Orthogonal Polynomials in the Normal Matrix Model

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Orthogonal polynomials

- **Orthogonal polynomials on the real line**

$$\int_{-\infty}^{\infty} P_n(x)x^k w(x)\,dx = 0, \qquad k = 0, 1, \ldots, n-1$$

- **OP satisfy three term recurrence**

$$xP_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x)$$

- **What about polynomials satisfying a longer recurrence (but finite) ?**

$$\begin{aligned} xQ_n(x) = Q_{n+1}(x) + a_{n,0}Q_n(x) + a_{n,1}Q_{n-1}(x) + \\ + a_{n,2}Q_{n-2}(x) + \cdots + a_{n,r}Q_{n-r}(x) \end{aligned}$$

Multiple orthogonal polynomials

- **Multiple orthogonal polynomial (MOP) is a monic polynomial of degree** $n_1 + n_2$

$$P_{n_1,n_2}(x) = x^{n_1+n_2} + \cdots$$

characterized by

$$\int_{-\infty}^{\infty} P_{n_1,n_2}(x) x^k w_1(x)\, dx = 0, \qquad k = 0, 1, \ldots, n_1 - 1,$$
$$\int_{-\infty}^{\infty} P_{n_1,n_2}(x) x^k w_2(x)\, dx = 0, \qquad k = 0, 1, \ldots, n_2 - 1.$$

- w_1, w_2 **are two given weight functions.**
- $(n_1, n_2) \in \mathbb{N}^2$ **is a multi-index.**

- **Immediate extension to** r **weights** $w_1, \ldots, w_r$ **and** $(n_1, \ldots, n_r) \in \mathbb{N}^r$.

Short recurrence

- **Given MOPs P_{n_1,n_2} with two weight functions.**
- **The polynomials Q_n defined by**

$$Q_{2k} = P_{k,k}, \qquad Q_{2k+1} = P_{k+1,k}$$

have a four term recurrence

$$xQ_n(x) = Q_{n+1}(x) + a_n Q_n(x) + b_n Q_{n-1}(x) + c_n Q_{n-2}(x)$$

- **MOPs with r weight functions and near-diagonal multi-indices satisfy an $r+2$-term recurrence.**

2. MOP in random matrix theory

- **MOPs appeared first in Hermite's proof of the transcendence of the number e.**
- **MOPs were later used in analytic number theory, and approximation theory (simultaneous rational approximation).**

- **MOPs appear in random matrix theory and related stochastic processes**

(a) **Random matrices with external source**
(b) **Non-intersecting Brownian motions**
(c) **Non-intersecting squared Bessel paths**
(d) **Coupled random matrices (two matrix model)**

Non-intersecting squared Bessel paths

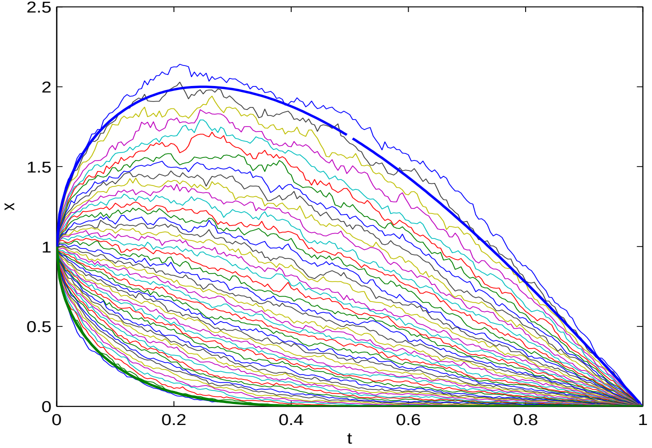
- **Squared Bessel process** is a Markov process on $[0, \infty)$ depending on a parameter $\alpha > -1$, with transition probabilities

$$p_t^\alpha(x, y) = \frac{1}{2t} \left(\frac{y}{x}\right)^{\alpha/2} e^{-(x+y)/(2t)} I_\alpha \left(\frac{\sqrt{xy}}{t}\right), \qquad x, y > 0,$$

  where I_α is the **modified Bessel function**

- **Assume n independent squared Bessel paths conditioned so that**
  - (a) **the paths start at time $t = 0$ at $a > 0$**
  - (b) **the paths end at time $t = 1$ at 0**
  - (c) **the paths do not intersect**

