kernel, Image

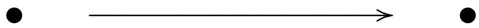
Irreducibility

Decomposition

Two examples from finite Quiver Theory



Jordan blocks

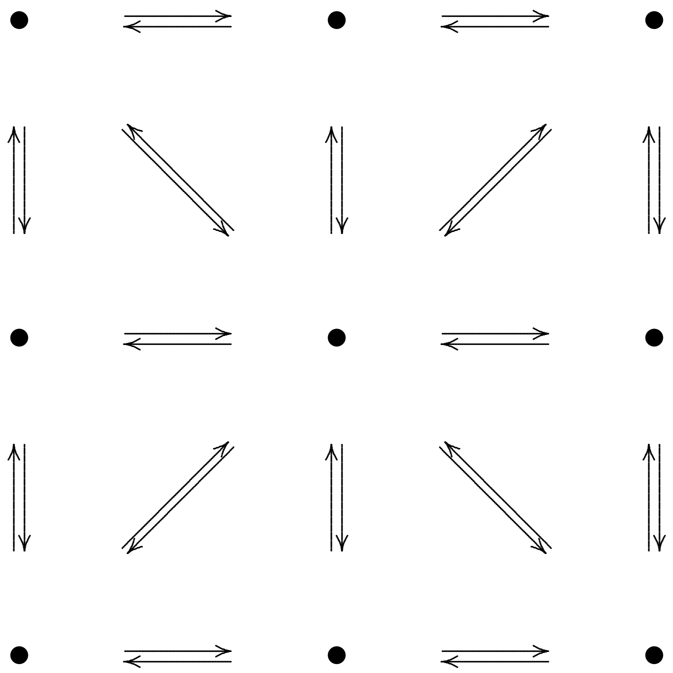


$$\{0\} \longrightarrow \mathbb{R}$$

$$\mathbb{R} \xrightarrow{\;1\;} \mathbb{R}$$

$$\mathbb{R} \longrightarrow \{0\}$$

More Specialized Situation



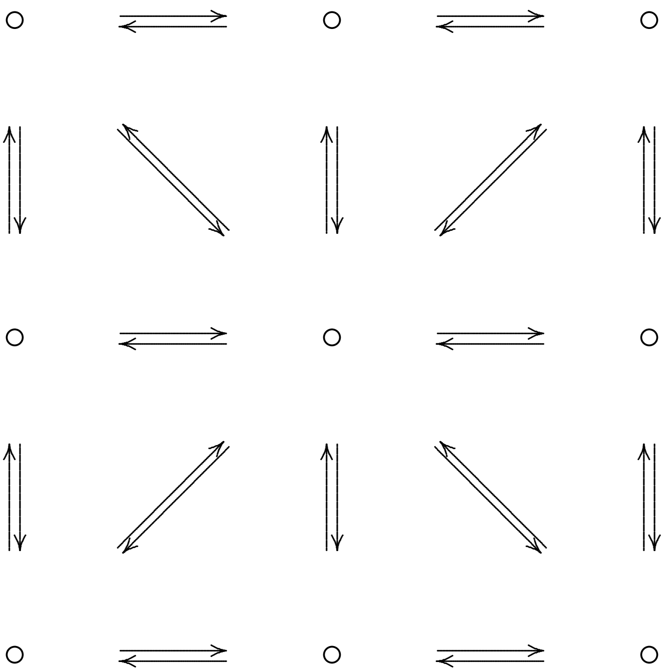
Only double arrows

Periodic infinite structure

"Dimension" 1: Ladders
"Dimension" 2: Grids
"Dimension" 3: Cube

Narrow Loops
Wide Loops

SIE Subrepresentations



All loops act as

Scalar **I**ntrinsic **E**ndomorphisms

The Heisenberg–Weyl Algebra I

$\mathbb{C}$ algebra with unit Two Generators D, X

Relation $[D, X] = DX - XD = 1.$

Standard Representation on $\mathcal{C}^\infty(\mathbb{R})$ (or $\mathcal{P}$, $\mathcal{S}$, $\mathcal{D}$, $\mathcal{D}'$,...)

$$D \rightsquigarrow \partial \qquad X \rightsquigarrow x.$$

Representation on $\mathcal{C}^\infty(\mathbb{R})$ after Transformation

$f(x) \mapsto \varrho(x) f(\sigma(x))$

$$D \rightsquigarrow \frac{1}{\sigma'}(\partial - \frac{\varrho'}{\varrho}) \qquad X \rightsquigarrow \sigma(x).$$

The Heisenberg–Weyl Algebra II — Some algebraic observations

Group of Algebra automorphisms

$$\begin{pmatrix} D \\ X \end{pmatrix} \quad \to \quad \underbrace{\begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}}_{\text{det}=1} \begin{pmatrix} D \\ X \end{pmatrix} + \begin{pmatrix} \bullet \\ \bullet \end{pmatrix}$$

Involution (= Star operator)

$$\begin{pmatrix} D \\ X \end{pmatrix} \quad \stackrel{*}{\to} \quad \begin{pmatrix} -D \\ X \end{pmatrix} \qquad\qquad (AB)^* \;=\; B^* A^*$$

Algebraic Differentiation

$$\begin{aligned} [D, X^k] \;&=\; DX^k - X^k D \;=\; k \cdot X^{k-1} \\ [D^\ell, X] \;&=\; D^\ell X - X D^\ell \;=\; \ell \cdot D^{\ell-1} \end{aligned}$$

The h–deformed Heisenberg–Weyl Algebra I

$\mathbb{C}$ algebra with unit / Three Generators D, M, X

Relations
$$[D,M] = 0$$
$$[D,X] = M \qquad M^2 - h^2 D^2 = 1$$
$$[M,X] = h^2 D$$

Pythagroas Relation

Two Standard Representations

	on $\mathcal{F}(\mathrm{h}\mathbb{Z})$	on $\mathcal{C}^\infty(\mathbb{S}_{1/\mathrm{h}})$
	complex sequences	$\frac{2\pi}{h}$–periodcic functions
$D \rightsquigarrow$	$f(x) \mapsto \frac{f(x+h)-f(x-h)}{2h}$	$f(x) \mapsto i \cdot \frac{\sin(hx)}{h} \cdot f(x)$
$M \rightsquigarrow$	$f(x) \mapsto \frac{f(x+h)+f(x-h)}{2}$	$f(x) \mapsto \cos(hx) \cdot f(x)$
$X \rightsquigarrow$	$f(x) \mapsto x \cdot f(x)$	$f(x) \mapsto i \cdot f'(x)$

