F ormal and

A symptotic (not necessarily analytic)

S olutions of some classes of linear

D ifference

E quations (actually recurrence relations)

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The equations we want to consider have the form

$$y(n+1) + a(n)y(n) + b(n)y(n-1)=0,$$

which are called either

3-term linear recurrence relations or

second order linear difference equations.

We will treat them as 2-dimensional, first order linear vector/matrix systems of the form

$$y(n+1)=A(n)y(n), \quad A(n)= \begin{bmatrix} 0 & 1 \\ -b(n) & -a(n) \end{bmatrix}$$

where we always assume A is invertible, so that a fundamental solution exists and can be given by

$$Y(n)= A(n-1)A(n-2)\ldots A(1) .$$

We ask for the asymptotic behavior of Y(n) as $n \to \infty$.

Such equations are especially studied due to

1. **applications to orthogonal polynomials, e. g.**

$$(n+1)p_{n+1}(x)-(2n+1)xp_n(x)+np_{n-1}(x)=0$$

$$h_{n+1}(x)-2xh_n(x)+2nh_{n-1}(x)=0$$

Our emphasis will be on classes of asymptotically equivalent equations instead of special examples.

2. **second order equations or 2-d systems are the simplest ones for which there are not generally closed form solutions. Close analogies exist with second order or 2-d systems of linear differential equations. The terms Liouville/Green or WKB asymptotics are associated with the kinds of problems we consider. When "realistic" error terms are also included, they are also often called Liouville/Green/Olver formulas.**

3. **Applications to spectral problems of Jacobi operators.**

Our own interest is motivated by

4. **a book we are writing on asymptotic methods for solving differential and difference equations; we are looking for some nice examples for applying a class of general theorems leading to what we call "asymptotic factorization" of a fundamental matrix.**

Many ad hoc techniques have been applied to study the asymptotic behavior of classes of orthogonal polynomials.

Some of these include:

1. **Riccati methods. If y(n)/y(n-1)=u(n), the equation is equivalent to**

   $$u(n+1)u(n) + a(n)u(n) + b(n) = 0,$$ **which can**

   ***sometimes* be approximated by**

   $$u^2(n) + a(n)u(n) + b(n) = 0.$$

2. **Continued fraction expansions. These can lead to some interesting types of formal solutions.**

3. **Expanding difference operators in terms of differential operators, using the difference calculus and considering difference equations as "differential equations of infinite order". See Dingle, etal.**

While these techniques have lead to some useful results, our goal is to see whether or not more general asymptotic methods can be applied to yield at least equivalent (or possibly better) results. If, not then maybe we can learn some new procedure that would improve the general approach. In doing so, we also hope to understand the various assumptions from a more general perspective.

Asymptotic Factorization:

$$Y(n)=P(n)[I+Q(n)][I+E(n)]\prod_{1}^{n-1}\Lambda(k),$$

where P(n) is a normalizing or preliminary transformation,

I+Q is a so-called conditioning transformation,

$\Lambda(n)$ is a diagonal matrix, and E =o(1) and can be

"realistically estimated" in the sense of Olver.

The role of the transformation

$$y(n)=P(n)[I+Q(n)]z(n)$$

is to take the system y(n+1)=A(n)y(n) into a normalized form

$$z(n+1)=[\Lambda(n)+R(n)]z(n),$$

so that a discrete analogue of Levinson's Fundamental Theorem on asymptotic integration can be applied to yield

$$Z(n)= [I+E(n)]\prod_{1}^{n-1}\Lambda(k).$$

Compare with formal and asymptotic solutions of meromorphic linear differential systems.

A discrete analog of Levinson's Asymptotic Integration Theorem.

Consider $z(n+1) = [\Lambda(n) + R(n)]z(n)$,

where $\Lambda(n) = \text{diag}\{\lambda_1(n), \ldots, \lambda_d(n)\}$ is invertible for all n and

satisfies ordinary dichotomy conditions involving partial

quotients $$\prod_m^n \left| \frac{\lambda_j(k)}{\lambda_i(k)} \right|.$$

If also $R(n)/\lambda_i(n)$ is in l^1, then there exists a solution vector of the form

$$\mathbf{z_i(n)=[e_i + o(1)]} \prod_1^{n-1} \lambda_i(k) \ ,$$

where the o(1) terms can be estimated in terms of R and Λ.

