Analytic Solutions of Iterative Functional Equations

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Iterative Functional Equation

Functional equations containing the compositon of unknown functions

Examples: $f(f(x)) = e^x, \; f(x) + f(x + f(x)) = x$

Functional Equations of Several Variables

Functional equations whose arguments containing several variables

Examples: $f(x) + f(y) = f(x + y)$ (Cauchy's functional equation)

$$\begin{cases} f(x+y) = f(x)g(y) + g(x)f(y) \\ g(x+y) = g(x)g(y) - f(x)f(y) \end{cases}$$

Iterative root

N: natural number

A solution of the iterative functional equation

$$f^N(x) = g(x)$$

is called N-th iterative root of $g(x)$.

No general theory to obtain iterative roots.

Iteration group

I : interval

$f : I \to I$: given function

A family $\{f^t \mid t \in \mathbb{R}\}$ of maps $f^t : I \to I$ is called the iteration group of f if

(1) $f^t(f^s(x)) = f^{t+s}(x), \; t, \; s \in \mathbb{R}$

(2) $f^1(x) = f(x)$

(3) The map $t \in \mathbb{R} \mapsto f^t(x)$ is continuous for all $x \in I$.

Note that

$$f^{1/N}(x)$$

is an N-th iterative root of $f(x)$.

Abel's equation

$f : I \to I$: given function
Abel's equation for f is

$$\Phi(f(x)) = \Phi(x) + 1.$$

If the solution of Abel's equation exists, then we can construct the iteration group of f by

$$f^t(x) = \Phi^{-1}(\Phi(x) + t).$$

The analytic solution of Abel's equation

$$\Phi(e^x) = \Phi(x) + 1.$$

is obtained by Kneser(1950), Belitskii et al.(1993).

Prompter

Consider iterative functional equations of the form

$$\sum_{i=1}^{n} a_i f^i(x) = bx + \sum_{j=0}^{\infty} c_j e^{-d_j x} \quad (1)$$

where $a_i,\ b,\ c_j,\ d_j \in \mathbb{R}$ and $0 = d_0 < d_1 < d_2 < \cdots \to \infty$.

Note that this equation contains a linear term bx on the right-hand side.

Suppose that the solutions of equation (1) is of the form

$$f(x) = \lambda x + \sum_{j=0}^{\infty} p_j e^{-q_j x}, \quad (2)$$

where $p_j,\ q_j \in \mathbb{R}$ and $0 = q_0 < q_1 < q_2 < \cdots \to \infty$.
The term λx is called prompter of the equation.

Characteristic equation

If we substitute (2) into (1), we have

$$\sum_{i=1}^{n} a_i \lambda^i x + (\text{Dirichlet series}) = bx + (\text{Dirichlet series}).$$

Comparing linear terms of both sides, we have

$$\sum_{i=1}^{n} a_i \lambda^i = b. \tag{3}$$

Equation (3) is called characteristic equation of (1).

Main theorem A

Consider iterative functional equation

$$\sum_{i=1}^{n} a_i f^i(x) = bx + \sum_{j=0}^{\infty} c_j e^{-d_j x}, \qquad (1)$$

where all $a_i \geq 0$ and $a_n \neq 0$.
For each prompter of positive coefficient, namely, for each positive root λ of characteristic equation of (1), we have a unique formal solution of the form

$$f(x) = \lambda x + \sum_{j=0}^{\infty} p_j e^{-q_j x}. \qquad (2)$$

The coefficients $p_j,\ q_j$ are determined recursively.

Initial exponent

First, we must determine the constant term. Substituting (2) into (1), and comparing the constant term of both sides, we have

$$\sum_{i=1}^{n} a_i (1 + \lambda + \lambda^2 + \cdots + \lambda^{i-1}) p_0 = c_0 \,.$$

Since $\sum_{i=1}^{n} a_i > 0, \ \lambda > 0$, p_0 is determined uniquely.
Note that if $c_0 = 0$, then constant term of the solution vanishes.
Next, we must determine the minimal exponent among nonzero q_j's. This is called initial exponent and determined by

$$q_1 = \max\{d_1, \ \frac{d_1}{\lambda}\}.$$

Substitute (2) into (1) and expanding double exponentials by using Taylor expansion, and compare the coefficients of $e^{-q_1 x}$.
Then we have the value of p_1.
Next, the second smallest exponent is q_2, and we proceed similarly.