The h–HW Algebra II — Algebraic Observations

Group of Algebra automorphisms

$$\begin{pmatrix} M \\ D \\ X \end{pmatrix} \quad \to \quad \underbrace{\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}}_{\det=+1} \begin{pmatrix} M \\ D \\ X \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \bullet \end{pmatrix}$$

Involution (= Star operator)

$$\begin{pmatrix} M \\ D \\ X \end{pmatrix} \quad \stackrel{*}{\to} \quad \begin{pmatrix} M \\ -D \\ X \end{pmatrix} \qquad\qquad (AB)^* \;=\; B^* A^*$$

The h–HW Algebra III — Algebraic Observations

Automorphism ($\mathbb{Z}_4$ Action)

$$\begin{pmatrix} M \\ D \\ X \end{pmatrix} \quad \to \quad \begin{pmatrix} 0 & ih & 0 \\ \frac{i}{h} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} M \\ D \\ X \end{pmatrix}$$

An $h \searrow 0$ degeneration is not possible.

The h–HW Algebra IV — Self–Similarity

The h–HW algebra contains a $2h$–HW subalgebra.

Define

$$\begin{aligned} M' &:= 2M^2 - 1 \\ D' &:= DM \\ X' &:= X \end{aligned}$$

Relations

$$\begin{aligned} [D', M'] &= 0 \\ [D', X'] &= M' \\ [M', X'] &= 4h^2 D' \\ (M')^2 - 4h^2 (D')^2 &= 1 \end{aligned}$$

The h–HW Algebra V — Shift Generators

$$\begin{aligned} S &:= M + hD \qquad \text{(represented by forward shift)} \\ S^* &:= M - hD \qquad \text{(represented by backward shift)} \end{aligned}$$

h–HW Algebra generated by S, S^*, X.

Relations
$$\begin{aligned} [S, S^*] &= 0 \\ [S, X] &= hS \qquad\qquad SS^* = 1 \\ [S^*, X] &= -hS^* \end{aligned}$$

An $h \searrow 0$ degeneration is not possible.

The h–HW Algebra VI — Diff Op Generators

$$\Delta := \frac{1}{h}(S-1) \qquad \text{(represented by forward Diff Op)}$$

$$\Delta^* := \frac{1}{h}(S^*-1) \qquad \text{(represented by backward Diff Op)}$$

h–HW Algebra generated by Δ, Δ^*, X.

Relations
$$\begin{aligned} [\Delta, \Delta^*] &= 0 \\ [\Delta, X] &= 1 + h\Delta \qquad\qquad \Delta + \Delta^* + h\Delta\Delta^* = 0 \\ [\Delta^*, X] &= -(1 + h\Delta^*) \end{aligned}$$

$h \searrow 0$ degeneration:

$$\Delta \to D \qquad \Delta^* \to -D$$

The h–HW Algebra VII — Deformization Polynomials

Within the binomial identity we define two sequences of polynomials ($k \in \mathbb{N}_0$)

$$(X+h)^k \;=\; \underbrace{\sum_{\substack{i=0\\ i \text{ even}}}^{k} \binom{k}{i} h^i X^{k-i}}_{=:p_k(X)} + h \cdot \underbrace{\sum_{\substack{i=0\\ i \text{ odd}}}^{k} \binom{k}{i} h^{i-1} X^{k-i}}_{=:q_{k-1}(X)}$$

Alternatively

$$\begin{aligned} p_0 &\equiv 1 \\ q_{-1} &\equiv 0 \end{aligned} \qquad \begin{pmatrix} p_{k+1}(X) \\ q_k(X) \end{pmatrix} = \begin{pmatrix} X & h^2 \\ 1 & X \end{pmatrix} \begin{pmatrix} p_k(X) \\ q_{k-1}(X) \end{pmatrix}$$

All polynomials p_k and q_k ...

- only depend on h^2
- have degree k (in X)
- parity k (in X)
- homogeneous in (X, h).

k	p_k	q_k
0	1	1
1	X	$2X$
2	$X^2 + h^2$	$3X^2 + h^2$
3	$X^3 + 3h^2X$	$4X^3 + 4h^2X$
4	$X^4 + 6h^2X^2 + h^4$	$5X^4 + 10h^2X^2 + h^4$
5	$X^5 + 10h^2X^3 + 5h^4X$	$6X^5 + 20h^2X^3 + 6h^4X$
$h \searrow 0$	Monomials	Derivatives

Observation

$$\begin{array}{lrcl} \text{H.O Pythagoras} & MX^kM - h^2DX^kD & = & p_k(X) \\ \text{H.O Algebraic Differencing} & DX^kM - MX^kD & = & q_{k-1}(X) \\ & & & k \in \mathbb{N}_0 \end{array}$$

The q–deformed HW Algebra I

$\mathbb{C}$ algebra with unit

Two Generators D, X

Relation $DX - qXD = 1, \qquad q \neq 1$ fixed

Standard Representation on $\mathcal{F}(q\mathbb{Z})$

$$\begin{aligned} Df(x) &= \frac{f(qx) - f(x)}{qx - x} \\ Xf(x) &= x \cdot f(x) \end{aligned}$$

Ladders

$$\cdots \quad \bullet \underset{A^-_{n-1}}{\overset{A^+_{n-1}}{\rightleftarrows}} \bullet \underset{A^-_{n}}{\overset{A^+_{n}}{\rightleftarrows}} \bullet \underset{A^-_{n+1}}{\overset{A^+_{n+1}}{\rightleftarrows}} \bullet \quad \cdots$$

$$\alpha_{n-1} \qquad \alpha_n \qquad \alpha_{n+1}$$

A_i^+ Creation (Ascending / Raising) Linear Operators

A_i^- Annihilation (Descending / Lowering) Linear Operators

Given the above ladder and a number sequence α_n, define

$A_n^{\sqsupset}$	$:=$	$A_n^- A_n^+$	Right loop			
$A_n^{\sqsubset}$	$:=$	$A_{n-1}^+ A_{n-1}^-$	Left loop			
A_n^{Δ}	$:=$	$\frac{A_n^{\sqsupset} - A_n^{\sqsubset}}{2}$	Commutator	α_n^{Δ}	$:=$	$\frac{\alpha_n - \alpha_{n-1}}{2}$
$A_n^{\Diamond}$	$:=$	$\frac{A_n^{\sqsupset} + A_n^{\sqsubset}}{2}$	Anticommutator	$\alpha_n^{\Diamond}$	$:=$	$\frac{\alpha_n + \alpha_{n-1}}{2}$

