Self-similar solutions of semilinear wave equations

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Motivation

Based mainly on [Kycia].

Semilinear wave equation

$$\Box U(t,x) - U(t,x)^p = 0, \qquad \Box = \partial_{tt} - \triangle,$$

where $x \in R^n$, $n \geq 3$, p-even to preserve reflection symmetry or $U^p \rightarrow |U|^{p-1}U$.

Spherical symmetry

$$U_{tt} - U_{rr} - \frac{n-1}{r}U_r - U^p = 0,$$

where $r = |x|$.

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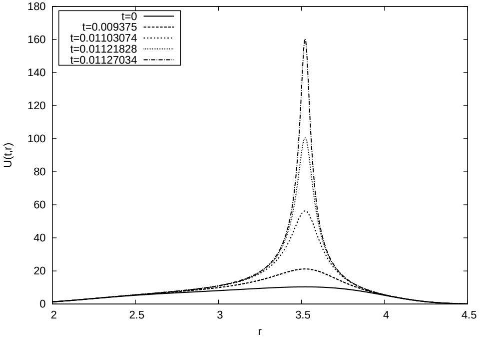
There exist 'smooth' initial data that develop singularity when $t \to T < \infty$.



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- Common behavior for many nonlinear PDEs, see [Eggers, Fontelos].
- Blowup dynamics is governed by self-similar solutions.

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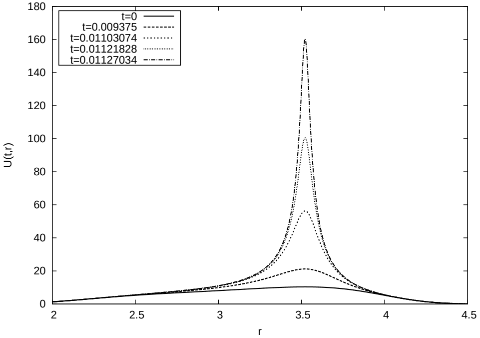

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Self-similar solutions

$$U(t,r) = \frac{u(\rho)}{(T-t)^{\alpha}}, \qquad \rho = \frac{r}{T-t}, \qquad \alpha = \frac{2}{p-1}(>0).$$

Self-similar solution

1. $U_0(t) = \frac{b_0}{(T-t)^{\alpha}}$
2. $U_\infty(r) = b_\infty r^{-\alpha}$

Self-similar profile

1. $u_0 = b_0$
2. $u_\infty(\rho) = b_\infty \rho^{-\alpha}$

1. $b_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}$
2. $b_\infty = \left(\frac{2(p(n-2)-n)}{(p-1)^2}\right)^{\frac{1}{p-1}}$

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ODE for self-similar profiles

$$(1-\rho^2)u'' + \left(\frac{n-1}{\rho} - \frac{2(p+1)}{p-1}\rho\right)u' - \frac{2(p+1)}{(p-1)^2}u + u^p = 0,$$

where $' = \frac{d}{d\rho}$.

- Fixed singularities at 0, ± 1, ∞.
- Question: Is there a global (on $[0;1]$) analytic solution ?
- Method of attack [Bizoń, Maison, Wasserman]:
  - Construct local analytic solution at 0.
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Local analytic solution at $\rho = 0$

- On substituting the formal power series $u(\rho) = \sum_{l=0}^{\infty} a_l \rho^l$
- and using Cauchy product

$$\left(\sum_{l=0}^{\infty} a_l (x - x_0)^l\right)^p = \sum_{l=0}^{\infty} c_l (x - x_0)^l,$$

$$c_0 = a_0^p, \qquad c_m = \frac{1}{m a_0} \sum_{l=1}^{m} (lp - m + l) a_l c_{m-l},$$

that simplifies nonlinear term u^p

- we obtain unique recurrence for $\{a_l\}_{l=1}^{\infty}$
- and the solution

$$u(\rho) = c + \frac{1}{n}\left[c\frac{p+1}{(p-1)^2} - \frac{1}{2}c^p\right]\rho^2 + O(\rho^4),$$

where c is initial data at $\rho = 0$.

Local analytic solution at $\rho = 0$

Question: Is the formal solution a solution ?

