Analytic solutions of moment-PDEs

Sławomir Michalik

Cardinal Stefan Wyszyński University
Warsaw, Poland

Formal and Analytic Solutions of Differential and Difference Equations
Będlewo, August 8–13, 2011

Abstract

We consider the Cauchy problem for linear moment partial differential equations in two complex variables with constant coefficients

$$\begin{cases} P(\partial_{m_1,t}, \partial_{m_2,z})u(t,z) = 0 \\ \partial_{t,m_1}^{j} u(0,z) = \varphi_n(z) \in \mathcal{O}(D) \quad \text{for} \quad j = 0, ..., n-1 \end{cases},$$

where $\partial_{m_1,t}$ and $\partial_{m_2,z}$ are moment-differential operators introduced recently by W. Balser and M. Yoshino, $n \in \mathbb{N}$ and

$$P(\lambda, \zeta) = P_0(\zeta)\lambda^n - \sum_{j=1}^{n} P_j(\zeta)\lambda^{n-j}$$

is a polynomial of order n with respect to λ.

Abstract

We consider the Cauchy problem for linear moment partial differential equations in two complex variables with constant coefficients

$$\begin{cases} P(\partial_{m_1,t}, \partial_{m_2,z})u(t,z) = 0 \\ \partial^j_{t,m_1} u(0,z) = \varphi_n(z) \in \mathcal{O}(D) \quad \text{for} \quad j = 0, ..., n-1 \end{cases},$$

where $\partial_{m_1,t}$ and $\partial_{m_2,z}$ are moment-differential operators introduced recently by W. Balser and M. Yoshino, $n \in \mathbb{N}$ and

$$P(\lambda, \zeta) = P_0(\zeta)\lambda^n - \sum_{j=1}^{n} P_j(\zeta)\lambda^{n-j}$$

is a polynomial of order n with respect to λ.

We construct the integral representation of the solution of this problem and we show when this solution is analytic. As a consequence we also obtain the characterization of summable formal solutions of the Cauchy problem.

Moment functions

Definition

A pair of functions $e(z)$ and $E(z)$ is said to be kernel functions of order k $(k > 1/2)$ if they have the following properties:

1. $e(z) \in \mathcal{O}(S(0, \pi/k))$, $e(z)/z$ is integrable at the origin, $e(x) \in \mathbb{R}_+$ for $x \in \mathbb{R}_+$ and $e(z)$ is exponentially flat of order k in $S(0, \pi/k)$ (i.e. $\forall_{\varepsilon>0}\exists_{A,B>0}$ such that $|e(z)| \leq Ae^{-(|z|/B)^k}$ for $z \in S(0, \pi/k - \varepsilon)$).

Moment functions

Definition

A pair of functions $e(z)$ and $E(z)$ is said to be kernel functions of order k ($k > 1/2$) if they have the following properties:

1. $e(z) \in \mathcal{O}(S(0, \pi/k))$, $e(z)/z$ is integrable at the origin, $e(x) \in \mathbb{R}_+$ for $x \in \mathbb{R}_+$ and $e(z)$ is exponentially flat of order k in $S(0, \pi/k)$ (i.e. $\forall_{\varepsilon>0}\exists_{A,B>0}$ such that $|e(z)| \leq Ae^{-(|z|/B)^k}$ for $z \in S(0, \pi/k - \varepsilon)$).
2. $E(z) \in \mathcal{O}^k(\mathbb{C})$ (i.e. $E(z) \in \mathcal{O}(\mathbb{C})$ and $\exists_{A,B>0}$ such that $|e(z)| \leq Ae^{B|z|^k}$ for $z \in \mathbb{C}$) and $E(1/z)/z$ is integrable at the origin in $S(\pi, 2\pi - \pi/k)$.