Ladders

Given a ladder with representation V of the underlying algebra

$$\cdots \qquad V \underset{A_{n-1}^{-}}{\overset{A_{n-1}^{+}}{\rightleftarrows}} V \underset{A_{n}^{-}}{\overset{A_{n}^{+}}{\rightleftarrows}} V \underset{A_{n+1}^{-}}{\overset{A_{n+1}^{+}}{\rightleftarrows}} V \qquad \cdots$$

$$\alpha_{n-1} \qquad \alpha_n \qquad \alpha_{n+1}$$

define sequence of subspaces

$$\begin{aligned} \mathcal{E}_n &:= \operatorname{eig}(A_n^{\sqsupset}, \alpha_n) \cap \operatorname{eig}(A_n^{\sqsubset}, \alpha_{n-1}) \\ &= \operatorname{eig}(A_n^{\Delta}, \alpha_n^{\Delta}) \cap \operatorname{eig}(A_n^{\Diamond}, \alpha_n^{\Diamond}) \end{aligned}$$

Is this sequence $(\mathcal{E}_n)$ a subladder?

Observation

IF

$(A_n^\Delta - \alpha_n^\Delta)$ is a ladder endomorphism

AND

$(A_n^\Diamond - \alpha_n^\Diamond)$ is a ladder endomorphism,

THEN

$(\mathcal{E}_n)$ is an SIE subladder.

Ladder Theorem

IF

$(A_n^{\Delta} - \alpha_n^{\Delta})$ is a ladder endomorphism

OR

$(A_n^{\Diamond} - \alpha_n^{\Diamond})$ is a ladder endomorphism,

THEN

$(\mathcal{E}_n)$ is an SIE subladder.

Corollary

$$\cdots \qquad \bullet \; \underset{A^-_{n-1}}{\overset{A^+_{n-1}}{\rightleftarrows}} \; \bullet \; \underset{A^-_{n}}{\overset{A^+_{n}}{\rightleftarrows}} \; \bullet \; \underset{A^-_{n+1}}{\overset{A^+_{n+1}}{\rightleftarrows}} \; \bullet \qquad \cdots$$

$$\alpha_{n-1} \qquad \alpha_n \qquad \alpha_{n+1}$$

If $A_n^\Delta = \alpha_n^\Delta$, then

$$\cdots \quad \mathcal{E}_{n-1} \underset{A^-_{n-1}}{\overset{A^+_{n-1}}{\rightleftarrows}} \mathcal{E}_n \underset{A^-_{n}}{\overset{A^+_{n}}{\rightleftarrows}} \mathcal{E}_{n+1} \underset{A^-_{n+1}}{\overset{A^+_{n+1}}{\rightleftarrows}} \mathcal{E}_{n+2} \quad \cdots$$

with $\mathcal{E}_n = \ker(A_n^\Diamond - \alpha_n^\Diamond)$ is an SIE subladder.

$$A_n^\Delta \text{ scalar} \qquad \curvearrowright \qquad \begin{cases} \text{Put } \alpha_n^\Delta := A_n^\Delta \\ \text{Solve for } \alpha_n \\ \text{Compute } \alpha_n^\Diamond \\ \text{Compute } \mathcal{E}_n \text{ (depending on rep)} \end{cases}$$

Additional Observation

$$\cdots \quad \mathcal{E}_{n-1} \underset{A_{n-1}^-}{\overset{A_{n-1}^+}{\rightleftarrows}} \mathcal{E}_n \underset{A_n^-}{\overset{A_n^+}{\rightleftarrows}} \mathcal{E}_{n+1} \underset{A_{n+1}^-}{\overset{A_{n+1}^+}{\rightleftarrows}} \mathcal{E}_{n+2} \quad \cdots$$

$$\alpha_{n-1} \qquad \alpha_n \qquad \alpha_{n+1}$$

If $\alpha_n \neq 0$, then $\mathcal{E}_n \cong \mathcal{E}_{n+1}$

If $\alpha_n = 0$, then "ladder broken" between $\mathcal{E}_n$ and $\mathcal{E}_{n+1}$
$\ker A_n^+$ generates SIE subladder in left direction
$\ker A_n^-$ generates SIE subladder in right direction

Heisenberg Ladder

$$\cdots \quad \bullet \quad \underset{D}{\overset{X}{\rightleftarrows}} \quad \bullet \quad \underset{D}{\overset{X}{\rightleftarrows}} \quad \bullet \quad \cdots$$

$$A_n^\Delta = \tfrac{1}{2}(DX - XD) = \tfrac{1}{2} \quad \Longrightarrow \quad \mathcal{E}_n \text{ is SIE subladder.}$$

$$\begin{pmatrix} D \\ X \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial \\ x \end{pmatrix} & \curvearrowright \quad \text{Monomials} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \partial \\ x \end{pmatrix} & \curvearrowright \quad \text{Hermite functions} \\ & \qquad \text{(Dirac Harmonic Oscillator)} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \partial \\ x \end{pmatrix} & \curvearrowright \quad \text{Hermite polynomials} \end{cases}$$

Legendre Ladder

$$\cdots \quad \bullet \quad \underset{nX - ZD}{\overset{nX + ZD}{\rightleftarrows}} \quad \bullet \quad \underset{(n+1)X - ZD}{\overset{(n+1)X + ZD}{\rightleftarrows}} \quad \bullet \quad \cdots$$

$$\begin{aligned} Z &:= X^2 - 1 \\ A_n^{\Delta} &= n + \frac{1}{2} \\ A_n^{\Diamond} &= Z \underbrace{\left[-ZD^2 - 2XD + n^2 + n\right]}_{\text{Legendre Differential Operator}} + n^2 + n + \frac{1}{2} \\ \alpha_n &= (n+1)^2 \\ \mathcal{E}_n &= \langle \text{ Legendre Polynomial, degree } n \ \rangle \end{aligned}$$