Proposition [Breitenlohner, Forgács, Maison]

Consider a system of differential equations for $i + j$ functions $u = (u_1, \ldots, u_i)$ and $v = (v_1, \ldots, v_j)$,

$$t\frac{du_l}{dt} = t^{\mu_l} f_l(t, u, v), \qquad t\frac{dv_l}{dt} = -\lambda_l v_l + t^{\nu_l} g_l(t, u, v),$$

with constants $\lambda_l > 0$ and integers $\mu_l, \nu_l \geq 1$ and let U be an open subset of R^n such that the functions f and g are analytic in a neighborhood of $t = 0$, $u = c$, $v = 0$ for all $c \in U$. Then there exists an i-parameter family of solutions of that system such that

$$u_l(t) = c_l + O(t^{\mu_l}), \qquad v_l(t) = O(t^{\nu_l}),$$

where $u_l(t)$ and $v_l(t)$ are defined for $c \in U$, $|t| < t_0(c)$ and are analytic in t and c.

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The Lane-Emden equation

$$U'' + \frac{n-1}{x}U' + U^p = 0, \qquad U = U(x)$$

- Widely used in astrophysics to model star structures.
- The simplest equation with power type nonlinearity and fixed singularities at 0 and ∞.
- Formal power series solution at $x = 0$ is of the form $U(x) = 1 - \frac{1}{2n}x^2 + O(x^4)$ with $U(0) = 1$, $U'(0) = 0$.
- Formal solutions are solutions - apply the Proposition [Breitenlohner, Forgács, Maison]:

$$\begin{cases} U' = V \\ xV' = -(n-1)V - xU^p, \end{cases}$$

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Movable singularities decrease the radius of convergence. They can be determined numerically - scanning complex plane [Hunter].

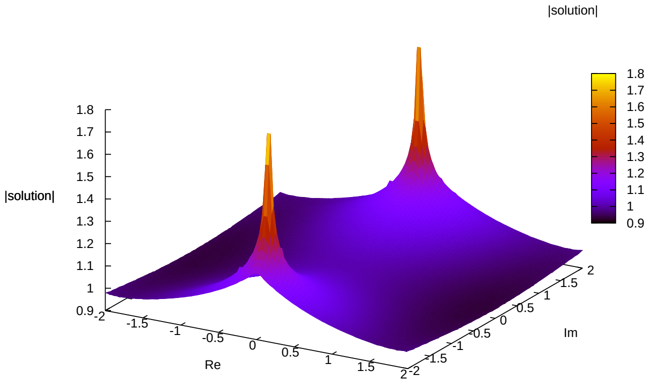

Figure: The Lane-Emden solution for $n = 3$, $p = 5$.

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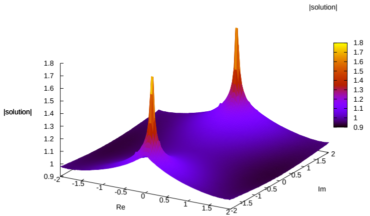

Figure: The Lane-Emden solution for $n = 3$, $p = 5$.

p	movable singularities
1	∞
2	$\pm 3.964581i$
3	$\pm 2.574840i$
4	$\pm 2.034896i$
5	$\pm 1.732051i$

Table: Numerical estimates of the location of movable singularities for $n = 3$ [Hunter].

Local analytic solution at $\rho = 0$

Question: How large is the circle of convergence - where are the closest movable singularities ?

- Rescaling local solution at $\rho = 0$ using $\rho = \frac{x}{c^{\frac{p-1}{2}}}$ and $U = u/c$
- we get

$$u(\rho) = cU\left(\frac{x}{c^{\frac{p-1}{2}}}\right) = c\left[(1 - \frac{1}{2n}x^2 + O(x^4)) + o\left(\frac{1}{c}\right)\right].$$

- The equation transforms as

$$U'' + \frac{n-1}{x}U' + U^p = \frac{1}{c^{p-1}}\left(x^2U'' + (2\alpha+2)xU' + \alpha(\alpha+1)U\right).$$

- For large c asymptotic form of local solution at $\rho = 0$ obey the Lane-Emden equation.
- Connection between location of movable singularities (the radius of convergence):

$$R_W \approx \frac{R_{LE}}{c^{\frac{p-1}{2}}}.$$

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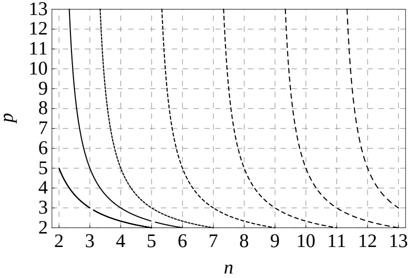
Local analytic solution at $\rho = 1$

ODE for self-similar profile at $\rho = 1$

$$y(2-y)u''(y) - \left[\frac{n-1}{1-y} - \frac{2(p+1)}{p-1}(1-y)\right]u'(y) - \frac{2(p+1)}{(p-1)^2}u(y) + u^p(y) = 0,$$

where $y = 1 - \rho$ and $' = \frac{d}{dy}$.