Simulation of 50 non-intersecting paths



Average polynomial

- **Random positions** $x_1(t) < x_2(t) < \cdots < x_n(t)$ **at time** $t \in (0,1)$ **give rise to random polynomial**

$$\prod_{j=1}^{n} (x - x_j(t))$$

- **Average polynomial**

$$P_n(x) = \mathbb{E}\left[\prod_{j=1}^{n} (x - x_j(t))\right]$$

is MOP on $[0,\infty)$ **with** $(n_1, n_2) = (\lceil n/2 \rceil, \lfloor n/2 \rfloor)$ **and**

$$w_1(x) = x^{\alpha/2} e^{-\frac{x}{2t(1-t)}} I_\alpha\left(\frac{\sqrt{ax}}{t}\right)$$

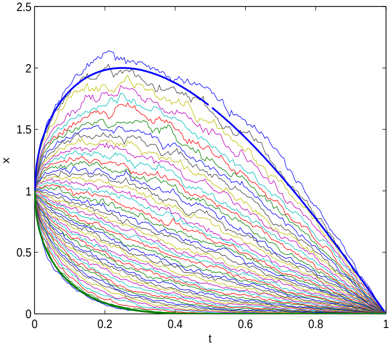
$$w_2(x) = x^{(\alpha+1)/2} e^{-\frac{x}{2t(1-t)}} I_{\alpha+1}\left(\frac{\sqrt{ax}}{t}\right)$$

K-Martínez Finkelshtein-Wielonsky (2009)

Recurrences

- Recurrence relation (four term) and differential equation (third order) for MOPs were found earlier

  Coussement-Van Assche (2003)



- Asymptotic analysis of MOPs leads to the limiting domain filled by the squared Bessel paths
- Local correlations at the critical time when the paths come to the wall at 0

K-Martínez Finkelshtein-Wielonsky (2009 + to appear)

3. Normal matrix model

- **Probability measure on** $n \times n$ **complex matrices**

$$\frac{1}{Z_n} e^{-\frac{n}{t_0} \mathrm{Tr}(MM^* - V(M) - \overline{V}(M^*))} dM, \qquad t_0 > 0,$$

where

$$V(M) = \sum_{k=1}^{\infty} \frac{t_k}{k} M^k.$$

- **Model depends on parameters**

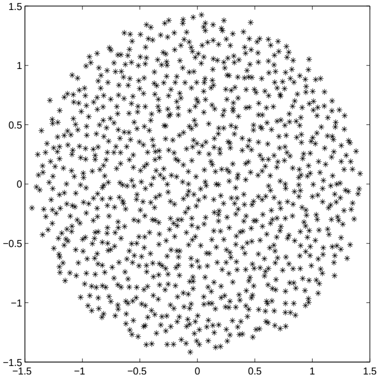
$$t_0 > 0, \qquad t_1, t_2, \dots, t_k, \dots .$$

- **For** $t_1 = t_2 = \cdots = 0$ **this is the Ginibre ensemble.**

Ginibre (1965)

Ginibre ensemble

- **Eigenvalues in the Ginibre ensemble have a limiting distribution as $n \to \infty$ that is uniform in a disk around 0 with radius $\sqrt{t_0}$.**



Laplacian growth

- **For general $t_1, t_2, \ldots$, the eigenvalues of M fill out a two-dimensional domain**

$$\Omega = \Omega(t_0, t_1, \ldots)$$

provided $t_0 > 0$ is sufficiently small.

- **Ω is characterized by**

$$t_0 = \frac{1}{\pi} \operatorname{area}(\Omega), \qquad t_k = -\frac{1}{\pi} \iint_{\mathbb{C} \setminus \Omega} \frac{dA(z)}{z^k}, \quad k \geq 1$$

- **As a function of t_0, the boundary of Ω evolves according to the model of Laplacian growth.**
- **The exterior harmonic moments t_k, $k \geq 1$, are constants of the motion.**

Wiegmann-Zabrodin (2000)
Teoderescu-Bettelheim-Agam-Zabrodin-Wiegmann (2005)

Unstable

- **Laplacian growth model is unstable.**
- **Singularities develop in finite time.**

4. Mathematical problem

- **Normal matrix model**

$$\frac{1}{Z_n} e^{-\frac{n}{t_0}\operatorname{Tr}(MM^* - V(M) - \overline{V}(M^*))} dM, \qquad t_0 > 0,$$

is **not well-defined** **if** V **is a polynomial of degree** ≥ 3

- **The normalization constant (partition function)**

$$Z_n = \int e^{-\frac{n}{t_0}\operatorname{Tr}(MM^* - V(M) - \overline{V}(M^*))} dM = +\infty.$$

is divergent.