Legendre Ladder → Three Term Recurrence Relation

$$\cdots \quad \bullet \quad \underset{nX - ZD}{\overset{nX + ZD}{\rightleftarrows}} \quad \bullet \quad \underset{(n+1)X - ZD}{\overset{(n+1)X + ZD}{\rightleftarrows}} \quad \bullet \quad \cdots$$

$$((n+1)X + ZD)p_n = \underbrace{(n+1)}\, p_{n+1}$$

depends on norming

$$(nX - ZD)p_n = \overbrace{\;n\;}\, p_{n-1}$$

$$(2n+1)Xp_n = (n+1)p_{n+1} + np_{n-1}$$

Laguerre Ladder

$$\cdots \qquad \bullet \quad \underset{n - XD}{\overset{n + XD - X}{\rightleftarrows}} \quad \bullet \quad \underset{(n+1) - XD}{\overset{(n+1) + XD - X}{\rightleftarrows}} \quad \bullet \qquad \cdots$$

$$\begin{aligned}
A_n^{\Delta} &= n + \frac{1}{2} \\
A_n^{\Diamond} &= -X \underbrace{\left[XD^2 + (1 - X)D + n \right]}_{\text{Laguerre Differential Operator}} + n^2 + n + \frac{1}{2} \\
\alpha_n &= (n+1)^2 \\
\mathcal{E}_n &= \langle \text{ Laguerre Polynomial, degree } n \ \rangle
\end{aligned}$$

Bessel Ladder

$$\cdots \qquad \bullet \quad \underset{nX^{-1}+D}{\overset{(n-1)X^{-1}-D}{\rightleftarrows}} \quad \bullet \quad \underset{(n+1)X^{-1}+D}{\overset{nX^{-1}-D}{\rightleftarrows}} \quad \bullet \qquad \cdots$$

$$\begin{aligned} & \text{Adjoin } X^{-1} \\ A_n^{\Delta} &= 0 \\ \alpha_n &= 1 \\ A_n^{\Diamond} &= -X^2 \underbrace{\left[X^2D^2 + XD + X^2 - n^2\right]}_{\text{Bessel Differential Operator}} \\ \mathcal{E}_n &= \langle\ n\text{–th order Bessel Function}\ \rangle \end{aligned}$$

Heat Ladder

$$\cdots \qquad \bullet \quad \underset{nM+DX}{\overset{M}{\rightleftarrows}} \quad \bullet \quad \underset{(n+1)M+DX}{\overset{M}{\rightleftarrows}} \quad \bullet \qquad \cdots$$

$$\begin{aligned}
A_n^{\Delta} &= \frac{1}{2} \\
A_n^{\Diamond} &= (n+\frac{1}{2})^2 M^2 + \frac{1}{2} D(XM+MX) \\
\alpha_n &= n+1 \\
\mathcal{E}_n &= \operatorname{eig}(A_n^{\Diamond}, n+\frac{1}{2}) \\
&= \langle \text{ Centralized Binomial distributions, } \#\,\text{supp} = n+1 \rangle
\end{aligned}$$

Discrete Harmonic Oscillator Ladder

$$\cdots \quad \bullet \quad \underset{MX+[1+nh^2]D}{\overset{XM-[1+nh^2]D}{\rightleftarrows}} \quad \bullet \quad \underset{MX+[1+(n+1)h^2]D}{\overset{XM-[1+(n+1)h^2]D}{\rightleftarrows}} \quad \bullet \quad \cdots$$

$$A_n^{\Delta} = 1 + nh^2$$

$$A_n^{\Diamond} = MX^2M - (1+nh^2)D^2$$

$$\mathcal{E}_n = \langle \text{ Centralized Binomial distributions } \times \text{ Kravchuk polynomials } \rangle$$

Discrete Analog of Heisenberg Ladder

$$\cdots \quad \bullet \quad \underset{MX + nh^2 D}{\overset{-D}{\rightleftarrows}} \quad \bullet \quad \underset{MX + (n+1)h^2 D}{\overset{-D}{\rightleftarrows}} \quad \bullet \quad \cdots$$

$$\begin{aligned}
A_n^{\Delta} &= \frac{1}{2} \\
A_n^{\Diamond} &= -[M(DX + XD) + (n + \frac{1}{2})h^2 D^2] \\
\alpha_n &= n + 1 \\
\mathcal{E}_n &= \mathrm{eig}(A_n^{\Diamond}, n) \\
&= \langle \text{ Centralized Binomial distributions,} \\
&\qquad\qquad \text{alternating sign }, \#\,\mathrm{supp} = n + 1\rangle
\end{aligned}$$

q–Heisenberg Ladder

$$\cdots \qquad \bullet \quad \underset{D}{\overset{q^{-(n-1)}X}{\rightleftarrows}} \quad \bullet \quad \underset{D}{\overset{q^{-n}X}{\rightleftarrows}} \quad \bullet \qquad \cdots$$

$$A_n^{\Delta} = \frac{1}{2}(Dq^{-n}X - q^{-(n-1)}XD) = \frac{q^{-n}}{2}(DX - qXD) = \frac{q^{-n}}{2}$$

$$A_n^{\Diamond} = \frac{q^{-n}}{2}(DX + qXD)$$

$$\alpha_n = \frac{q^{-(n+1)} - 1}{q^{-1} - 1} = (n+1)_{\frac{1}{q}}$$

$$\mathcal{E}_n = \langle \text{ monomial, degree } n \ \rangle$$

q–Harmonic Oscillator Ladder

$$\cdots \quad \bullet \quad \underset{q^{n}X+D}{\overset{q^{-(n-1)}X-D}{\rightleftarrows}} \quad \bullet \quad \underset{q^{n+1}X+D}{\overset{q^{-n}X-D}{\rightleftarrows}} \quad \bullet \quad \cdots$$

$$A_n^{\Delta} = \frac{q^n + q^{-n}}{2}$$

$$A_n^{\Diamond} = qX^2 + \frac{q^{-n} - q^n}{2}(DX + qXD) - D^2$$

$$\alpha_n = \frac{q^{n+1} - q^{-n}}{q-1}$$

$$\mathcal{E}_n = \langle \; n^{\text{th}} \; q\text{–Hermite function} \; \rangle$$