There exist the parameter $k = \frac{(n-1)p-n-3}{2(p-1)}$ which determines the form of analytic solution.



Local analytic solution at $\rho = 1$

Formal power series solutions $u(y) = \sum_{l=0}^{\infty} a_l y^l$:

- For noninteger k we get standard formal analytic solution

$$u(y) = b + \frac{2(p+1)b - (p-1)^2 b^p}{2(1-k)(p-1)^2} y + O(y^2),$$

  where $b = u(y=0)$ is a free parameter.
- For integer k we have:
  - First k recurrence equations for $\{a_0, \ldots, a_{k-1}\}$ give nonlinear algebraic system of equations...
  - ...which has (among others) the solution $a_0 = b_\infty, \ldots, a_{k-1} = (-1)^k \frac{u_\infty^{(k)}(\rho)|_{\rho=1}}{k!}$.
  - The free parameter b appears at y^k and then formal series proceed in usual way

$$u(y) = a_0 + \ldots + a_{k-1} y^{k-1} + b y^k + a_{k+1}(b, a_0, \ldots, a_{k-1}) y^{k+1} + \ldots$$

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Local analytic solution at $\rho = 1$

Movable singularity moves from $\rho = \infty$ to $\rho = 1$ along the real axis when $|b| > b_0$ increases.

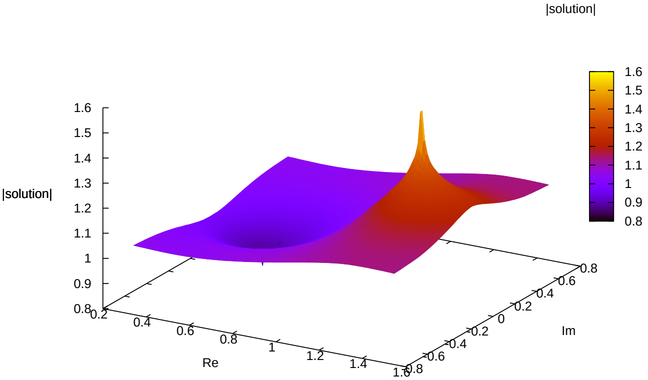

Figure: Local solution at $\rho = 1$ for $n = 3$, $p = 7$ and $b = 1.1 > b_0 \approx 0.8736$.

Global existence

Matching of two local asymptotics at some $\rho_0 \in (0;1)$ (details in [Bizoń, Maison, Wasserman],[Kycia]):

- Use of Lyapunov functions - local solutions can develop singularities only at endpoints of $[0;1]$.
- C^1 matching at ρ_0 (analytic continuation):
  - C_0 - curve of integration of local analytic solution at 0 to ρ_0 and parameterized by initial data c.
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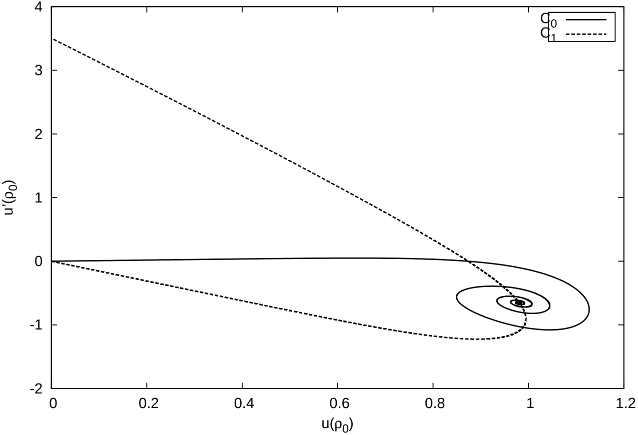

Figure: $n = 3,\ p = 7,\ \rho_0 = 0.5$

Global existence

- The curves have a countable number of intersections $\Rightarrow$ countable family of global analytic solutions.
- The family can be enumerated by the pairs of initial data at both endpoints $\{c_l, b_l\}_{l=0}^{\infty}$.
- For integer k only solutions with asymptotics at $\rho = 1$ for which $u(1) = b_\infty$ $(a_0 = b_\infty, \ldots, a_{k-1} = (-1)^k \frac{u_\infty^{(k)}(\rho)|_{\rho=1}}{k!})$ gives countable family of global solutions; the others give only one global solution for each solution. These constraints lead to simple numerical methods to study global solutions [Kycia].

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Thank You for Your Attention

Any suggestions and comments would be extremely useful.

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Classification

Equation's symmetry

$$U_\lambda(t,r) = \lambda^\alpha U\left(\frac{t}{\lambda}, \frac{r}{\lambda}\right), \qquad \alpha = \frac{2}{p-1}.$$

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