Mean-values and analytic solutions of the heat equation

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1. Introduction

It is well known that harmonic functions posses the mean-value property.
If u is harmonic on $\Omega \subset \mathbb{R}^d$, then for every closed ball $B(\mathring{x}, R) \subset \Omega$ (or sphere $S(\mathring{x}, R) \subset \Omega$) of a center at $\mathring{x} \in \Omega$ and radius $R > 0$ the average of u over $B(\mathring{x}, R)$ (or over $S(\mathring{x}, R)$) equals to $u(\mathring{x})$ i.e.

$$
\begin{aligned}
u(\mathring{x}) &= \frac{1}{\sigma(d)R^d} \int_{B(\mathring{x},R)} u(x)dx \\
&= \frac{1}{d\sigma(d)R^{d-1}} \int_{S(\mathring{x},R)} u(x)dS(x),
\end{aligned}
$$

where $\sigma(d) = \pi^{d/2}/\Gamma(d/2+1)$ is the volume of the unit ball in $\mathbb{R}^d$ and dS denotes the surface measure on $S(\mathring{x}, R)$.

On the other hand if a continuous function u satisfies the above equality for every ball (for every sphere) in Ω, then u is twice continuously differentiable and harmonic on Ω.

Clearly, for polyharmonic functions, i.e. the solutions of the iterated Laplace operator Δ^m, $m \in \mathbb{N}$, or more generally for real analytic functions, the integral means over balls or spheres need not to be equal the value of a function at the center of a ball or sphere. It appears however that this means can be expressed by some polynomials of the radius of the ball or sphere.

In fact, the mean-value formula for polyharmonic functions for spherical means in dimensions $d = 2, 3$ was established already in 1909 by Pizzetti [15, 16]. The inverse to the mean-value property for polyharmonic functions in dimension $d = 2$ was first proved by Sbrana [17]. The Pizzetti mean-value property for polyharmonic functions and its inverse was extended to the case of spherical and solid means in arbitrary dimension by Nicolesco [14].

Afterwards some other theorems on mean-value properties for polyharmonic and real analytic functions have been obtained by Ghermanesco [9], Friedman [8], Bramble and Payne [5], Bojanov [4], Zalcman [18], and others.

In the lecture we first derive differential relations between the spherical and solid means of functions. Next we extend the mean-value formulas to the case of real analytic functions we obtain a characterization of that functions in terms of integral means over balls or spheres. We also obtain similar characterization of functions of Laplacian growth.

As an application we study the problem of analyticity in time of solutions of the initial value problem to the heat equations $\partial_t u = \Delta u$ with real analytic initial data $u(0, \cdot) = u_0$. We prove that the solution u is analytic in time at $t = 0$ if and only if the integral means of u_0 over balls or spheres of radius R can be extended to entire functions of R of exponential order at most 2.

2. Relations between spherical and solid means.

Let $u \in C^0(\Omega)$, $\mathring{x} \in \Omega$, $0 < R < \operatorname{dist}(\mathring{x}, \partial\Omega)$. Define

$$M(u, \mathring{x}; R) = \frac{1}{\sigma(d)R^d} \int\limits_{B(\mathring{x},R)} u(x)dx,$$

$$N(u, \mathring{x}; R) = \frac{1}{d\sigma(d)R^{d-1}} \int\limits_{S(\mathring{x},R)} u(x)dS(x).$$

Lemma 1 *Let* $u \in C^0(\Omega)$*. Then for any* $\mathring{x} \in \Omega$ *and* $0 < R < \operatorname{dist}(\mathring{x}, \partial\Omega)$,

$$\Big(\frac{R}{d}\frac{\partial}{\partial R} + 1\Big) M(u, \mathring{x}; R) = N(u, \mathring{x}; R).$$

If $u \in C^2(\Omega)$, *then*

$$\frac{d}{R}\frac{\partial}{\partial R} N(u, \mathring{x}; R) = M(\Delta u, \mathring{x}; R).$$

The proof of the first formula is done by computation of $M(u, R)$ in the spherical coordinates, while that of second by using the Green formula.