Ladder complexes

$$\cdots \quad \bullet \quad \underset{A^-_{n-1}}{\overset{A^+_{n-1}}{\rightleftarrows}} \quad \bullet \quad \underset{A^-_{n}}{\overset{A^+_{n}}{\rightleftarrows}} \quad \bullet \quad \cdots$$

$$A^+_n A^+_{n-1} = 0 \qquad A^-_{n-1} A^-_n = 0 \qquad \text{for all } n.$$

$\curvearrowright$ $(A^\Diamond_n)$ is a Ladder Endomorphism

$\curvearrowright$ $\mathcal{E}_n(\lambda) = \operatorname{eig}(A^\Diamond_n, \lambda)$ is a subladder /

splits into two SIE subladders $\mathcal{E}'_n(\lambda)$ and $\mathcal{E}''_n(\lambda)$

$$\cdots\ \mathcal{E}^\Diamond_{n-1}(\lambda) \underset{A^-_{n-1}}{\overset{A^+_{n-1}}{\rightleftarrows}} \mathcal{E}^\Diamond_{n}(\lambda) \underset{A^-_{n}}{\overset{A^+_{n}}{\rightleftarrows}} \mathcal{E}^\Diamond_{n+1}(\lambda) \underset{A^-_{n+1}}{\overset{A^+_{n+1}}{\rightleftarrows}} \mathcal{E}^\Diamond_{n+2}(\lambda)\ \cdots \quad =$$

$$\cdots\ \mathcal{E}'_{n-1}(\lambda) \overset{0}{\rightleftarrows} \mathcal{E}'_{n}(\lambda) \overset{\lambda}{\rightleftarrows} \mathcal{E}'_{n+1}(\lambda) \overset{0}{\rightleftarrows} \mathcal{E}'_{n+2}(\lambda)\ \cdots$$

$$\oplus$$

$$\cdots\ \mathcal{E}''_{n-2}(\lambda) \overset{\lambda}{\rightleftarrows} \mathcal{E}''_{n}(\lambda) \overset{0}{\rightleftarrows} \mathcal{E}''_{n+2}(\lambda) \overset{\lambda}{\rightleftarrows} \mathcal{E}''_{n+4}(\lambda)\ \cdots$$

Example: De Rham complex of a compact Riemann manifold

$$\cdots \quad \bullet \quad \overset{d}{\underset{\delta}{\rightleftarrows}} \quad \bullet \quad \overset{d}{\underset{\delta}{\rightleftarrows}} \quad \bullet \quad \cdots$$

$$A_n^\diamondsuit = \frac{\delta d + d\delta}{2} \qquad \text{Laplace–Beltrami operator}$$

$$\mathcal{E}_n(\lambda) = \operatorname{eig}(A_n^\diamondsuit, \lambda) \qquad \text{subladder}$$

$\boxed{\lambda \neq 0}$ the above splitting shows: $\mathcal{E}_n(\lambda)$ is exact

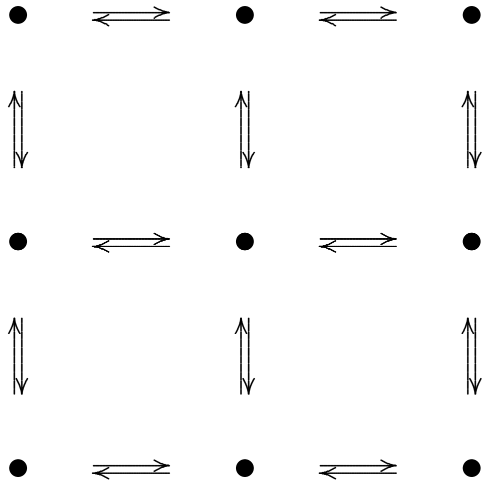
$\curvearrowright$ Cohomology yields the zero complex

$\boxed{\lambda = 0}$ $\mathcal{E}_n(0)$ all operators zero

$\curvearrowright$ Cohomology reproduces $\mathcal{E}_n(0)$

Altogether: $H^n(\Omega) \cong \mathcal{E}_n(0) = \underbrace{\operatorname{eig}(A_n^\diamondsuit, 0)}_{\text{harmonic forms}}$ (Theorem of Hodge)

Grids



Theorem:

Assume that ...

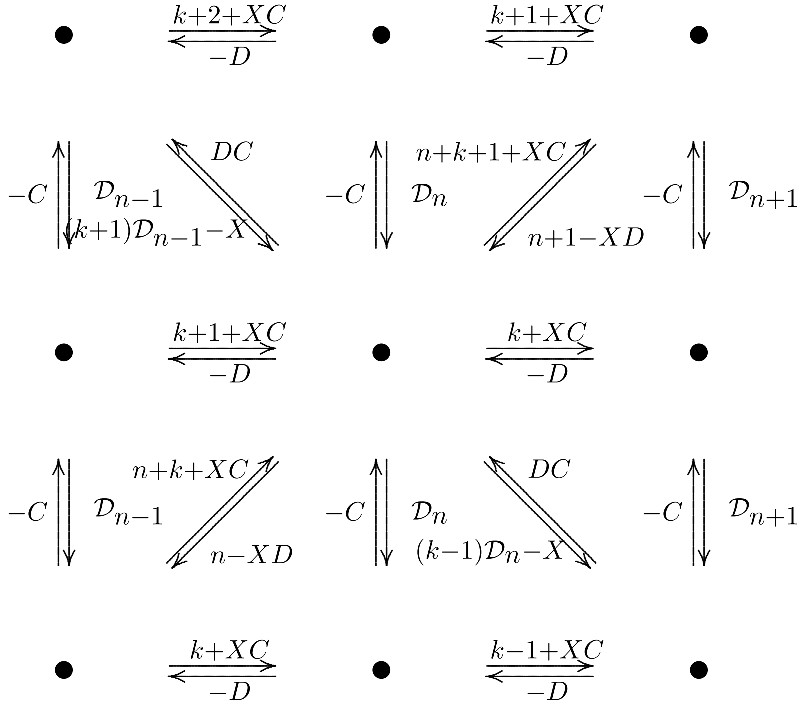
All horizontal ladders are SIE

All vertical ladders are SIE

Each square contain at least one "Scalar Commutative Loop"

Then all loops are SIE.