Moment functions

Definition

A pair of functions $e(z)$ and $E(z)$ is said to be kernel functions of order k ($k > 1/2$) if they have the following properties:

1. $e(z) \in \mathcal{O}(S(0, \pi/k))$, $e(z)/z$ is integrable at the origin, $e(x) \in \mathbb{R}_+$ for $x \in \mathbb{R}_+$ and $e(z)$ is exponentially flat of order k in $S(0, \pi/k)$ (i.e. $\forall_{\varepsilon>0}\exists_{A,B>0}$ such that $|e(z)| \leq Ae^{-(|z|/B)^k}$ for $z \in S(0, \pi/k - \varepsilon)$).
2. $E(z) \in \mathcal{O}^k(\mathbb{C})$ (i.e. $E(z) \in \mathcal{O}(\mathbb{C})$ and $\exists_{A,B>0}$ such that $|e(z)| \leq Ae^{B|z|^k}$ for $z \in \mathbb{C}$) and $E(1/z)/z$ is integrable at the origin in $S(\pi, 2\pi - \pi/k)$.
3. The connection between $e(z)$ and $E(z)$ is given by the corresponding moment function $m(u)$ as follows. The function $m(u)$ is defined in terms of $e(z)$ by
$$m(u) := \int_0^{\infty} x^{u-1} e(x) dx \ \text{ for } \ \operatorname{Re} u \geq 0$$
and the kernel function $E(z)$ has the power series expansion $E(z) = \sum_{n=0}^{\infty} \frac{z^n}{m(n)}$ for $z \in \mathbb{C}$.

Moment functions

Remark

The moments $m(n)$ are of the same order as $\Gamma(1+n/k)$. It means that there exists constants $c, C > 0$ such that

$$c^n\Gamma(1+n/k) \leq m(n) \leq C^n\Gamma(1+n/k) \quad \text{for every} \quad n \in \mathbb{N}.$$

Moment functions

Remark

The moments $m(n)$ are of the same order as $\Gamma(1+n/k)$. It means that there exists constants $c, C > 0$ such that

$$c^n\Gamma(1+n/k) \leq m(n) \leq C^n\Gamma(1+n/k) \quad \text{for every} \quad n \in \mathbb{N}.$$

Example

The most important examples of kernel functions of order k are

- $e(z) = kz^k e^{-z^k}$
- $m(u) = \Gamma(1+u/k)$
- $E(z) = \sum_{n=0}^{\infty} z^n/\Gamma(1+n/k) =: E_{1/k}(z)$, where $E_{1/k}$ is the Mittag-Leffler function of index $1/k$.

Gevrey order and Borel summability

According to the properties of moment functions and general theory of moment summability we may define the Gevrey order and the Borel summability of formal power series as follows

Gevrey order and Borel summability

According to the properties of moment functions and general theory of moment summability we may define the Gevrey order and the Borel summability of formal power series as follows

Definition

Let $m(u)$ be a moment function of order k, $\hat{u}(t,z) = \sum_{n=0}^{\infty} u_n(z)t^n$ be a formal power series with coefficients $u_n(z) \in \mathcal{O}(D)$ and $v(t,z) := \sum_{n=0}^{\infty} \frac{u_n(z)}{m(n)} t^n$. Then we say that:

Gevrey order and Borel summability

According to the properties of moment functions and general theory of moment summability we may define the Gevrey order and the Borel summability of formal power series as follows

Definition

Let $m(u)$ be a moment function of order k, $\hat{u}(t,z)=\sum_{n=0}^{\infty}u_n(z)t^n$ be a formal power series with coefficients $u_n(z)\in\mathcal{O}(D)$ and $v(t,z):=\sum_{n=0}^{\infty}\frac{u_n(z)}{m(n)}t^n$. Then we say that:

1. $\hat{u}$ is a Gevrey series of order $1/k$ if and only if $v\in\mathcal{O}(D^2)$, where $D\subset\mathbb{C}$ is a complex neighbourhood of the origin.