Example 1

$$f(f(x)) = x - 3e^{-x} \tag{4}$$

Characteristic equation: $\lambda^2 = 1$
Positive characteric: $\lambda = 1$
Initial exponent: 1
Putting

$$f(x) = x + p_1 e^{-x} + p_2 e^{-2x} + p_3 e^{-3x} + \cdots$$

and substituting into (4), we have

$$\begin{aligned} f(f(x)) &= f(x) + p_1 e^{-f(x)} + p_2 e^{-2f(x)} + p_3 e^{-3f(x)} + \cdots \\ &= x + p_1 e^{-x} + p_2 e^{-2x} + p_3 e^{-3x} + \cdots \\ &\quad + p_1 e^{-x} \cdot e^{-p_1 e^{-x}} \cdot e^{-p_2 e^{-2x}} \cdot e^{-p_3 e^{-3x}} \cdots \\ &\quad + p_2 e^{-2x} \cdot e^{-2p_1 e^{-x}} \cdot e^{-2p_2 e^{-2x}} \cdot e^{-2p_3 e^{-3x}} \cdots \\ &\quad + p_3 e^{-3x} \cdot e^{-3p_1 e^{-x}} \cdot e^{-3p_2 e^{-2x}} \cdot e^{-3p_3 e^{-3x}} \cdots \end{aligned}$$

Example 1 (cont'd)

$$= x + p_1 e^{-x} + p_2 e^{-2x} + p_3 e^{-3x} + \cdots$$

$$+ p_1 e^{-x} \left(1 - p_1 e^{-x} + \frac{p_1^2}{2!} e^{-2x} - \cdots\right)\left(1 - p_2 e^{-2x} + \frac{p_2^2}{2!} e^{-4x} - \cdots\right)($$

$$+ p_2 e^{-2x} \left(1 - 2p_1 e^{-x} + \frac{(2p_1)^2}{2!} e^{-2x} - \cdots\right)\left(1 - 2p_2 e^{-2x} + \frac{(2p_2)^2}{2!} e^{-4}$$

$$+ p_3 e^{-3x} \left(1 - 3p_1 e^{-x} + \frac{(3p_1)^2}{2!} e^{-2x} - \cdots\right)\left(1 - 3p_2 e^{-2x} + \frac{(3p_2)^2}{2!} e^{-4}$$

$$= x + 2p_1 e^{-x} + (2p_2 - p_1^2) e^{-2x} + \left(2p_3 + \frac{p_1^3}{2} - 3p_1 p_2\right) e^{-3x} + \cdots$$

$$= x - 3e^{-x}.$$

Comparing coefficients, we have $p_1 = -\frac{3}{2}$, $p_2 = \frac{9}{8}$, $p_3 = -\frac{27}{16}, \ldots$ recursively.

Example 1 (cont'd 2)

Up to 10 terms:

$$fa(x) = x - \frac{3}{2}e^{-x} + \frac{9}{8}e^{-2x} - \frac{27}{16}e^{-3x} + \frac{189}{64}e^{-4x} - \frac{567}{128}e^{-5x} + \frac{3159}{2560}e^{-6x}\frac{143613}{5120}e^{-7x} - \frac{4877739}{35840}e^{-8x} + \frac{8636463}{28672}e^{-9x} + \frac{79218243}{229376}e^{-10x}$$