Corollary 1 (Beckenbach-Radó-Reade). *Let* $u \in C^0(\Omega)$*. If for any* $\mathring{x} \in \Omega$ *and* $0 < R < \operatorname{dist}(\mathring{x}, \partial\Omega)$,

$$M(u, \mathring{x}; R) = N(u, \mathring{x}; R),$$

then u *is harmonic on* Ω*.*

3. Mean-value properties for real-analytic functions.

Let $\mathcal{A}(\Omega)$ be the set of real-analytic functions i.e. $u \in \mathcal{A}(\Omega)$ if for any $\mathring{x} \in \Omega$ and $\|x - \mathring{x}\| < r$ with some $r > 0$,

$$u(x) = \sum_{\ell \in \mathbb{N}_0^d} \frac{1}{\ell_1! \cdots \ell_d!} \frac{\partial^{|\ell|}}{\partial x^\ell} u(\mathring{x})(x - \mathring{x})^\ell, \tag{1}$$

where $|\ell| = \ell_1 + \cdots + \ell_d$ and $x^\ell = x_1^{\ell_1} \cdots x_d^{\ell_d}$.

Theorem 1 (Mean-value property). *Let $u \in \mathcal{A}(\Omega)$, $\mathring{x} \in \Omega$. Then $M(u,\mathring{x};R)$ and $N(u,\mathring{x};R)$ are analytic functions at the origin and for R small*

$$(2) \qquad M(u,\mathring{x};R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(\mathring{x})}{4^k \left(\frac{d}{2}+1\right)_k k!} R^{2k},$$

$$(3) \qquad N(u,\mathring{x};R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(\mathring{x})}{4^k \left(\frac{d}{2}\right)_k k!} R^{2k}.$$

Here $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhamer symbol.

Proof. For simplicity assume $\mathring{x} = 0$ and let (1) holds for $x \in B(0,\rho)$. Set $B(R) = B(0,R)$ with $R < \rho$. Note that if at least one of the exponents $\ell_1, \ldots, \ell_d$ is odd, then the integral of

$$x^\ell = x_1^{\ell_1} \cdots x_d^{\ell_d}$$

over $B(R)$ vanishes.
Next for $\ell = 2\kappa$ with $\kappa \in \mathbb{N}_0^d$, $|\kappa| = k$, using [7, formula 676, 11)]

we derive

$$\begin{aligned}
&\frac{1}{\sigma(d)R^d}\int_{B(R)} x_1^{2\kappa_1}\cdots x_d^{2\kappa_d}dx \\
&= \frac{R^{2k}}{\sigma(d)}\int_{B(1)} y_1^{2\kappa_1}\cdots y_d^{2\kappa_d}dy \\
&= \frac{\left(\frac{1}{2}\right)_{\kappa_1}\cdots\left(\frac{1}{2}\right)_{\kappa_d}}{\left(\frac{d}{2}+1\right)_k} R^{2k}.
\end{aligned}$$

$$
\begin{aligned}
M(u,\mathring{x};R) &= \cdots \\
&= \sum_{k=0}^{\infty} \frac{R^{2k}}{4^k \left(\frac{d}{2}+1\right)_k k!} \sum_{\kappa\in\mathbb{N}_0^d, |\kappa|=k} \frac{k!}{\kappa_1!\cdots\kappa_d!} \frac{\partial^{2k} u(\mathring{x})}{\partial x^{2\kappa}} \\
&= \sum_{k=0}^{\infty} \frac{\Delta^k u(\mathring{x})}{4^k \left(\frac{d}{2}+1\right)_k k!} R^{2k}.
\end{aligned}
$$

The series converges for $R < \rho/\sqrt{d}$.

Finally, applying Lemma 1 we get (3). $\square$

Theorem 2 (Converse to the mean value property). *Let* $\rho \in C^0(\Omega, \mathbb{R}^+)$, $u \in C^\infty(\Omega)$. *If*

$$\widetilde{N}(x;R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{d}{2}\right)_k k!} \, R^{2k}$$

is convergent locally uniformly in
$\{(x,R) : x \in \Omega, |R| < \rho(x)\}$,
then $u \in \mathcal{A}(\Omega)$ *and* $N(u,x;R) = \widetilde{N}(x;R)$ *for* $x \in \Omega$, $R < \min\big(\rho(x), \mathrm{dist}(x, \partial\Omega)\big)$.