Ref: Rappoport, Evgrafov, Gel'fond/Kubenskaya, Benzaid/Lutz, Bodine/Lutz.

In Benzaid/Lutz, two further extensions were given which we will also use. They correspond to special choices of $[I+Q(n)]$ or conditioning transformations.

These results reduce the problem from analyzing the asymptotic behavior of products of two-dimensional matrices to that of products of scalars, which can often be treated using standard techniques or even sometimes having explicit representations using elementary functions.

Some applications to classes of 3 term recursions.

1.(Spigler/Vianello) [1992])
2.(Mate/Nevai/Totik [1985])
3.(Mate/Nevai [1988])
4.(Geronimo/Smith [1992])
5.(Stepin/Titov [2006])
6.(van Assche/Geronimo [1988])

We emphasize that we are only concerned here with deriving some basic asymptotic formulas, not with more specialized results deduced from them. Our main point will be that the general approach can yield at least as good or better results in most cases.

The main focus now is to find suitable preliminary or normalizing transformations P(n), followed by (if necessary) conditioning transformations (I+Q(n)). Then the discrete analog of Levinson's Theorem and estimation of operators yield the asymptotic results.

A. l^1 perturbations of asymptotically constant equations.

Let $y(n+1) = [A + R(n)]y(n)$, where A is

invertible and diagonalizable and R is in l^1. Then $P^{-1}AP = \Lambda$ satisfies Levinson's dichotomy conditions and there exists a fundamental matrix satisfying

$$Y(n) = P[I + E(n)]\,\mathrm{diag}\{\lambda_1^n, \ldots, \lambda_d^n\}.$$

Example 1: (see Spigler/Vianello [1992])

$$\Delta(\Delta y(n)) + (\alpha + g(n))y(n) = 0,$$

where α is unequal to **-1**. This is equivalent to the

above system with $A = \begin{bmatrix} 0 & 1 \\ -(1+\alpha) & 2 \end{bmatrix}$ and $R(n) = \begin{bmatrix} 0 & 0 \\ -g(n) & 0 \end{bmatrix}$.

If α is also not zero and g is in l^1, then the analog of Levinson's Theorem immediately applies. If $\alpha = 0$, then A has equal roots, but is not diagonalizable and stronger decay conditions such as $ng(n)$ or $n^2 g(n)$ in l^1 lead to asymptotic formulas.

Error estimates for o(1) terms can be given in terms of g and the eigenvalues of A.

Example 2: (Spigler/Vianello)[1994])

$$y(n+2) + a(n)y(n+1) + b(n)y(n) = 0,$$

where is $b(n)/a(n)a(n-1) = L + h(n)$, where h is in l^1 and $L \neq 0, 1/4$. Then

$$y(n) = \mathrm{diag}\{1, a(n-1)\} \prod_{1}^{n-2} a(k)\ z(n)$$

transforms the system into $z(n+1) = [A + R(n)]z(n)$,

where $A = \begin{bmatrix} 0 & 1 \\ L & -1 \end{bmatrix}$ and $R(n) = \begin{bmatrix} 0 & 0 \\ h(n) & 0 \end{bmatrix}$ is in l^1.

So a fundamental matrix $Y(n) = \begin{bmatrix} y_1(n-1) & y_2(n-1) \\ y_1(n) & y_2(n) \end{bmatrix}$

can be represented as

$$Y(n) = \mathrm{diag}\{1, a(n-1)\} \prod_{1}^{n-2} a(k)\ [P + E(n)]\mathrm{diag}\{\lambda_1^{n-1}, \lambda_2^{n-1}\},$$

with $\lambda_{1,2} = \left(-1 \pm \sqrt{1-4L}\right)/2)$ and $P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}$.

If $L = 1/4$, then stronger decay conditions on h lead to some corresponding asymptotic representations.

B. Asymptotically constant systems with regularly-varying perturbations;

$$y(n+1) = [A + V(n)]y(n),$$

where A has distinct, non-zero eigenvalues, and $V = o(1)$.

If $\Lambda(n)$, the diagonal matrix of eigenvalues of $A+V(n)$ satisfies Levinson's dichotomy conditions and if also $V(n+1)-V(n)$ is in l^1, then

$$Y(n) = P(n)[I+E(n)]\prod_{1}^{n-1}\Lambda(k)$$

where $P(n)$ diagonalizes $A+V(n)$.