Gevrey order and Borel summability

According to the properties of moment functions and general theory of moment summability we may define the Gevrey order and the Borel summability of formal power series as follows

Definition

Let $m(u)$ be a moment function of order k, $\hat{u}(t,z) = \sum_{n=0}^{\infty} u_n(z)t^n$ be a formal power series with coefficients $u_n(z) \in \mathcal{O}(D)$ and $v(t,z) := \sum_{n=0}^{\infty} \frac{u_n(z)}{m(n)} t^n$. Then we say that:

1. $\hat{u}$ is a Gevrey series of order $1/k$ if and only if $v \in \mathcal{O}(D^2)$, where $D \subset \mathbb{C}$ is a complex neighbourhood of the origin.
2. $\hat{u}$ is k-summable in a direction d ($d \in \mathbb{R}$) if and only if $v \in \mathcal{O}^k(\hat{S}_d \times D)$ (i.e. $v \in \mathcal{O}(\hat{S}_d \times D)$ and $|v(t,z)| \leq Ae^{C|t|^k}$ for some $A, C > 0$), where $S_d \subset \mathbb{C}$ is a sector in a direction d and $\hat{S}_d := D \cup S_d$.

Moment-differential operators

Definition

For every moment functions $m_1(u)$ and $m_2(u)$ the linear operators $\partial_{m_1,t}, \partial_{m_2,z} \colon \mathbb{C}[[t,z]] \to \mathbb{C}[[t,z]]$ defined by

$$\partial_{m_1,t}\Big(\sum_{n=0}^{\infty}\frac{u_n(z)}{m_1(n)}t^n\Big) := \sum_{n=0}^{\infty}\frac{u_{n+1}(z)}{m_1(n)}t^n$$

and

$$\partial_{m_2,z}\Big(\sum_{n=0}^{\infty}\frac{\tilde{u}_n(t)}{m_2(n)}z^n\Big) := \sum_{n=0}^{\infty}\frac{\tilde{u}_{n+1}(t)}{m_2(n)}z^n$$

are called moment-differential operators $\partial_{m_1,t}$ and $\partial_{m_2,z}$.

Moment-differential operators

Definition

For every moment functions $m_1(u)$ and $m_2(u)$ the linear operators $\partial_{m_1,t}, \partial_{m_2,z} \colon \mathbb{C}[[t,z]] \to \mathbb{C}[[t,z]]$ defined by

$$\partial_{m_1,t}\Big(\sum_{n=0}^{\infty}\frac{u_n(z)}{m_1(n)}t^n\Big) := \sum_{n=0}^{\infty}\frac{u_{n+1}(z)}{m_1(n)}t^n$$

and

$$\partial_{m_2,z}\Big(\sum_{n=0}^{\infty}\frac{\tilde{u}_n(t)}{m_2(n)}z^n\Big) := \sum_{n=0}^{\infty}\frac{\tilde{u}_{n+1}(t)}{m_2(n)}z^n$$

are called moment-differential operators $\partial_{m_1,t}$ and $\partial_{m_2,z}$.

Remark

- For $m_1(u) = \Gamma(1+u)$, the operator $\partial_{m_1,t}$ coincides with differentiation ∂_t.

Moment-differential operators

Definition

For every moment functions $m_1(u)$ and $m_2(u)$ the linear operators $\partial_{m_1,t}, \partial_{m_2,z}\colon \mathbb{C}[[t,z]] \to \mathbb{C}[[t,z]]$ defined by

$$\partial_{m_1,t}\Big(\sum_{n=0}^{\infty}\frac{u_n(z)}{m_1(n)}t^n\Big) := \sum_{n=0}^{\infty}\frac{u_{n+1}(z)}{m_1(n)}t^n$$

and

$$\partial_{m_2,z}\Big(\sum_{n=0}^{\infty}\frac{\tilde{u}_n(t)}{m_2(n)}z^n\Big) := \sum_{n=0}^{\infty}\frac{\tilde{u}_{n+1}(t)}{m_2(n)}z^n$$

are called moment-differential operators $\partial_{m_1,t}$ and $\partial_{m_2,z}$.

Remark

- For $m_1(u) = \Gamma(1+u)$, the operator $\partial_{m_1,t}$ coincides with differentiation ∂_t.
- For $m_1(u) = \Gamma(1+u/k)$, the operator $\partial_{m_1,t}$ is related to $1/k$-fractional differentiation $\partial_t^{1/k}$.