Proof. Fix a compact set $K \Subset \Omega$ and set $\rho = \inf_{x\in K} \rho(x) > 0$. Then the assumption implies

$$\frac{\Delta^k u(x)}{4^k \left(\frac{d}{2}\right)_k k!} R^{2k} \to 0 \quad \text{as} \quad k \to \infty$$

uniformly on $K \times \{|R| \le \rho_1\}$ with any $\rho_1 < \rho$. So for any $\rho_1 < \rho$ there exists a constant $C(\rho_1) < \infty$ such that for $k \in \mathbb{N}_0$

$$\sup_{x\in K} |\Delta^k u(x)| \le C(\rho_1) \cdot 4^k \big(d/2\big)_k k!\, \rho_1^{-2k}.$$

Applying inequalities $(a)_k \le \big(\max(1,a)\big)^k k!$ for $a > 0$ and $2^k k! k! \le (2k)!$ we see that for any compact set $K \Subset \Omega$ one can find $C < \infty$ and $L < \infty$ such that for $k \in \mathbb{N}_0$

$$\sup_{x \in K} |\Delta^k u(x)| \le C(2k)! L^{2k}.$$

But by [1, Thm 2.2 in Chapter II] this inequality implies that $u \in \mathcal{A}(\Omega)$. Finally, by Theorem 1 we get $\widetilde{N}(x; R) = N(u, x; R)$. $\square$

Corollary 2. *Let* $u \in C^\infty(\Omega)$. *If* u *is polyharmonic in* Ω, *then* $u \in \mathcal{A}(\Omega)$.

Proof. Indeed polyharmonicity of u implies that the series in (2) is finite and so convergent. The application of Theorem 2 gives the thesis. $\square$

4. Functions of Laplacian growth.

In order to control the growth of iterated Laplacians of smooth functions Aronszajn et al. [1] introduced the notion of the Laplacian growth.

Definition. Let $\varrho > 0$ and $\tau \geq 0$. A function u smooth on $\Omega \subset \mathbb{R}^d$ is of *Laplacian growth* (ϱ, τ) if for every $K \Subset \Omega$ and $\varepsilon > 0$ one can find $C = C(K, \varepsilon) < \infty$ such that for $k \in \mathbb{N}_0$,

$$\sup_{x \in K} |\Delta^k u(x)| \leq C (2k)!^{1-1/\varrho} (\tau + \varepsilon)^{2k}. \tag{4}$$

Definition. ([3]) Let $\varrho > 0$ and $\tau \geq 0$. An entire function F is said to be of *exponential growth* (ϱ, τ) if for every $\varepsilon > 0$ one can find C_ε such that for any $R < \infty$

$$\sup_{|z| \leq R} |F(z)| \leq C_\varepsilon \exp\{(\tau + \varepsilon) R^\varrho\}.$$

The exponential growth of an entire function can be expressed in terms of estimations of its Taylor coefficients.

It appears that a function u of Laplacian growth (ϱ, τ) on Ω is in fact real-analytic on Ω (see [1, Theorem 2.2 in Chapter II]). So the spherical and solid means $N(u, x; R)$ and $M(u, x; R)$ are well defined for $x \in \Omega$ and R small enough. However due to estimation (4) both functions $N(u, x; R)$ and $M(u, x; R)$ can be extended to entire functions of exponential growth.

Theorem 3 *Let $\varrho > 0$ and $\tau \geq 0$. If u is of Laplacian growth (ϱ, τ), then $N(u, x; R)$ and $M(u, x; R)$ extend holomorphically to entire functions of exponential growth $(\varrho, \tau^{\varrho}/\varrho)$ locally uniformly in Ω.*

Theorem 4 *Let* $u \in \mathcal{A}(\Omega)$. *If* $M(u,x;R)$ *defined for* $x \in \Omega$ *and* $0 \leq R < \operatorname{dist}(x,\partial\Omega)$ *extends to an entire function* $\widetilde{M}(u,x;z)$ *of exponential growth* (ϱ,τ) *locally uniformly in* Ω,
then u *is of Laplacian growth* $\big(\varrho,(\varrho\tau)^{1/\varrho}\big)$.

The same holds for $N(u,x;R)$.

5. Convergent solutions of the heat equation.

Let us consider the initial value problem for the heat equation

$$\begin{cases} \partial_t u - \Delta_x u &= 0, \\ u\big|_{t=0} &= u_0, \end{cases} \tag{5}$$

where $u_0 \in \mathcal{A}(\Omega)$, $\Omega \subset \mathbb{R}^d$.