A function V satisfying the condition $V(n+1)-V(n)$ in l^1 is said to be *regularly varying*. Examples include such functions as $1/n$, $1/\log n$, etc.

Application 1. (Mate/Nevai/Totik[1985]); Let

$$xp(n,x)=a(n+1)p(n+1,x)+b(n)p(n,x)+a(n)p(n-1,x)=0,$$

where x is a complex parameter with values in K, a compact set avoiding {-1,+1} and a(n)>0 for all n.

Assume that a(n) = ½ + o(1) as n tends to infinity and b(n) = o(1) as well. Finally assume that a(n) and b(n) are "regularly varying" in the sense that both a(n+1)-a(n) and b(n+1)-b(n) are in l^1 .

Then we have p(n+1)=[A + V(n,x)]p(n), where

$$A=\begin{bmatrix} 0 & 1 \\ -1 & 2x \end{bmatrix} \quad \text{and}$$

$$V(n,x)=\begin{bmatrix} 0 & 0 \\ 1-a(n)/a(n+1) & (x-b(n))/a(n+1)-2x \end{bmatrix}.$$

A is diagonalizable provided x is not in {-1, +1} and has eigenvalues

$$\rho(x)=x+\sqrt{x^2-1} \text{ and } 1/\rho(x).$$

Also, it is easy to check that V is regularly varying.

We take the positive branch of the root for x not in [-1,+1]. Then A+V(n,x) has eigenvalues

$$\lambda_1(n,x) = \sqrt{\frac{a(n)}{a(n+1)}}\frac{1}{\rho\left(\frac{x-b(n)}{2\sqrt{a(n)a(n+1)}}\right)} = 1/\rho(x) + o(1)$$

and $\lambda_2(n,x) = \sqrt{\frac{a(n)}{a(n+1)}}\rho\left(\frac{x-b(n)}{2\sqrt{a(n)a(n+1)}}\right) = \rho(x) + o(1)$.

Then provided the eigenvalues satisfy dichotomy conditions, the discrete analogue of Levinson's theorem applies and there exists a fundamental matrix for the original system of the form

$$\begin{bmatrix} 1 & 1 \\ 1/\rho(x) & \rho(x) \end{bmatrix} \begin{bmatrix} 1+\varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & 1+\varepsilon_{22} \end{bmatrix} \prod_{1}^{n-1} \begin{bmatrix} \lambda_1(k,x) & 0 \\ 0 & \lambda_2(k,x) \end{bmatrix}$$

and the ε_{ij} can be estimated. For x to lie in certain subsets of C, this yields the most of the asymptotic formulae in Mate/Nevai/Totik. If x is in a compact subset of the complex plane bounded away from [-1,1], then L even satisfies an exponential dichotomy condition. This can be used to extend those results as in the next section.

A related asymptotic factorization result: (Mate/Nevai[1988].

Let $Ey(n)=y(n+1)$ and consider the operator

$$E^2 + a(n)E + b(n),$$

where a(n) and b(n) are asymptotically constant and regularly varying in the above sense. Assume the limiting characteristic equation has roots λ_1 and λ_2 with distinct moduli and let $|\lambda_1/\lambda_2| > 1$.

Let $\lambda_1(n)$ and $\lambda_2(n)$ be corresponding roots for the perturbed equation. Then there exist sequences $\mu_1(n)$ and $\mu_2(n)$ such that the operator can be factored as

$$(E- \mu_2(n))(E-\mu_1(n),$$

where $\mu_1(n)-\lambda_1(n)$ is in l^1. This implies that

$$y(n+1)=(\lambda_1(n) + [\mu_1(n)-\lambda_1(n)])y(n)$$

has a solution $y_1(n)=[1+o(1)]\Pi\, \lambda_1(k)$.

This leads to the following natural questions:

a. What can be said about the second factor μ_2?
b. Is there a corresponding factorization with the order of the factors reversed?
c. Is it essential that the roots have distinct moduli?
d. Do such factorization results also hold for higher order equations?

The answers to these questions as well as an alternate proof of the original result can be obtained from the asymptotic factorization

$$P(I+Q(n))(I+E(n))\prod_{1}^{n-1}\begin{bmatrix}\lambda_1(k) & 0\\ 0 & \lambda_2(k)\end{bmatrix},\text{ where } E(n)$$

can be shown to satisfy $E(n+1)-E(n)$ is in l^1.