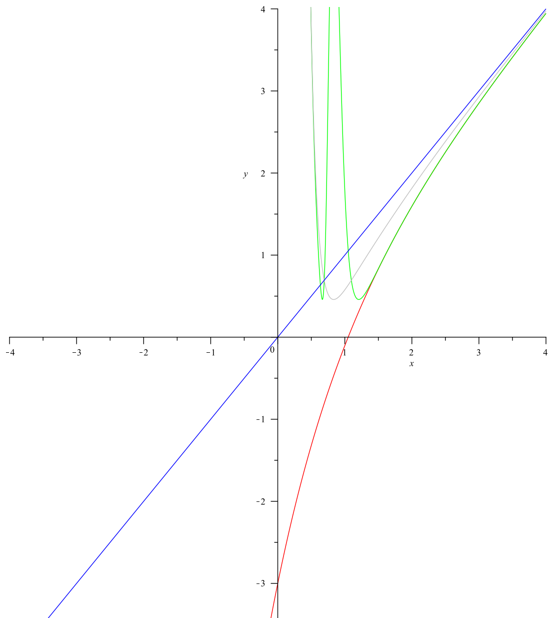
Graphs:

$y = x$

$y = fa(fa(x))$

$y = x - 3e^{-x}$

$y = fa(x)$



Example 2

$$f(f(x)) = x + \int_1^2 e^{-tx} dt = x + \frac{e^{-2x} - e^{-x}}{x} \tag{5}$$

Putting

$$f(x) = x + \int_1^\infty p(s) e^{-sx} ds$$

and substituting into (5), and applying Taylor expansion of exp, we can compare the Laplace preimages of both sides.

$$p(s) = \begin{array}{ll} 0, & 0 \le s < 1 \\ \frac{1}{2}, & 1 \le s < 2 \\ \frac{s^2 - 2s}{16}, & 2 \le s < 3 \cdots \end{array}$$

Example 3

$$f(f(x)) = x + \sin x \tag{6}$$

Putting

$$f(x) = x + \sum_{i,\,j\geq 0,\ (i,\,j)\neq(0,\,0)} a_{i,\,j} \sin^i x \cos^j x$$

and substituting into (6), we have

$$f(f(x)) = f(x) + \sum_{i,\,j\geq 0,\ (i,\,j)\neq(0,\,0)} a_{i,\,j} \sin^i f(x) \cos^j f(x)$$

$$= x + \sum_{i,\,j\geq 0,\ (i,\,j)\neq(0,\,0)} a_{i,\,j} \sin^i x \cos^j x + \sum_{i,\,j\geq 0,\ (i,\,j)\neq(0,\,0)} a_{i,\,j}$$

$$\times\{\sin(x + \sum_{k,\,l\geq 0,\ (k,\,l)\neq(0,\,0)} a_{k,\,l} \sin^k x \cos^l x)\}^i$$

$$\times\{\cos(x + \sum_{k,\,l\geq 0,\ (k,\,l)\neq(0,\,0)} a_{k,\,l} \sin^k x \cos^l x)\}^j$$

Example 3 (cont'd)

$$= x + \sum_{i,\, j \geq 0,\, (i,\, j) \neq (0,\, 0)} a_{i,\, j} \sin^i x \cos^j x + \sum_{i,\, j \geq 0,\, (i,\, j) \neq (0,\, 0)} a_{i,\, j}$$

$$\times (\sin x \cos B + \cos x \sin B)^i (\cos x \cos B - \sin x \sin B)^j$$

$$= x + \sum_{i,\, j \geq 0,\, (i,\, j) \neq (0,\, 0)} a_{i,\, j} \sin^i x \cos^j x + \sum_{i,\, j \geq 0,\, (i,\, j) \neq (0,\, 0)} a_{i,\, j}$$

$$\times \left(\sin x \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} B^{2k} + \cos x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} B^{2k+1} \right)^i$$

$$\times \left(\cos x \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} B^{2k} - \sin x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} B^{2k+1} \right)^j$$

$$= x + \sin x$$

where $B = \sum_{k,\, l \geq 0,\, (k,\, l) \neq (0,\, 0)} a_{k,\, l} \sin^k x \cos^l x,$

Example 3 (cont'd 2)