Elbau-Felder approach

- **Elbau and Felder use a cut-off.**
- **They restrict to matrices with eigenvalues in a well-chosen bounded domain D.**
- **Then the induced probability measure on eigenvalues is a determinantal point process on D.**
- **Eigenvalues fill out a domain Ω that evolves according to Laplacian growth provided t_0 is small enough.**

Elbau-Felder (2005)

Orthogonal polynomials

- **Average characteristic polynomial**

$$P_n(z) = \mathbb{E}\left[zI_n - M\right]$$

in the cut-off model is an orthogonal polynomial for scalar product

$$\langle f, g \rangle = \iint_D f(z)\overline{g(z)}e^{-\frac{n}{t_0}(|z|^2 - V(z) - \overline{V(z)})} dA(z)$$

Elbau (ETH thesis, arXiv 2007)

- **Orthogonality does not make sense if $D = \mathbb{C}$, since integrals would diverge if f and g are polynomials**

Zeros of OPs

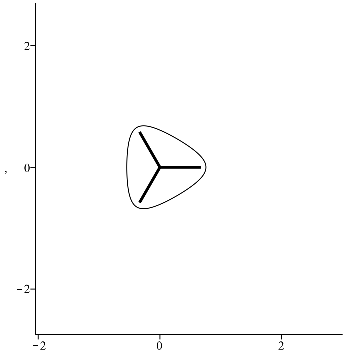
- **Conjecture: The zeros of P_n do not fill out the domain Ω as $n \to \infty$, but instead accumulate along a contour Σ_1.**
- **In the cubic case**

$$V(z) = \frac{t_3}{3} z^3, \qquad t_3 > 0,$$

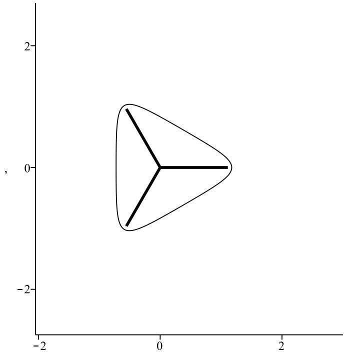
the contour is a three-star

$$\Sigma_1 = [0, x^*] \cup [0, e^{2\pi i/3} x^*] \cup [0, e^{-2\pi i/3} x^*].$$

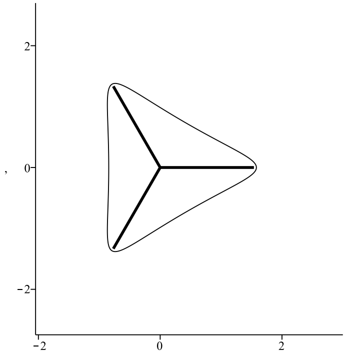
Cubic case



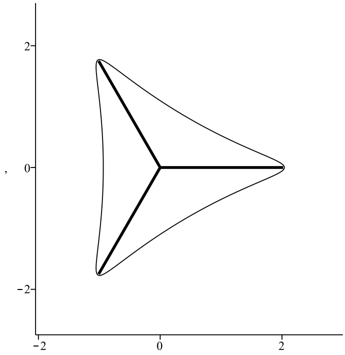
Cubic case



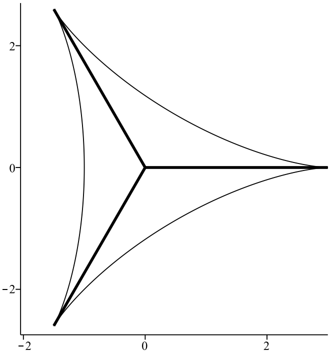
Cubic case



Cubic case



Cubic case



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Recurrence relation

- **OPs in the cut-off model satisfy a recurrence relation**
- **If** $\deg V = r + 1$ **then**

$$zP_n(z) = P_{n+1}(z) + a_{n,0}P_n(z) + \cdots + a_{n,r}P_{n-r}(z) \\ + \text{“remainder term”}$$

- **Remainder term comes from boundary integrals that are due to the cut-off.**
- **Remainder term is exponentially small for** $t_0 > 0$ **sufficiently small.**