Then the functions μ_1 and μ_2 can be expressed in terms of the above functions and the results follow.

C. l^p perturbations of diagonal systems having an exponential dichotomy.

Consider $y(n+1) = [\Lambda(n) + R(n)]y(n)$, where

$\Lambda(n) = \text{diag}\{\lambda_1(n), \ldots, \lambda_d(n)\}$ satisfies an <u>exponential dichotomy condition</u> and for some fixed index i,

$R(n)/\lambda_i(n)$ is in l^p for p at most 2.

Then there exists a solution $y_i(n)$ satisfying

$$y_i(n) = (e_i + o(1))\prod_{1}^{n-1}(\lambda_i(k) + r_{ii}(k)).$$

For finite $p > 2$, analogous results also hold with further modifications to the eigenvalues using an iteration of normalizing and conditioning transformations.

Application: (Geronimo/Smith[1992]):

Let $\quad d(n+1)y(n+1)-q(n,x)y(n)+y(n-1)=0,$

where $d(n)=1-\delta(n)$ and $\delta(n)=o(1)$ and q is a complex sequence depending upon a complex parameter x which will range over some compact set K. They considered a corresponding Riccati equation

$$d(n+1)u(n+1)u(n)-q(n,x)u(n)+1=0$$

and the quadratic limiting equation

$$u^2(n)-q(n,x)u(n)+1=0$$

with solutions $u_0(n,x)$, $v_0(n,x)$ as first approximations to solutions of the Riccati equation. Some other ad hoc considerations lead them to two further approximations, called $u_1(n,x)$, $u_2(n,x)$, and $v_1(n,x)$, $v_2(n,x)$.

Their main results involve asymptotic formulas for solutions as products of $u_2(n,x)$ and $v_2(n,x)$ with error estimates for the remainders under the following assumptions:

$q(n,x)$ assumes values in K, a subset of the complex plane which is bounded away from $[-2,+2]$.

$\delta(n)$ and $\Delta(q(n,x))$ in l^2 and $\Delta(\delta(n))$ and $\Delta^2(q(n,x))$ in l^1.

Instead of using the Riccati approach, we consider

$$y(n+1)=\begin{bmatrix} 0 & 1 \\ -1/(1-\delta(n+1)) & q(n)/(1-\delta(n+1)) \end{bmatrix} y(n),$$

which we decompose as

$$y(n+1)=[A(n,x)+V(n,x)]y(n), \text{ where}$$

$$A(n,x)=\begin{bmatrix} 0 & 1 \\ -1 & q(n,x) \end{bmatrix},\quad V(n,x)=\begin{bmatrix} 0 & 0 \\ \dfrac{-\delta(n+1)}{1-\delta(n+1)} & \dfrac{q(n)\delta(n+1)}{1-\delta(n+1)} \end{bmatrix}.$$

$A(n,x) \cong diag\{\lambda_1(n,x), \lambda_2(n,x)\}$, where

$$\lambda_{1,2}(n,x)=(q(n)\pm\sqrt{q^2(n)-4})/2.$$

Then letting $y(n)=\begin{bmatrix} 1 & 1 \\ \lambda_1(n-1) & \lambda_2(n-1) \end{bmatrix} \hat{y}(n) =: T(n,x)\hat{y}(n)$

we obtain $\hat{y}(n+1)=\lfloor \Lambda(n,x)+\hat{V}(n,x) \rfloor \hat{y}(n)$, where

$$\hat{V}(n,x)=T^{-1}(n+1)A(n)[T(n)-T(n+1)]+T^{-1}(n+1)V(n)T(n),$$

$$=\Lambda(n)T^{-1}(n+1)[T(n)-T(n+1)]+T^{-1}(n+1)V(n)T(n).$$

If q(n,x) is bounded away from [-2,+2], then Λ(n,x) satisfies an exponential dichotomy condition. It can be easily checked that

$$\frac{\hat{V}(n,x)}{\lambda_1(n,x)}$$ is in l^2 provided δ(n) and Δ(q(n,x)) are in l^2.