The Laguerre grid — local



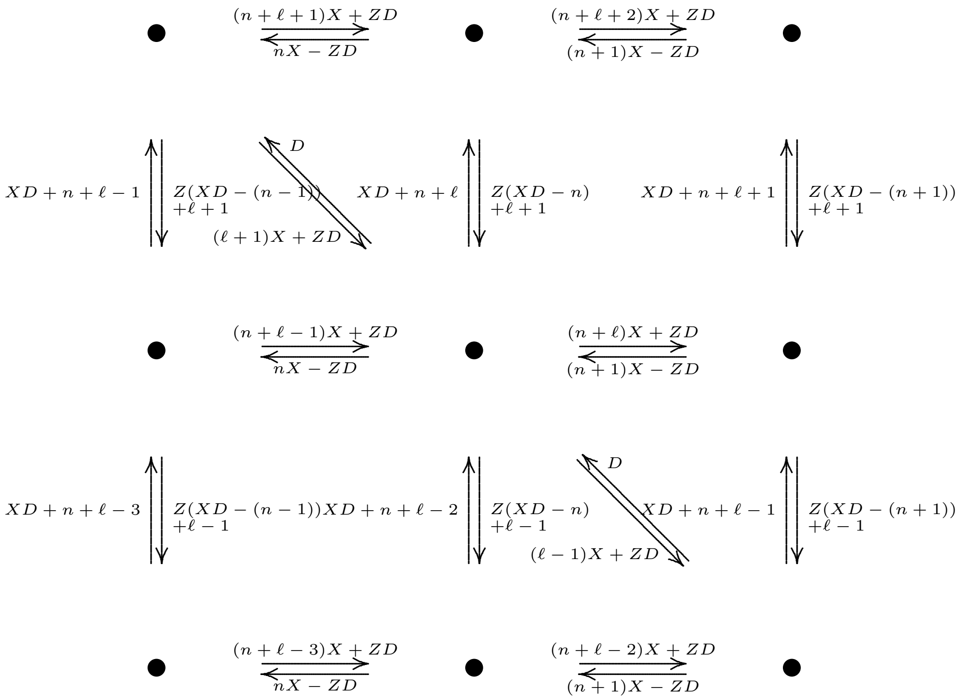
$$C := D - 1$$

$$\mathcal{D}_\ell := \sum_{k=0}^{\ell} D^k$$

The Laguerre grid — SIE subrepresentation

$$
\begin{array}{lccccccccccc}
 & -C\uparrow\downarrow\mathcal{D}_0 & & -C\uparrow\downarrow\mathcal{D}_1 & & -C\uparrow\downarrow\mathcal{D}_2 & & -C\uparrow\downarrow\mathcal{D}_3 & & -C\uparrow\downarrow\mathcal{D}_4 & & -C\uparrow\downarrow\mathcal{D}_5 \\
 & \langle 1\rangle & \overset{2+XC}{\underset{-D}{\rightleftarrows}} & \langle x-2\rangle & \overset{1+XC}{\underset{-D}{\rightleftarrows}} & \langle x^2-4x+2\rangle & \overset{0+XC}{\underset{-D}{\rightleftarrows}} & \langle x^3-6x^2+6x\rangle & \overset{-1+XC}{\underset{-D}{\rightleftarrows}} & \langle x^4-8x^3+12x^2\rangle & \overset{-2+XC}{\underset{-D}{\rightleftarrows}} & \langle x^5-10x^4+20x^3\rangle \\
k=2\nearrow & & & & & & & & & & & \\
 & -C\uparrow\downarrow\mathcal{D}_0 & & -C\uparrow\downarrow\mathcal{D}_1 & & -C\uparrow\downarrow\mathcal{D}_2 & & -C\uparrow\downarrow\mathcal{D}_3 & & -C\uparrow\downarrow\mathcal{D}_4 & & -C\uparrow\downarrow\mathcal{D}_5 \\
 & \langle 1\rangle & \overset{1+XC}{\underset{-D}{\rightleftarrows}} & \langle x-1\rangle & \overset{0+XC}{\underset{-D}{\rightleftarrows}} & \langle x^2-2x\rangle & \overset{-1+XC}{\underset{-D}{\rightleftarrows}} & \langle x^3-3x^2\rangle & \overset{-2+XC}{\underset{-D}{\rightleftarrows}} & \langle x^4-4x^3\rangle & \overset{-3+XC}{\underset{-D}{\rightleftarrows}} & \langle x^5-5x^4\rangle \\
k=1\nearrow & & & & & & & & & & & \\
 & -C\uparrow\downarrow\mathcal{D}_0 & & -C\uparrow\downarrow\mathcal{D}_1 & & -C\uparrow\downarrow\mathcal{D}_2 & & -C\uparrow\downarrow\mathcal{D}_3 & & -C\uparrow\downarrow\mathcal{D}_4 & & -C\uparrow\downarrow\mathcal{D}_5 \\
 & \langle 1\rangle & \overset{0+XC}{\underset{-D}{\rightleftarrows}} & \langle x\rangle & \overset{-1+XC}{\underset{-D}{\rightleftarrows}} & \langle x^2\rangle & \overset{-2+XC}{\underset{-D}{\rightleftarrows}} & \langle x^3\rangle & \overset{-3+XC}{\underset{-D}{\rightleftarrows}} & \langle x^4\rangle & \overset{-4+XC}{\underset{-D}{\rightleftarrows}} & \langle x^5\rangle \\
k=0\nearrow & n=0 & & n=1 & & n=2 & & n=3 & & n=4 & & n=5
\end{array}
$$