Elbau (ETH thesis, arXiv 2007)

5. Different approach

- **Scalar product**

$$\langle f, g\rangle = \iint_D f(z)\overline{g(z)} e^{-\frac{n}{t_0}(|z|^2 - V(z) - \overline{V(z)})} dA(z)$$

satisfies (due to Green's theorem)

$$\begin{aligned} n\langle zf, g\rangle = t_0\langle f, g'\rangle + n\langle f, V'g\rangle \\ - \frac{t_0}{2i}\oint_{\partial D} f(z)\overline{g(z)} e^{-\frac{n}{t_0}\left(|z|^2 - V(z) - \overline{V(z)}\right)} dz. \end{aligned}$$

- **Drop the boundary term.**

Hermitian form

- **We consider an a priori abstract sesquilinear form on the space of polynomials satisfying**

$$n\langle zf, g\rangle = t_0\langle f, g'\rangle + n\langle f, V'g\rangle$$

- **We also want to keep the Hermitian form condition**

$$\langle g, f\rangle = \overline{\langle f, g\rangle}.$$

- **What can we say about the orthogonal polynomials (OPs)**

$$\langle P_k, z^j\rangle = 0, \qquad j = 0, 1, \ldots, n-1$$

for such an Hermitian form ?

Short recurrence

Lemma

If $\deg V = r+1$ **then OPs for an Hermitian form satisfying**

$$n\langle zf, g\rangle = t_0\langle f, g'\rangle + n\langle f, V'g\rangle$$

satisfy an $r+2$**-term recurrence relation**

Proof: Suppose $$zP_k(z) = P_{k+1}(z) + \sum_{j=0}^{k} a_{k,j} P_{k-j}(z)$$

- **Then** $a_{k,j} = \dfrac{\langle zP_k, P_{k-j}\rangle}{\langle P_{k-j}, P_{k-j}\rangle}$
- $n\langle zP_k, P_{k-j}\rangle = t_0\langle P_k, P'_{k-j}\rangle + n\langle P_k, V'P_{k-j}\rangle$
- **First term** $\langle P_k, P'_{k-j}\rangle = 0$
- **Second term** $\langle P_k, V'P_{k-j}\rangle = 0$ **if** $j > r$.

Short recurrence

Lemma

If $\deg V = r + 1$ **then OPs for an Hermitian form satisfying**

$$n\langle zf, g\rangle = t_0\langle f, g'\rangle + n\langle f, V'g\rangle$$

satisfy an $r + 2$**-term recurrence relation**

Is there a multiple orthogonality?

Double integral representations

Theorem (Bleher-K, arXiv 2011)

(a) **The real vector space of Hermitian forms satisfying**

$$n\langle zf, g\rangle = t_0\langle f, g'\rangle + n\langle f, V'g\rangle$$

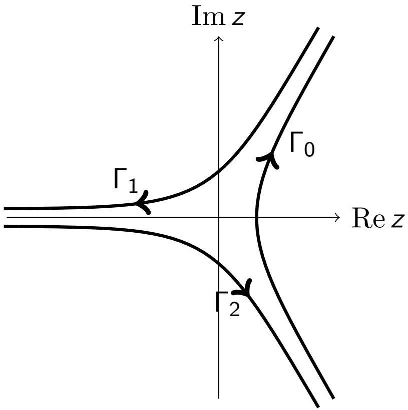
is r^2 dimensional, where $r = \deg V - 1$.

(b) **Any such Hermitian form is of the form**

$$\sum_{j,k=0}^{r} C_{j,k} \int_{\Gamma_j} dz \int_{\overline{\Gamma}_k} ds\, f(z)\overline{g}(s) e^{-\frac{n}{t_0}\left(zs - V(z) - \overline{V}(s)\right)}$$

- $(C_{j,k})_{j,k=0,\ldots,r}$ **is a Hermitian matrix with zero row and column sums,**
- $\Gamma_0, \ldots, \Gamma_r$ **is a system of unbounded contours along which the integrals converge.**

Contours Γ_j for cubic potential



- **Contours Γ_0, Γ_1, Γ_2 for $V(z) = \frac{t_3}{3} z^3$ with $t_3 > 0$, extending to infinity at asymptotic angles $\pm\pi/3$ and π.**

Hermitian form

- **The Hermitian form**

$$\sum_{j,k=0}^{r} C_{j,k} \int_{\Gamma_j} dz \int_{\overline{\Gamma}_k} ds\, f(z)\overline{g}(s) e^{-\frac{n}{t_0}(zs - V(z) - \overline{V}(s))}$$

is similar to the bilinear form for the biorthogonal polynomials in the two-matrix model.