Therefore there exists a solution vector of the form

$$\hat{y}_1(n,x) = (\vec{e}_1 + \varepsilon_1(n,x))\prod_{1}^{n-1}[\lambda_1(k,x) + diag_1\hat{V}(k,x)].$$

This leads to an asymptotic formula that is apparently different from that obtained by Geronimo/Smith. Also, we do not require their further conditions that Δ(δ(n)) and $\Delta^2 q(n,x)$ are in l^1. However, with those extra conditions, the two expressions can be shown to produce asymptotically equivalent results since the terms in the products differ by l^1 functions.

Moreover, other conditioning transformations could be employed, leading to many other results of this type.

Another result stated in G/S *strengthens* the assumptions on δ and q to just

$$\delta(n) \text{ and } \Delta(q(n,x)) \text{ in } l^1,$$

while *weakening* the assumption on q(n,x) to just lie in a region bounded away from the *points* {−2,+2}. This allows for oscillation of solutions of the unperturbed equation when the eigenvalues have equal moduli, which occurs when q(n,x) can take on values inside the so-called oscillatory interval (−2,+2).

The assumptions on δ and q imply that $\hat{V}$ is in l^1, but the weakened condition on q means that exponential dichotomy is lost and should be replaced by ordinary dichotomy for the eigenvalues of the unperturbed equation (in order to apply the analog of Levinson's theorem).

Without such an assumption on q, such as q not crossing the interval (−2,+2), the eigenvalues could fail to satisfy such an ordinary dichotomy condition. This could lead to the possibility of a change in the asymptotic behavior, even when δ and q satisfy the other conditions above.

Also in this case, $u_0(n,x)$ is already a good approximation to $\lambda_1(n,x)$, which leads to a simpler result.

D. Some "averaged" decay conditions; see (Stepin/Titov[2006]).

Consider $y(n+1) - q(n)y(n) + y(n-1) = 0$, where $q(n) > 0$.

This is a normalized version of the equation

$$w(n+1) - a(n)w(n) + b(n)w(n-1) = 0,$$

where both a(n) and b(n) are positive sequences.

Theorem 1. If $\{1/q(n)q(n+1)\}$ is in l^1, then there exist solutions satisfying

$$y_1(n) \sim \prod_1^{n-1} q(k) \text{ and } y_2(n) \sim \prod_1^{n} 1/q(k).$$

Example: $q(2n) = 1/n$, $q(2n+1) = 2^n$.

Theorem 2. If $q(n) > 2$ and $\{1/q(n-1)q^2(n)q(n+1)\}$ is in l^1, then there exist solutions satisfying

$$y_1(n) \sim \prod_1^{n-1}\left(q(k) - \frac{1}{q(k-1)}\right) \text{ and } y_2(n) \sim \prod_1^{n}\left(q(k) - \frac{1}{q(k+1)}\right)^{-1}.$$

We first (in the case of Theorem 1) treat the equation as

$$y(n+1)=\begin{bmatrix} 0 & 1 \\ -1 & q(n) \end{bmatrix} y(n), \text{ and let } y(n)=P(n)(I+Q(n))z(n),$$

with

$$P(n)=\begin{bmatrix} 1 & 0 \\ 1/q(n) & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/q(n) \end{bmatrix} \text{ and}$$

$$(I+Q(n))=\begin{bmatrix} 1 & 1/q(n)q(n-1) \\ 1/q(n)q(n+1) & 1 \end{bmatrix}.$$

This yields $z(n+1)=(\Lambda(n)+R(n))z(n)$, where

$$\Lambda(n)=\mathrm{diag}\{1/q(n), q(n+1)\}.$$

Since the assumptions imply that for n sufficiently large, $q(n+1)q(n) > 2$, then Λ satisfies exponential dichotomy conditions, and both $q(n)R(n)$ and $R(n)/q(n+1)$ are in l^1.

So the analog of Levinson's theorem applies and yields the asymptotic representations for solutions $y_1(n)$ and $y_2(n)$, also with error estimates not present in Stepin/Titov. Since exponential dichotomy conditions are satisfied, the l^1 decay condition can be relaxed to an l^2 condition with the corresponding modification by diagonal terms from R(n). Furthermore, the condition $q(n)>0$ can also be relaxed to just q(n) not zero together with $\{q(n)q(n+1)\}$ in l^1.