Then its formal power series solution is given by

$$\widehat{u}(t,x) = \sum_{k=0}^{\infty} \frac{\Delta^k u_0(x)}{k!} t^k. \tag{6}$$

We ask when the solution u is an analytic function of time variable at $t = 0$. In the dimension $d = 1$ the problem was solved by Kowalevskaya [10].

She proved that the solution u is analytic in time if and only if the initial data u_0 can be analytically extended to an entire function of exponential order 2. In the multidimensional case the solution of the problem was given by Aronszajn at al. [1] in terms of the growth of iterates of the Laplacian of the initial data.

Theorem 5 *Let* $0 < T \leq \infty$. *If formal power series solution (6) of the heat equation (5) is convergent for* $|t| < T$ *locally uniformly in* Ω, *then* $M(u_0, x; R)$ *and* $N(u_0, x; R)$ *extend to an entire function of exponential growth* $(2, 1/(4T))$ *locally uniformly in* Ω.

Conversely, if $M(u_0, x; R)$ or $N(u_0, x; R)$ can be extended to an entire function of exponential growth $(2, 1/(4T))$ locally uniformly in Ω, then the solution $\widehat{u}$ of the heat equation (5) is convergent for $|t| < T$ locally uniformly in Ω.

Proof. Assume that $\widehat{u}(t,x)$ is convergent for $|t| < T$ loc. unif. in Ω. Then $\forall K \Subset \Omega, \varepsilon > 0\ \exists C$ s.t.

$$\begin{aligned}\sup_{x\in K}|\Delta^k u_0(x)| &\le C\Big(\frac{1}{T}+\varepsilon\Big)^k \cdot k! \\ &\le C_\varepsilon\Big(\frac{1}{T}+\varepsilon\Big)^k\Big(\frac{1}{2}+\varepsilon\Big)^k \cdot (2k)!^{1/2} \\ &\le C_\varepsilon\Big((2T)^{-1/2}+\varepsilon\Big)^{2k} \cdot (2k)!^{1/2}.\end{aligned}$$

Hence u_0 is of Laplacian growth $(2, 1/\sqrt{2T})$ and by Theorem 3, $M(u_0, x; R)$ and $N(u_0, x; R)$ extend to entire functions of exponential growth $(2, 1/(4T))$ locally uniformly in Ω.

On the other hand let $M(u_0, x; R)$ or $N(u_0, x; R)$ can be extended to an entire function of exponential growth $(2, 1/(4T))$ loc. unif. in Ω. Then by Theorem 4, u_0 is of Laplacian growth $(2, 1/\sqrt{2T})$ loc. unif. in Ω.

Hence for $|t| < T$ and small $\varepsilon > 0$

$$\begin{aligned}\sup_{x\in K} \sum_{k=0}^{\infty} \frac{|\Delta^k u_0(x)|}{k!}|t|^k &\le \dots \\ &\le C_\varepsilon \sum_{k=0}^{\infty} \Big[\Big(\frac{1}{T} + \varepsilon\Big)|t|\Big]^k < \infty.\end{aligned}$$

So $\widehat{u}(t, x)$ is convergent for $|t| < T$ locally uniformly in Ω. $\square$

Finally let me say that using the modified Borel transformation Sławek Michalik obtained a characterization of Borel summable solutions of the heat equation (5).

Theorem 6 ([13]). *Let $\widehat{u}$ be the formal power series solution (6) of the heat equation (5) with $u_0 \in \mathcal{O}(D^d)$.*
Then the following conditions are equivalent

- $\widehat{u}$ *is* 1*-summable in a direction* θ*;*
- $M(u_0; z, R) \in \mathcal{O}^2\big(D^d \times (\widehat{S}_{\theta/2} \cup \widehat{S}_{\theta/2+\pi})\big)$*;*
- $N(u_0; z, R) \in \mathcal{O}^2\big(D^d \times (\widehat{S}_{\theta/2} \cup \widehat{S}_{\theta/2+\pi})\big)$*.*

Furthermore, the 1*-sum of* $\widehat{u}$ *is given by*

$$u^\theta(t, z) = \frac{1}{(4\pi t)^{d/2}} \int_{(e^{i\theta/2}\mathbb{R})^d} \exp\Big\{ - \frac{e^{i\theta}|x|^2}{4t} \Big\} u_0(x + z)dx.$$

Thank you for your attention!

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