- **The integrals for the biorthogonal polynomials are over the real line, instead of over Γ_j and $\overline{\Gamma}_k$.**

Mehta (1994), Eynard-Mehta (1998)
Ercolani-McLaughlin (2001)
Bertola-Eynard-Harnad (2002, 2003)

Multiple orthogonal polynomials

- **The biorthogonal polynomials are multiple orthogonal polynomials in the case of polynomial potentials.**
  K-McLaughlin (2005)
- **Same argument carries over to orthogonal polynomials for the Hermitian forms. They are multiple orthogonal polynomials with r weights.**
- **Weights are on**
  $$\Gamma = \bigcup_{j=0}^{r} \Gamma_j$$
  instead of on the real line.

MOP in cubic case

- **For** $V(z) = \frac{t_3}{3} z^3$ **the two weights are**

$$\begin{cases} w_0(z) = e^{\frac{nt_3}{3t_0} z^3} \sum_{k=0}^{2} C_{j,k} \int_{\overline{\Gamma}_k} e^{-\frac{n}{t_0}\left(zs - \frac{t_3}{3} s^3\right)} ds \\ w_1(z) = e^{\frac{nt_3}{3t_0} z^3} \sum_{k=0}^{2} C_{j,k} \int_{\overline{\Gamma}_k} s e^{-\frac{n}{t_0}\left(zs - \frac{t_3}{3} s^3\right)} ds \end{cases} \qquad z \in \Gamma_j,$$

- **Multiple orthogonality on** $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$

$$\int_\Gamma P_n(z) z^k w_0(z) dz = 0, \quad k = 0, \dots, \lceil \tfrac{n}{2} \rceil - 1,$$

$$\int_\Gamma P_n(z) z^k w_1(z) dz = 0, \quad k = 0, \dots, \lfloor \tfrac{n}{2} \rfloor - 1,$$

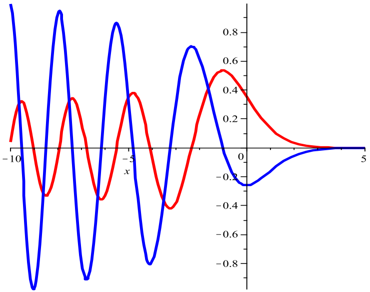
Airy functions

- **Weight w_0 is expressed in terms of the Airy function**

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\int_{\Gamma_0} e^{\frac{1}{3}s^3 - zs}\,ds$$

and weight w_1 in terms of the derivative

$$\mathrm{Ai}'(z) = -\frac{1}{2\pi i}\int_{\Gamma_0} s e^{\frac{1}{3}s^3 - zs}\,ds$$



Airy functions

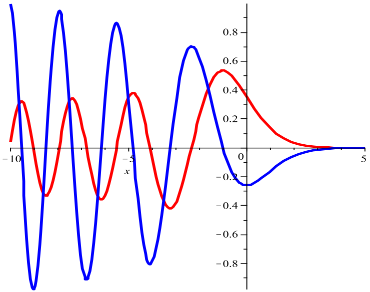
- **The Airy function** Ai(x) **is the solution of the Airy differential equation**

$$y''(x) = xy(x)$$

that satisfies

$$\mathrm{Ai}(x) = \frac{1}{2\sqrt{\pi}x^{1/4}} e^{-\frac{2}{3}x^{3/2}}(1 + O(1/x))$$

as $x \to +\infty$**.**



- **We want to choose Hermitian matrix $(C_{j,k})$ in such a way that we can find the large n asymptotics of the MOP P_n for the n-dependent weights**

$$\begin{cases} w_0(z) = e^{\frac{nt_3}{3t_0}z^3} \sum_{k=0}^{2} C_{j,k} \int_{\overline{\Gamma}_k} e^{-\frac{n}{t_0}\left(zs - \frac{t_3}{3}s^3\right)} ds \\ w_1(z) = e^{\frac{nt_3}{3t_0}z^3} \sum_{k=0}^{2} C_{j,k} \int_{\overline{\Gamma}_k} s e^{-\frac{n}{t_0}\left(zs - \frac{t_3}{3}s^3\right)} ds \end{cases} \qquad z \in \Gamma_j,$$