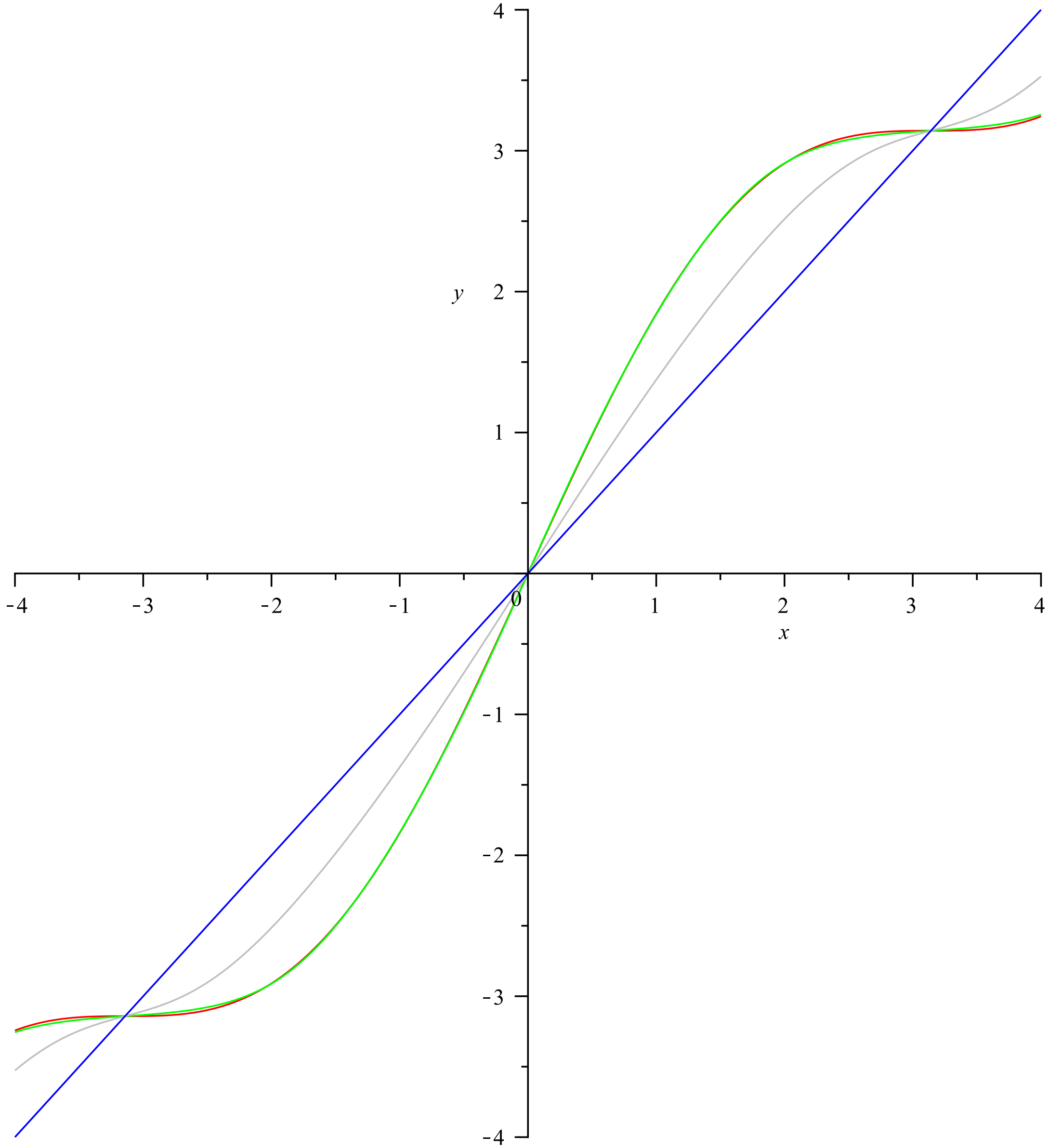
by using addition theorems and Taylor expansions of sin and cos.