The Legendre–Gegenbauer grid — local

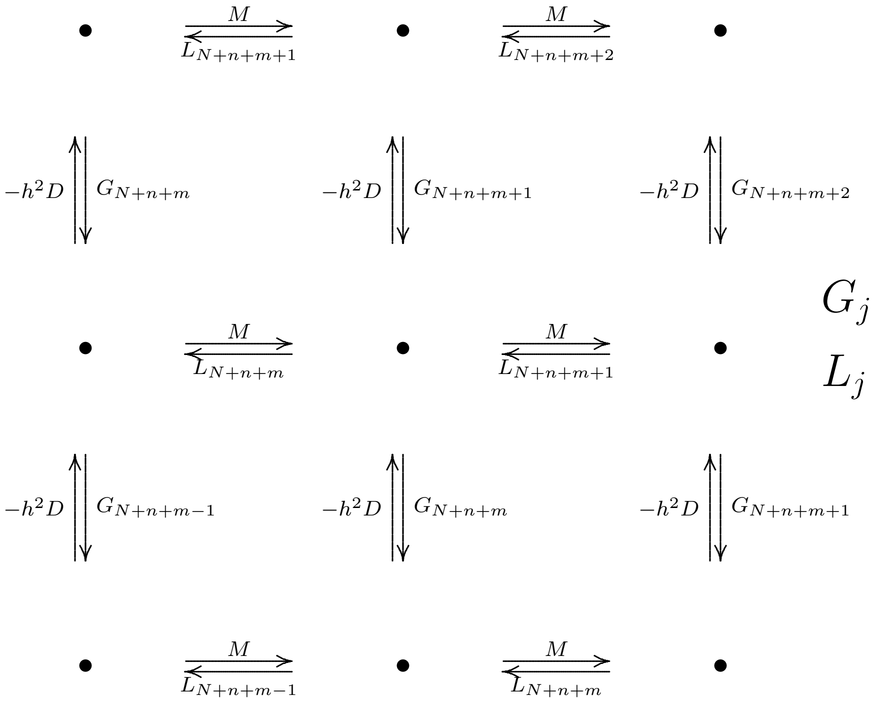


$$Z := X^2 - 1$$

The Legendre–Gegenbauer grid — SIE subrepresentation

	$XD+3 \uparrow\downarrow Z(XD)+4$		$XD+4 \uparrow\downarrow Z(XD-1)+4$		$XD+5 \uparrow\downarrow Z(XD-2)+4$		$XD+6 \uparrow\downarrow Z(XD-3)+4$		$XD+7 \uparrow\downarrow Z(XD-4)+4$		$XD+8 \uparrow\downarrow Z(XD-5)+4$
$\ell{=}3$	$\langle 1 \rangle$	$\underset{1X-ZD}{\overset{3X+ZD}{\rightleftarrows}}$	$\langle x \rangle$	$\underset{2X-ZD}{\overset{4X+ZD}{\rightleftarrows}}$	$\langle 5x^2{-}1 \rangle$	$\underset{3X-ZD}{\overset{5X+ZD}{\rightleftarrows}}$	$\langle 7x^3{-}3x \rangle$	$\underset{4X-ZD}{\overset{6X+ZD}{\rightleftarrows}}$	$\langle 21x^4{-}14x^2{+}1 \rangle$	$\underset{5X-ZD}{\overset{7X+ZD}{\rightleftarrows}}$	$\langle 231x^5{-}210x^3{+}35x \rangle$
	$XD+1 \uparrow\downarrow Z(XD)+2$		$XD+2 \uparrow\downarrow Z(XD-1)+2$		$XD+3 \uparrow\downarrow Z(XD-2)+2$		$XD+4 \uparrow\downarrow Z(XD-3)+2$		$XD+5 \uparrow\downarrow Z(XD-4)+2$		$XD+6 \uparrow\downarrow Z(XD-5)+2$
$\ell{=}1$	$\langle 1 \rangle$	$\underset{1X-ZD}{\overset{1X+ZD}{\rightleftarrows}}$	$\langle x \rangle$	$\underset{2X-ZD}{\overset{2X+ZD}{\rightleftarrows}}$	$\langle 3x^2{-}1 \rangle$	$\underset{3X-ZD}{\overset{3X+ZD}{\rightleftarrows}}$	$\langle 5x^3{-}3x \rangle$	$\underset{4X-ZD}{\overset{4X+ZD}{\rightleftarrows}}$	$\langle 35x^4{-}30x^2{+}3 \rangle$	$\underset{5X-ZD}{\overset{5X+ZD}{\rightleftarrows}}$	$\langle 63x^5{-}70x^3{+}15x \rangle$
	$XD-1 \uparrow\downarrow Z(XD)$		$XD \uparrow\downarrow Z(XD-1)$		$XD+1 \uparrow\downarrow Z(XD-2)$		$XD+2 \uparrow\downarrow Z(XD-3)$		$XD+3 \uparrow\downarrow Z(XD-4)$		$XD+4 \uparrow\downarrow Z(XD-5)$
$\ell{=}{-}1$	$\langle 1 \rangle$	$\underset{1X-ZD}{\overset{-1X+ZD}{\rightleftarrows}}$	$\langle x \rangle$	$\underset{2X-ZD}{\overset{0X+ZD}{\rightleftarrows}}$	$\langle x^2{-}1 \rangle$	$\underset{3X-ZD}{\overset{1X+ZD}{\rightleftarrows}}$	$\langle 3x^3{-}3x \rangle$	$\underset{4X-ZD}{\overset{2X+ZD}{\rightleftarrows}}$	$\langle 5x^4{-}6x^2{+}1 \rangle$	$\underset{5X-ZD}{\overset{3X+ZD}{\rightleftarrows}}$	$\langle 7x^5{-}10x^3{+}3x \rangle$
	$n{=}0$		$n{=}1$		$n{=}2$		$n{=}3$		$n{=}4$		$n{=}5$