Moment-differentiation of analytic functions

Proposition

Let e_{m_2} and E_{m_2} be kernel functions of order k_2 with corresponding moment function m_2. Then for every $\varphi \in \mathcal{O}(D_r)$ ($D_r := \{z \in \mathbb{C}\colon |z| < r\}$), $|z| < \varepsilon < r$ and $n \in \mathbb{N}$ we have

$$\partial^n_{m_2,z}\varphi(z) = \frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_0^{\infty(\theta)}\zeta^n E_{m_2}(z\zeta)\frac{e_{m_2}(w\zeta)}{w\zeta}\,d\zeta\,dw,$$

where $\theta \in (-\arg w - \frac{\pi}{2k_2}, -\arg w + \frac{\pi}{2k_2})$.

Moment-differentiation of analytic functions

Proof.

By the Cauchy integral formula and the definition of moment function we have

$$\partial^n_{m_2,z}\varphi(0)=\frac{m_2(n)}{n!}\varphi^{(n)}(0)=\frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_0^{\infty(\theta)}\zeta^n\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\,dw,$$

where $\theta\in(-\arg w-\frac{\pi}{2k_2},-\arg w+\frac{\pi}{2k_2})$.

Moment-differentiation of analytic functions

Proof.

By the Cauchy integral formula and the definition of moment function we have

$$\partial^n_{m_2,z}\varphi(0) = \frac{m_2(n)}{n!}\varphi^{(n)}(0) = \frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_0^{\infty(\theta)}\zeta^n\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\,dw,$$

where $\theta \in (-\arg w - \frac{\pi}{2k_2}, -\arg w + \frac{\pi}{2k_2})$. Hence

$$\varphi(z) = \sum_{n=0}^{\infty}\frac{\partial^n_{m_2,z}\varphi(0)}{m_2(n)}z^n = \frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_0^{\infty(\theta)}E_{m_2}(\zeta z)\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\,dw.$$

Moment-differentiation of analytic functions

Proof.

By the Cauchy integral formula and the definition of moment function we have

$$\partial^n_{m_2,z}\varphi(0)=\frac{m_2(n)}{n!}\varphi^{(n)}(0)=\frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_0^{\infty(\theta)}\zeta^n\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\,dw,$$

where $\theta\in(-\arg w-\frac{\pi}{2k_2},-\arg w+\frac{\pi}{2k_2})$. Hence

$$\varphi(z)=\sum_{n=0}^{\infty}\frac{\partial^n_{m_2,z}\varphi(0)}{m_2(n)}z^n=\frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_0^{\infty(\theta)}E_{m_2}(\zeta z)\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\,dw.$$

Since $\partial^n_{m_2,z}E_{m_2}(\zeta z)=\zeta^n E_{m_2}(\zeta z)$, we finally obtain

$$\partial^n_{m_2,z}\varphi(z)=\frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_0^{\infty(\theta)}\zeta^n E_{m_2}(\zeta z)\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\,dw.$$

$\square$

Moment-pseudodifferential operators

The formula for moment-differentiation of analytic functions motivates the introduction of the following moment-pseudodifferential operators

Definition

Let $\lambda(\zeta)$ be an analytic function for $|\zeta| > |\zeta_0|$ of polynomial growth at infinity. Then the moment-pseudodifferential operator $\lambda(\partial_{m_2,z})$ is defined by

1. $$\lambda(\partial_{m_2,z})E_{m_2}(\zeta z) := \lambda(\zeta)E_{m_2}(\zeta z)$$

Moment-pseudodifferential operators

The formula for moment-differentiation of analytic functions motivates the introduction of the following moment-pseudodifferential operators

Definition

Let $\lambda(\zeta)$ be an analytic function for $|\zeta| > |\zeta_0|$ of polynomial growth at infinity. Then the moment-pseudodifferential operator $\lambda(\partial_{m_2,z})$ is defined by

1. $$\lambda(\partial_{m_2,z})E_{m_2}(\zeta z) := \lambda(\zeta)E_{m_2}(\zeta z)$$

2. $$\lambda(\partial_{m_2,z})\varphi(z) := \frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_{\zeta_0}^{\infty(\theta)}\lambda(\zeta)E_{m_2}(\zeta z)\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\,dw$$

   for every $\varphi \in \mathcal{O}(D_r)$.