Q1: **Can we find the limiting behavior of zeros of $P_{n,n}$ as $n \to \infty$?**

Q2: **Can we find the connection with Laplacian growth ?**

Q3: **What happens in the critical case ?**

Existence of OP

Theorem (Bleher-K, arXiv 2011)

With the choice

$$C = (C_{j,k}) = \frac{1}{2\pi i}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

the following hold. Assume $0 < t_0 < t_{0,crit} = \frac{1}{8t_3^2}$

(a) **Then the orthogonal polynomials** P_n **for the Hermitian form exist if** n **is sufficiently large.**

(b) **The zeros of** P_n **accumulate as** $n \to \infty$ **on the set**

$$\Sigma_1 = [0, x^*] \cup [0, \omega x^*] \cup [0, \omega^2 x^*], \qquad \omega = e^{2\pi i/3},$$

$$x^* = \frac{3}{4t_3}\left(1 - \sqrt{1 - 8t_0 t_3^2}\right)^{2/3}$$

Theorem to be continued...

Main tool: Riemann-Hilbert problem

- **MOPs with two weight functions have a Riemann-Hilbert problem of size 3×3**

  (1) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{3\times 3}$ **is analytic,**

  (2) $Y_+ = Y_- \begin{pmatrix} 1 & w_0 & w_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **on $\mathbb{R}$,**

  (3) $Y(z) = (I_3 + O(1/z)) \begin{pmatrix} z^{n_1+n_2} & 0 & 0 \\ 0 & z^{-n_1} & 0 \\ 0 & 0 & z^{-n_2} \end{pmatrix}$ **as $z \to \infty$.**

  Van Assche-Geronimo-K (2001)

- **RH problem has a unique solution if and only if the MOP P_{n_1,n_2} uniquely exists and in that case**

$$Y_{11}(z) = P_{n_1,n_2}(z)$$

- **MOPs with r weight functions have a RH problem of size $(r+1) \times (r+1)$.**

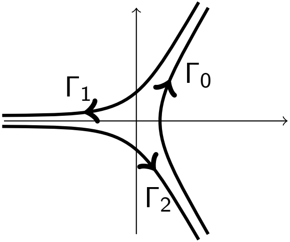
- **There is a** **RH problem** **of size** 3×3 **with jumps on** Γ **that characterizes the MOPs**

(1) $Y : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{3\times 3}$ **is analytic,**

(2) $Y_+ = Y_- \begin{pmatrix} 1 & w_0 & w_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **on** Γ,

(3) $Y(z) = (I_3 + O(1/z)) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} & 0 \\ 0 & 0 & z^{-n/2} \end{pmatrix}$ **as** $z \to \infty$.