Binomial grid — local

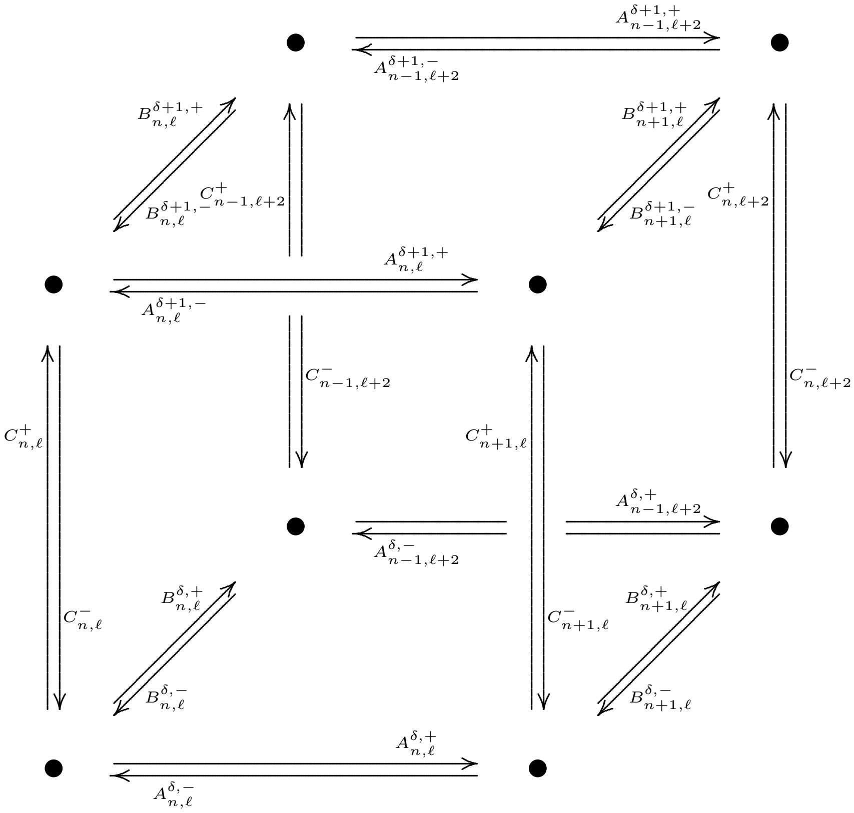


$$G_j := (1+jh^2)D + XM$$
$$L_j := (1+jh^2)M + h^2XD$$

Binomial grid — SIE subrepresentation

$\langle\delta_{-3}-3\delta_{-1}+3\delta_1-\delta_3\rangle$	$\overset{M}{\underset{L_{N+4}}{\rightleftarrows}}$	$\langle\delta_{-4}-2\delta_{-2}+2\delta_2-\delta_4\rangle$	$\overset{M}{\underset{L_{N+5}}{\rightleftarrows}}$	$\langle\delta_{-5}-\delta_{-3}-2\delta_{-1}-2\delta_1+\delta_3-\delta_5\rangle$	$\overset{M}{\underset{L_{N+6}}{\rightleftarrows}}$	$\langle\delta_{-6}-3\delta_{-2}+3\delta_2-\delta_6\rangle$	$\overset{M}{\underset{L_{N+6}}{\rightleftarrows}}$	$\langle\delta_{-7}-\delta_{-5}-3\delta_{-3}-3\delta_{-1}+3\delta_1+3\delta_3-\delta_5-\delta_7\rangle$
$-h^2D\uparrow\ \downarrow G_{N+3}$		$-h^2D\uparrow\ \downarrow G_{N+4}$		$-h^2D\uparrow\ \downarrow G_{N+5}$		$-h^2D\uparrow\ \downarrow G_{N+6}$		$-h^2D\uparrow\ \downarrow G_{N+6}$
$\langle\delta_{-2}-2\delta_0+\delta_2\rangle$	$\overset{M}{\underset{L_{N+3}}{\rightleftarrows}}$	$\langle\delta_{-3}-\delta_{-1}-\delta_1+\delta_3\rangle$	$\overset{M}{\underset{L_{N+4}}{\rightleftarrows}}$	$\langle\delta_{-4}-2\delta_0+\delta_4\rangle$	$\overset{M}{\underset{L_{N+5}}{\rightleftarrows}}$	$\langle\delta_{-5}+\delta_{-3}-2\delta_{-1}-2\delta_1+\delta_3+\delta_5\rangle$	$\overset{M}{\underset{L_{N+6}}{\rightleftarrows}}$	$\langle\delta_{-6}+2\delta_{-4}-\delta_{-2}-4\delta_0-\delta_2+2\delta_4+\delta_6\rangle$
$-h^2D\uparrow\ \downarrow G_{N+2}$		$-h^2D\uparrow\ \downarrow G_{N+3}$		$-h^2D\uparrow\ \downarrow G_{N+4}$		$-h^2D\uparrow\ \downarrow G_{N+5}$		$-h^2D\uparrow\ \downarrow G_{N+6}$
$\langle\delta_{-1}-\delta_1\rangle$	$\overset{M}{\underset{L_{N+2}}{\rightleftarrows}}$	$\langle\delta_{-2}-\delta_2\rangle$	$\overset{M}{\underset{L_{N+3}}{\rightleftarrows}}$	$\langle\delta_{-3}+\delta_{-1}-\delta_1-\delta_3\rangle$	$\overset{M}{\underset{L_{N+4}}{\rightleftarrows}}$	$\langle\delta_{-4}+2\delta_{-2}-2\delta_2-\delta_4\rangle$	$\overset{M}{\underset{L_{N+5}}{\rightleftarrows}}$	$\langle\delta_{-5}+3\delta_{-3}+2\delta_{-1}-2\delta_{-1}-3\delta_{-3}-\delta_{-5}\rangle$
$-h^2D\uparrow\ \downarrow G_{N+1}$		$-h^2D\uparrow\ \downarrow G_{N+2}$		$-h^2D\uparrow\ \downarrow G_{N+3}$		$-h^2D\uparrow\ \downarrow G_{N+4}$		$-h^2D\uparrow\ \downarrow G_{N+5}$
$\langle\delta_0\rangle$	$\overset{M}{\underset{L_{N+1}}{\rightleftarrows}}$	$\langle\delta_{-1}+\delta_1\rangle$	$\overset{M}{\underset{L_{N+2}}{\rightleftarrows}}$	$\langle\delta_{-2}+2\delta_0+\delta_2\rangle$	$\overset{M}{\underset{L_{N+3}}{\rightleftarrows}}$	$\langle\delta_{-3}+3\delta_{-1}+3\delta_1+\delta_3\rangle$	$\overset{M}{\underset{L_{N+4}}{\rightleftarrows}}$	$\langle\delta_{-4}+4\delta_{-2}+6\delta_0+4\delta_2+\delta_4\rangle$

The Jacobi cube



$$
\begin{aligned}
Z &:= X^2 - 1 \\
A^{\delta+}_{n,\ell} &:= ZD + (n+\ell)(X + \tfrac{\delta}{2n+\ell+1}) \\
A^{\delta-}_{n,\ell} &:= ZD - (n+1)(X - \tfrac{\delta}{2n+\ell+1}) \\
B^{\delta+}_{n,\ell} &:= D \\
B^{\delta-}_{n,\ell} &:= ZD + (\ell+1)X + \delta \\
C^{+}_{0,\ell} &:= 1 \\
C^{+}_{n,\ell} &:= C^{+}_{n-1,\ell+2}[n(n+\ell) + D] \\
&\quad +[ZD + (\ell+1)X, C^{+}_{n-1,\ell+2}]D \\
C^{-}_{0,\ell} &:= 1 \\
C^{-}_{n,\ell} &:= C^{-}_{n-1,\ell+2}[n(n+\ell) - D] \\
&\quad +[ZD + (\ell+1)X, C^{-}_{n-1,\ell+2}]D
\end{aligned}
$$

Reference

[1] S. Hilger, The Category of Ladders, *Results in Mathematics* 57, 3 (2010), 335–364.