Linear moment-PDEs

We consider the initial value problem for linear moment partial differential equation with constant coefficients

$$\begin{cases} P(\partial_{m_1,t}, \partial_{m_2,z})u = 0 \\ \partial_{m_1,t}^{j} u(0,z) = \varphi_j(z) \in \mathcal{O}(D) \quad \text{for} \quad j = 0, ..., n-1 \end{cases}.$$

Using the moment-pseudodifferential operators we factorize the moment-differential operator $P(\partial_{m_1,t}, \partial_{m_2,z})$ as follows

$$\begin{aligned} P(\partial_{m_1,t}, \partial_{m_2,z}) &= P_0(\partial_{m_2,z})(\partial_{m_1,t} - \lambda_1(\partial_{m_2,z}))^{n_1} ... (\partial_{m_1,t} - \lambda_l(\partial_{m_2,z}))^{n_l} \\ &=: P_0(\partial_{m_2,z})\tilde{P}(\partial_{m_1,t}, \partial_{m_2,z}) \end{aligned}$$

where $\lambda_1(\zeta), ..., \lambda_l(\zeta)$ are the characteristic roots of $P(\lambda, \zeta) = 0$ with multiplicity $n_1, ..., n_l$ $(n_1 + ... + n_l = n)$.

Decomposition of equation

By the factorization of operator $\tilde{P}(\partial_{m_1,t}, \partial_{m_2,z})$ we obtain

Theorem

If $\hat{u}$ is a formal solution of

$$\begin{cases} \tilde{P}(\partial_{m_1,t}, \partial_{m_2,z})u = 0 \\ \partial^j_{m_1,t}u(0,z) = 0 \ \ (j = 0, ..., n-2) \\ \partial^{n-1}_{m_1,t}u(0,z) = \varphi(z) \in \mathcal{O}(D), \end{cases}$$

then $\hat{u} = \sum_{\alpha=1}^{l} \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$, where $\hat{u}_{\alpha\beta}$ is a formal solution of

$$\begin{cases} (\partial_{m_1,t} - \lambda_\alpha(\partial_{m_2,z}))^\beta u_{\alpha\beta} = 0 \\ \partial^j_{m_1,t}u_{\alpha\beta}(0,z) = 0 \ \ (j = 0, ..., \beta-2) \\ \partial^{\beta-1}_{m_1,t}u_{\alpha\beta}(0,z) = c_{\alpha\beta}(\partial_{m_2,z})\varphi(z) =: \varphi_{\alpha\beta}(z) \in \mathcal{O}(D) \end{cases}$$

for some holomorphic function of polynomial growth $c_{\alpha\beta}(\zeta)$.

Moment-pseudodifferential equation

By the above theorem it is sufficient to study the moment-pseudodifferential equation

$$\begin{cases} (\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^{\beta} u = 0 \\ \partial^{j}_{m_1,t} u(0,z) = 0 \quad (n = 0, ..., \beta - 2) \\ \partial^{\beta-1}_{m_1,t} u(0,z) = \varphi(z) \in \mathcal{O}(D). \end{cases}$$

For simplicity we assume that $\beta = 1$.

Moment-pseudodifferential equation

By the above theorem it is sufficient to study the moment-pseudodifferential equation

$$\begin{cases} (\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^{\beta} u = 0 \\ \partial^{j}_{m_1,t} u(0,z) = 0 \quad (n = 0, ..., \beta - 2) \\ \partial^{\beta-1}_{m_1,t} u(0,z) = \varphi(z) \in \mathcal{O}(D). \end{cases}$$

For simplicity we assume that $\beta = 1$.