(assume n **is even)**

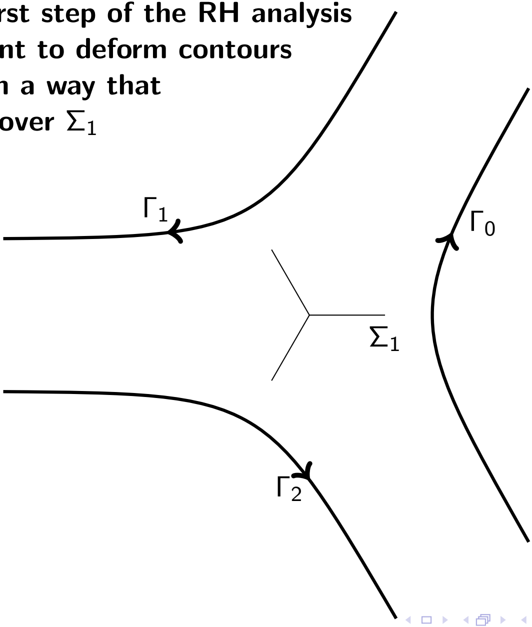


- **RH problem is ideal tool for** **asymptotic analysis...**

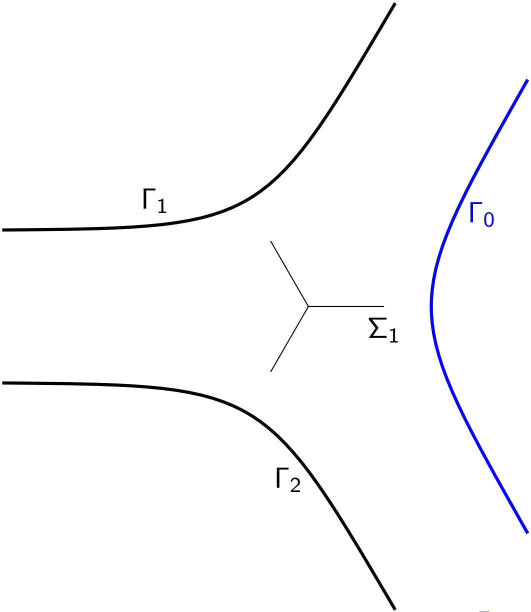
Deift-Zhou (1993)

Why this choice for C ?

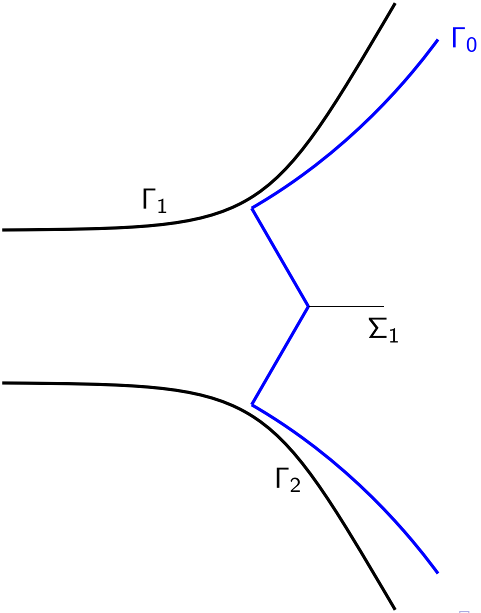
- **In a first step of the RH analysis we want to deform contours in such a way that they cover Σ_1**



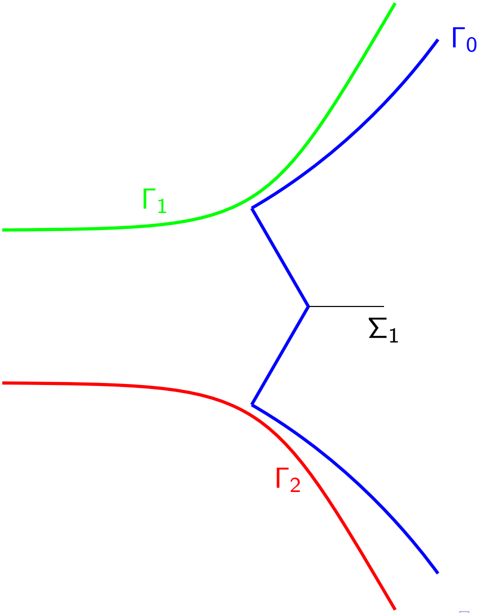
Deformation of contours



Deformation of contours

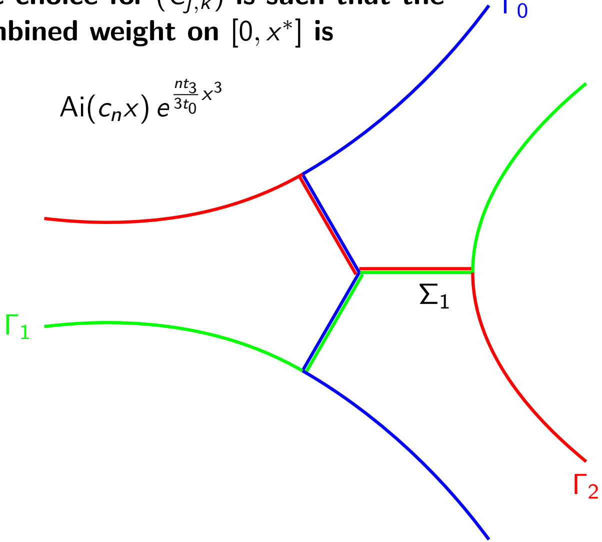


Deformation of contours



- **The choice for $(C_{j,k})$ is such that the combined weight on $[0, x^*]$ is**

$$\mathrm{Ai}(c_n x)\, e^{\frac{nt_3}{3t_0}x^3}$$



Multiple orthogonality with Airy weights

- **On Σ_1 the new weights are**

$$w_{0,n}(z) = \omega^{2j}\,\mathrm{Ai}(c_n|z|)\,e^{\frac{nt_3}{3t_0}z^3}, \quad z \in [0, \omega^j x^*],\; j = 0, 1, 2,$$
$$w_{1,n}(z) = \omega^{j}\,\mathrm{Ai}'(c_n|z|)\,e^{\frac{nt_3}{3t_0}z^3}, \quad c_n = \frac{n^{2/3}}{t_0^{2/3}t_3^{1/3}}.$$