For Theorem 2, we let

$$y(n) = \prod_{1}^{n-2} q(k) \begin{bmatrix} 1 & 0 \\ 0 & q(n-1) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tilde{y}(n)$$

to obtain $\tilde{y}(n+1) = [\Lambda(n) + V(n)]\tilde{y}(n)$,

where $\Lambda(n) = \begin{bmatrix} 1/q(n)q(n-1) & 0 \\ 0 & 1-1/q(n)q(n-1) \end{bmatrix}$ and

$$V(n) = \begin{bmatrix} 0 & 1/q(n)q(n-1) \\ -1/q(n)q(n-1) & 0 \end{bmatrix}.$$

If Λ satisfies an exponential dichotomy condition and $\sum_{1}^{\infty} \left(\frac{1}{q(n)q(n-1)} \right)^2 < \infty$, then the result would follow, but this condition is stronger (Cauchy-Schwarz) than

$\sum_{1}^{\infty} \left(\frac{1}{q(n-1)q^2(n)q(n+1)} \right) < \infty$ assumed in Theorem 2.

So make one further I+Q(n) transformation with

$$I + Q(n) = \begin{bmatrix} 1 & a(n-1)/(1-a(n-1)) \\ a(n)/(1-a(n)) & 1 \end{bmatrix}$$

and where $a(n) = \dfrac{1}{q(n)q(n-1)}$. This leads to a system

$\hat{y}(n+1) = [\Lambda(n) + R(n)]\hat{y}(n)$ with

$$\Lambda(n) = \{a(n)/(1-a(n)), 1-a(n)\}$$

which satisfies an exponential dichotomy condition provided that $q(n) > (3+\sqrt{5})/2$.

Moreover, $R(n)/\lambda_2(n)$ in l^1 and this yields part of Theorem 2, but with a slightly weaker assumption on q(n). To obtain the asymptotic for a recessive solution we use a modified conditioning transformation, and a more restrictive condition on q(n) (although still weaker than $q(n)>2$).

Theorem 2 as stated above is actually just the first case of a more general result in Stepin/Titov having assumptions which involve products of a fixed, finite number of consecutive terms in the sequence {q(n)}. Their more general result is proven using formal continued fractions. For that purpose they require the condition $q(n)>2$ for convergence. While we believe that an asymptotic factorization is possible in more general cases, there would be some unpleasant algebraic calculations and inductions using our approach, so we don't choose to try that now.

E. A case of unbounded, regularly-varying coefficients.

Similar to the case treated by Mate/Nevai/Totik for asymptotically constant, regularly-varying coefficients, van Assche and Geronimo (1989) treated a case of

$$xp_n(x) = a(n+1)p_{n+1}(x) + b(n)p_n(x) + a(n)p_{n-1}(x)$$

where $p_{-1} = 0, p_0 = 1$ and the coefficients satisfy the following conditions:

a. There exists a positive sequence $\{\lambda_n\}$ and a constant $\alpha > 0$ for which

$$\lim n\left(\frac{\lambda_{n+1}}{\lambda_n} - 1\right) = \alpha.$$

b. $\lim a(n)/\lambda_n = a > 0$ and $\lim n(a(n+1)-a(n))/\lambda_n = a\alpha$.

c. $\lim b(n)/\lambda_n = b$ and $\lim n(b(n+1)-b(n))/\lambda_n = b\alpha$.

Then

$$\lim \frac{p_n(\lambda_n x)}{\prod_1^n z_{k,n}} = \left(\frac{(x-b)^2 - 4a^2}{x^2}\right)^{-1/4} \exp\left(\frac{b}{2}\int_0^1 \frac{ds}{\sqrt{(x-bs)^2 - 4a^2 s^2}}\right)$$

as n tends to infinity, uniformly in x when it is restricted to lie in a compact subset of the complex plane bounded away from an interval [A,B], which is the smallest interval of R containing {0} and [b-2a, b+2a].

Here $z_{k,n}$ = $z_{k,n}(x)$ is defined by

$$z_{k,n} = \frac{\lambda_n x - b(k)}{2a(k)} + \sqrt{\left(\frac{\lambda_n x - b(k)}{2a(k)}\right)^2 - 1} \ .$$

<u>This result does not, so far, appear to be obtainable using the asymptotic factorization approach.</u>

First, asymptotic factorization cannot handle initial conditions and yield explicit connection formulae. So at best we would just want to show that the above limit exists (as a function of x).