Comparing the coefficients of $\sin^i x \cos^j x$ where $i + j \leq n$, we have a solution. For example, $n = 5$ case:

$$f(x) = x + \frac{1}{2}\sin x - \frac{1}{8}\sin x\cos x + \frac{1}{16}\sin x\cos^2 x - \frac{5}{128}\sin x\cos^3 x$$
$$+\frac{7}{256}\sin x\cos^4 x - \frac{1}{384}\sin^3 x\cos x + \frac{1}{192}\sin^3 x\cos^2 x - \frac{1}{256}\sin^5 x.$$

Example 3 (cont'd 3)

Graphs:



$y = x$

$y = f(f(x))$

$y = x + \sin x$

$y = f(x)$

Main theorem B

For a function of the form

$$f(x) = x + \sum_{j=0}^{\infty} c_j \, e^{-dj\,x},$$

there exists an algorithm to obtain iteration group of $f(x)$.

Example 4

$$f(x) = x + e^{-x}$$

Putting

$$f^t(x) = x + p_1(t)e^{-x} + p_2(t)e^{-2x} + p_3(t)e^{-3x} \cdots ,$$

and substituting into the condition

$$f^t(f^s(x)) = f^{t+s}(x), \ t, \ s \in \mathbb{R},$$

we have

$$x + \{p_1(t) + p_1(s)\}e^{-t} + \{p_2(t) - p_1(t)p_1(s) + p_2(s)\}e^{-2t}$$

$$+\{p_3(t) - p_2(t)p_1(s) + \frac{1}{2}p_1(t)^2 p_1(s) - 2p_1(t)p_2(s) + p_3(s)\}e^{-3t} \cdots$$

$$= x + p_1(t+s)e^{-x} + p_2(t+s)e^{-2x} + p_3(t+s)e^{-3x} \cdots .$$

Example 4 (cont'd)

Comparing the coefficients, we have

$$p_1(t+s) = p_1(t) + p_1(s)$$
$$p_2(t+s) = p_2(t) - p_1(t)p_1(s) + p_2(s)$$
$$p_3(t+s) = p_3(t) - p_2(t)p_1(s) + \frac{1}{2}p_1(t)^2 p_1(s) - 2p_1(t)p_2(s) + p_3(s)$$
$$\cdots$$

We can determine $p_i(t)$ one by one.

$$\begin{aligned} f^t(x) = x + te^{-x} &+ \left(\frac{t}{2} - \frac{1}{2}t^2\right)e^{-2x} + \left(\frac{5}{12}t - \frac{3}{4}t^2 + \frac{1}{3}t^3\right)e^{-3x} \\ &+ \left(\frac{5}{12}t - \frac{13}{12}t^2 + \frac{11}{12}t^3 - \frac{1}{4}t^4\right)e^{-4x} \\ &+ \left(\frac{107}{240}t - \frac{25}{16}t^2 + \frac{47}{24}t^3 - \frac{25}{24}t^4 + \frac{1}{5}t^5\right)e^{-5x} + \cdots \end{aligned}$$

Example 5

Riemann-zeta function:

$$\zeta(x) = 1 + 2^{-x} + 3^{-x} + 4^{-x} + \cdots\,, \ \Re x > 1$$

Term-by-term integration gives

$$Z(x) = x - \frac{1}{\log 2}2^{-x} - \frac{1}{\log 3}3^{-x} - \frac{1}{\log 4}4^{-x} - \cdots\,.$$

Then, the iteration group of $Z(x)$ is as follows:

$$Z^t(x) = x - \frac{t}{\log 2}2^{-x} - \frac{t}{\log 3}3^{-x} - \frac{t^2}{\log 4}4^{-x} - \frac{t}{\log 5}5^{-x}$$
$$+ \left\{ \left(-\frac{1}{2\log 2} - \frac{1}{2\log 3} \right) t^2 + \left(\frac{1}{2\log 2} + \frac{1}{2\log 3} - \frac{1}{\log 6} \right) t \right\} 6^{-x}$$
$$- \frac{t}{\log 7}7^{-x} - \frac{t^3}{\log 8}8^{-x} + \cdots$$