- **Large n behavior of the two weights for $z \in \Sigma_1 \setminus \{0\}$,**

$$w_{k,n}(z) \approx \exp(-nQ(z)),$$

$$Q(z) = \frac{1}{t_0}\left(\frac{2}{3\sqrt{t_3}}|z|^{3/2} - \frac{t_3}{3}z^3\right).$$

- **Next step is the characterization of the limiting zero distribution of the polynomials in terms of Q.**

Theorem (continued)

(c) **There is a limiting zero distribution μ_1^* on Σ_1.**

(d) **μ_1^* is characterized by a vector equilibrium problem from logarithmic potential theory.**

(e) **The function**

$$t_3 z^2 + t_0 \int \frac{1}{z-s} d\mu_1^*(s)$$

extends to a meromorphic function on a compact three sheeted Riemann surface

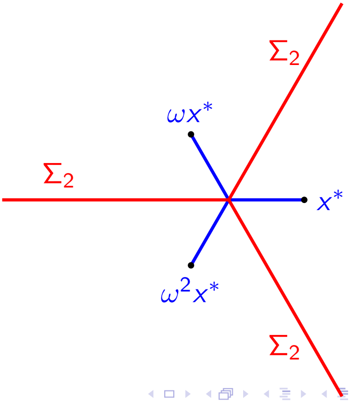
Vector equilibrium problem

Minimize the energy functional

$$\iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_1(y) - \iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_2(y) + \iint \log \frac{1}{|x-y|} d\mu_2(x) d\mu_2(y) + \int Q d\mu_1$$

over (μ_1, μ_2) **such that**

- μ_1 **is a measure on** Σ_1 **with** $\mu_1(\Sigma_1) = 1$
- μ_2 **is a measure on** Σ_2 **with** $\mu_2(\Sigma_2) = \frac{1}{2}$



Minimizer

- **There is a unique minimizer (μ_1^*, μ_2^*) of the vector equilibrium problem.**
- μ_1^* **is the limiting distribution of the zeros of P_n, that is,**

$$\lim_{n\to\infty} \frac{1}{n} \sum_{z:P_n(z)=0} \delta_z = \mu_1^*$$

(weak* convergence of measures)

The function

$$\xi_1(z) = t_3 z^2 + t_0 \int \frac{1}{z-s} d\mu_1^*(s), \qquad z \in \mathbb{C} \setminus \Sigma_1$$

extends to a **meromorphic function** **on a compact three sheeted Riemann surface whose only poles are at infinity.**

- $\xi_1(z)$ **is one of the solutions of a cubic equation (a.k.a. the** **spectral curve****)**

$$\xi^3 - t_3 z^2 \xi^2 - \left(t_0 t_3 + \frac{1}{t_3}\right) z\xi + z^3 + A = 0$$

$$A = \frac{1+20t_0t_3^2-8t_0^2t_3^4-(1-8t_0t_3^2)^{3/2}}{32t_3^3}.$$

Laplacian growth

$$\xi_1(z) = t_3 z^2 + t_0 \int \frac{1}{z-s} d\mu_1^*(s), \qquad z \in \mathbb{C} \setminus \Sigma_1$$

Theorem (continued)

(f) **The equation** $\boxed{\xi_1(z) = \bar{z}}$ **defines a simple closed curve** $\partial\Omega$ **that is the boundary of a domain** Ω **containing** Σ_1 **in its interior.**

(g) Ω **has exterior harmonic moments** $(0, 0, t_3, 0, 0, \ldots)$ **and**

$$\text{area}(\Omega) = \pi t_0$$

8. Outlook

Many open questions

- **Where are the eigenvalues ??**
- **Critical case** $t_0 = t_{0,crit}$ **?**
- **Supercritical case** $t_0 > t_{0,crit}$ **??**
- **More general potentials** V **?**

THANK YOU