We will study the Gevrey order of formal solution $\hat{u}$, which depends on the orders k_1 and k_2 of moment functions and depends on the characteristic root $\lambda(\zeta)$.

Moment-pseudodifferential equation

By the above theorem it is sufficient to study the moment-pseudodifferential equation

$$\begin{cases} (\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^{\beta} u = 0 \\ \partial_{m_1,t}^{j} u(0,z) = 0 \quad (n = 0, ..., \beta - 2) \\ \partial_{m_1,t}^{\beta-1} u(0,z) = \varphi(z) \in \mathcal{O}(D). \end{cases}$$

For simplicity we assume that $\beta = 1$.

We will study the Gevrey order of formal solution $\hat{u}$, which depends on the orders k_1 and k_2 of moment functions and depends on the characteristic root $\lambda(\zeta)$.

Definition

We define a pole order $q \in \mathbb{Q}$ and a leading term $\lambda \in \mathbb{C} \setminus \{0\}$ of $\lambda(\zeta)$ as the numbers satisfying the formula

$$\lim_{\zeta \to \infty} \frac{\lambda(\zeta)}{\zeta^q} = \lambda.$$

Gevrey order of formal solution

Observe that the formal solution $\hat{u}$ of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D)$$

is given by

$$\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{m_2,z})\varphi(z)}{m_1(n)} t^n.$$

Gevrey order of formal solution

Observe that the formal solution $\hat{u}$ of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D)$$

is given by

$$\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{m_2,z})\varphi(z)}{m_1(n)} t^n.$$

Estimating the coefficients of the formal solution $\hat{u}$ we have

Proposition

1. If $1/k_1 < q/k_2$ then $\hat{u}$ is a Gevrey series of order $q/k_2 - 1/k_1$ with respect to t.

Gevrey order of formal solution

Observe that the formal solution $\hat{u}$ of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D)$$

is given by

$$\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{m_2,z})\varphi(z)}{m_1(n)} t^n.$$

Estimating the coefficients of the formal solution $\hat{u}$ we have

Proposition

1. If $1/k_1 < q/k_2$ then $\hat{u}$ is a Gevrey series of order $q/k_2 - 1/k_1$ with respect to t.
2. If $1/k_1 = q/k_2$ then $u \in \mathcal{O}(D^2)$.

Gevrey order of formal solution

Observe that the formal solution $\hat{u}$ of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D)$$

is given by

$$\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{m_2,z})\varphi(z)}{m_1(n)} t^n.$$

Estimating the coefficients of the formal solution $\hat{u}$ we have

Proposition

1. If $1/k_1 < q/k_2$ then $\hat{u}$ is a Gevrey series of order $q/k_2 - 1/k_1$ with respect to t.
2. If $1/k_1 = q/k_2$ then $u \in \mathcal{O}(D^2)$.
3. If $1/k_1 > q/k_2$ then $u \in \mathcal{O}^{\frac{k_1 k_2}{k_2 - qk_1}}(\mathbb{C} \times D)$.

Integral representation of solution

By the definition of moment-pseudodifferential operators we have

Lemma

Let us assume that u is a solution of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D)$$

and $1/k_1 \geq q/k_2$. Then u is analytic in a complex neighbourhood of the origin and has the integral representation

$$u(t,z) \;=\; \frac{1}{2\pi i} \oint_{|w|=\varepsilon} \varphi(w) \int_{\zeta_0}^{\infty(\theta)} E_{m_1}(\lambda(\zeta)t) E_{m_2}(\zeta z) \frac{e_{m_2}(\zeta w)}{\zeta w}\, d\zeta\, dw.$$

Integral representation of solution

By the definition of moment-pseudodifferential operators we have

Lemma

Let us assume that u is a solution of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D)$$

and $1/k_1 \geq q/k_2$. Then u is analytic in a complex neighbourhood of the origin and has the integral representation

$$u(t,z) \quad = \quad \frac{1}{2\pi i} \oint_{|w|=\varepsilon} \varphi(w) \int_{\zeta_0}^{\infty(\theta)} E_{m_1}(\lambda(\zeta)t) E_{m_2}(\zeta z) \frac{e_{m_2}(\zeta w)}{\zeta w} \, d\zeta \, dw.$$

Remark

Observe that the function u satisfies also $(\partial_{m_2,z} - \lambda^{-1}(\partial_{m_1,t}))u = 0$.