Second, the sequence $\{q_n(x)\}$, where $q_n(x)=p_n(\lambda_n,x)$ does not necessarily satisfy a linear recurrence relation. So we would instead like to have an asymptotic representation which implies that

$$\lim p_n(x)/z_{k,n}(x/\lambda_n) \text{ exists,}$$

which would imply that the above limit exists (and is a function of x). This would suggest that the system corresponding to the equation be decomposed as

$$p(n+1) = [A(n,x) + V(n,x)]p(n),$$

with $$A(n,x) = \begin{bmatrix} 0 & 1 \\ -1 & (x-b(n))/a(n) \end{bmatrix},$$

(which has eigenvalues $z_n(x/\lambda_n)$) and

$$V(n,x) = \begin{bmatrix} 0 & 0 \\ 1-a(n)/a(n+1) & (x-b(n))\left(\dfrac{1}{a(n+1)} - \dfrac{1}{a(n)}\right) \end{bmatrix}.$$

For fixed x, the eigenvalues of A(n,x) tend to the eigenvalues of the limiting matrix $\begin{bmatrix} 0 & 1 \\ -1 & -b/a \end{bmatrix}$, which are unequal provided $b \neq \pm 2a$.

While V(n,x) tends to zero as n tends to infinity (when x lies in a compact domain), V just misses being regularly varying and their "dichotomy condition" concerning x to be bounded away from [A,B] does not yield any workable dichotomy condition for Λ(n,x) in our sense.

Conclusions:

1. In most cases the asymptotic factorization method produces at least as good results as the ad hoc techniques and gives improvements in some cases. The main challenge is in finding the “right” normalizing and conditioning transformations.

2. Because the asymptotic factorization approach preserves linearity and produces a set of equivalent linear systems, it is also quite flexible. One can use a wide variety of preliminary and conditioning transformations to yield many different kinds of new analogous extensions of known results.

3.The conditions which are required by the discrete analog of Levinson’s Theorem, while not necessary, are very sharp in a generic sense. So the results which are obtained using this approach often contain assumptions on the behavior of the coefficients which are sharper and closer to being “necessary”.

4. The approach is not restricted to 3-term recurrence relations or 2-dimensional systems as are most of the ad hoc techniques.

REFERENCES:

ASYMPTOTICS FOR SECOND ORDER LINEAR RECURRENCE RELATIONS; W/K/B AND L/G/O.

1. R. Spigler and M. Vianello, Discrete and continuous Liouville-Green-Olver approximations: a unified treatment via Volterra-Stieltjes integral equations. SIAM J. Math. Anal. 25 (1994), 720-732.
2. R. Spigler and M. Vianello, A survey on the Liouville-Green (WKB) approximation for linear difference equations of the second order, Advances in difference equations (Veszprem, 1995), Gordon & Breach, 1997, 565-577.
3. R. Spigler and M. Vianello, Liouville-Green approximations for a class of linear oscillatory difference equations of the second order, J. Comput. Appl. Math. 41 (1992), 105-116.
4. J. Geronimo and D. Smith, WKB (Liouville-Green) analysis of second order difference equations and applications, J. Approx. Theory, 69 (1992), 269-301.
5. A. Mate and P. Nevai, Factorization of second order difference equations and its applications to orthogonal polynomials, Lect. Notes in Math. #1329 (1988), 158-177.
6. A. Mate, P. Nevai, and V. Totik, Asymptotics for orthogonal polynomials defined by a recurrence relation, Constr. Approx. 1 (1985), 231-248.
7. S. Stepin and V. Titov, Dichotomy of WKB-solutions of discrete Schroedinger equations, Jour. Dyn. Control Systems, 12 (2006), 135-144.
8. W. van Assche and J. Geronimo, Asymptotics for orthogonal polynomials with regularly varying recurrence coefficients, Rocky Mt. Math. J.(1989).
9. W. van Assche, Asymptotics for orthogonal polynomials and three-term recurrences, in Orthogonal Polynomials, P. Nevai, ed. (1990), 435-462.
10. Z. Benzaid and D.A. Lutz, Asymptotic representation of solutions of perturbed systems of linear difference equations, Studies in Appl. Math. 77 (1987), 195-221.
11. R.B. Dingle and G.J. Morgan, WKB methods for difference equations; I,II, Appl. Sci. Res. 18 (1967), 221-237 and 238-245.
12. S. Bodine and D.A. Lutz, Asymptotic solutions and error estimates for linear systems of difference and differential equations, J. Math. Anal. Appl. 290 (2004), 343-362.