Analytic solution

Theorem (The main theorem)

Let us assume that u is a solution of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D)$$

and $1/k_1 = q/k_2$. *Then for every* $s > 1$ *and* $d \in \mathbb{R}$ *we have that*

$$\varphi \in \mathcal{O}^{qs}(\hat{S}_{(d+\arg\lambda+2n\pi)/q}) \text{ for every } n \in \mathbb{Z} \iff u \in \mathcal{O}^{s}(\hat{S}_d \times D).$$

Analytic solution

Theorem (The main theorem)

Let us assume that u is a solution of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D)$$

and $1/k_1 = q/k_2$. Then for every $s > 1$ and $d \in \mathbb{R}$ we have that

$$\varphi \in \mathcal{O}^{qs}(\hat{S}_{(d+\arg\lambda+2n\pi)/q}) \text{ for every } n \in \mathbb{Z} \iff u \in \mathcal{O}^s(\hat{S}_d \times D).$$

Idea of proof.

($\Longrightarrow$) To show analytic continuation we use the integral representation of u and deform the path of integration with respect to w in this formula.

Analytic solution

Theorem (The main theorem)

Let us assume that u is a solution of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D)$$

and $1/k_1 = q/k_2$. Then for every $s > 1$ and $d \in \mathbb{R}$ we have that

$$\varphi \in \mathcal{O}^{qs}(\hat{S}_{(d+\arg\lambda+2n\pi)/q}) \text{ for every } n \in \mathbb{Z} \iff u \in \mathcal{O}^s(\hat{S}_d \times D).$$

Idea of proof.

($\Longrightarrow$) To show analytic continuation we use the integral representation of u and deform the path of integration with respect to w in this formula.

($\Longleftarrow$) The function u satisfies $(\partial_{m_2,z} - \lambda^{-1}(\partial_{m_1,t}))u = 0$ with the initial condition $u(t,0) = \psi(t) \in \mathcal{O}^s(\hat{S}_d)$. Using the integral representation of solution to this problem and deforming the path of integration we obtain the assertion. $\square$

Application. Summable solution

Theorem

Let us assume that $\hat{u}$ is a formal solution of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D),$$

$1/k_1 < q/k_2$, $k := (q/k_2 - 1/k_1)^{-1}$ *and* $d \in \mathbb{R}$. *Then $\hat{u}$ is k-summable in a direction d if and only if* $\varphi \in \mathcal{O}^{qk}(\hat{S}_{(d+\arg\lambda+2n\pi)/q})$ *for every* $n \in \mathbb{N}$.

Application. Summable solution

Theorem

Let us assume that $\hat{u}$ is a formal solution of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D),$$

$1/k_1 < q/k_2$, $k := (q/k_2 - 1/k_1)^{-1}$ *and* $d \in \mathbb{R}$. *Then $\hat{u}$ is k-summable in a direction d if and only if* $\varphi \in \mathcal{O}^{qk}(\hat{S}_{(d+\arg\lambda+2n\pi)/q})$ *for every* $n \in \mathbb{N}$.

Proof.

$\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{m_2,z})\varphi(z)}{m_1(n)} t^n$ is k-summable in a direction $d \iff$
$v(t,z) := \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{m_2,z})\varphi(z)}{m_1(n)m(n)} t^n \in \mathcal{O}^k(\hat{S}_d \times D)$.

Application. Summable solution

Theorem

Let us assume that $\hat{u}$ is a formal solution of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D),$$

$1/k_1 < q/k_2$, $k := (q/k_2 - 1/k_1)^{-1}$ *and* $d \in \mathbb{R}$. *Then $\hat{u}$ is k-summable in a direction d if and only if* $\varphi \in \mathcal{O}^{qk}(\hat{S}_{(d+\arg\lambda+2n\pi)/q})$ *for every* $n \in \mathbb{N}$.

Proof.

$\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{m_2,z})\varphi(z)}{m_1(n)} t^n$ is k-summable in a direction $d \iff$
$v(t,z) := \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{m_2,z})\varphi(z)}{m_1(n)m(n)} t^n \in \mathcal{O}^k(\hat{S}_d \times D)$. Observe that $v(t,z)$ is a solution of $(\partial_{\tilde{m}_1,t} - \lambda(\partial_{m_2,z}))v = 0$ with $v(0,z) = \varphi(z) \in \mathcal{O}(D)$, where
$\tilde{m}_1(u) := m_1(u)m(u)$ is a moment function of order $\tilde{k}_1 := k_2/q$.

Application. Summable solution

Theorem

Let us assume that $\hat{u}$ is a formal solution of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D),$$

$1/k_1 < q/k_2$, $k := (q/k_2 - 1/k_1)^{-1}$ *and* $d \in \mathbb{R}$. *Then $\hat{u}$ is k-summable in a direction d if and only if* $\varphi \in \mathcal{O}^{qk}(\hat{S}_{(d+\arg\lambda+2n\pi)/q})$ *for every* $n \in \mathbb{N}$.

Proof.

$\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{m_2,z})\varphi(z)}{m_1(n)} t^n$ is k-summable in a direction $d \iff$ $v(t,z) := \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{m_2,z})\varphi(z)}{m_1(n)m(n)} t^n \in \mathcal{O}^k(\hat{S}_d \times D)$. Observe that $v(t,z)$ is a solution of $(\partial_{\tilde{m}_1,t} - \lambda(\partial_{m_2,z}))v = 0$ with $v(0,z) = \varphi(z) \in \mathcal{O}(D)$, where $\tilde{m}_1(u) := m_1(u)m(u)$ is a moment function of order $\tilde{k}_1 := k_2/q$. Hence, by the main theorem $v \in \mathcal{O}^k(\hat{S}_d \times D) \iff \varphi \in \mathcal{O}^{qk}(\hat{S}_{(d+\arg\lambda+2n\pi)/q})$ for every $n \in \mathbb{N}$. □

Application. Divergent Cauchy data

Theorem

Let us assume that $\hat{u}$ is a formal solution of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \hat{\varphi}(z) = \sum_{n=0}^{\infty} \varphi_n z^n,$$

where $\hat{\varphi}$ is a Gevrey series of order $1/k$ (i.e. $\tilde{\varphi}(z) := \sum_{n=0}^{\infty} \frac{\varphi_n}{m(n)} z^n \in \mathcal{O}(D)$). Moreover we assume that $1/\tilde{k} := q/k_2 + q/k - 1/k_1 > 0$. Then $\hat{u}$ is $\tilde{k}$-summable in a direction d if and only if $\tilde{\varphi} \in \mathcal{O}^{q\tilde{k}}(\hat{S}_{(d+\arg\lambda+2n\pi)/q})$ for every $n \in \mathbb{N}$.

Application. Divergent Cauchy data

Theorem

Let us assume that $\hat{u}$ is a formal solution of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \hat{\varphi}(z) = \sum_{n=0}^{\infty} \varphi_n z^n,$$

where $\hat{\varphi}$ is a Gevrey series of order $1/k$ (i.e. $\tilde{\varphi}(z) := \sum_{n=0}^{\infty} \frac{\varphi_n}{m(n)} z^n \in \mathcal{O}(D)$). Moreover we assume that $1/\tilde{k} := q/k_2 + q/k - 1/k_1 > 0$. Then $\hat{u}$ is $\tilde{k}$-summable in a direction d if and only if $\tilde{\varphi} \in \mathcal{O}^{q\tilde{k}}(\hat{S}_{(d+\arg\lambda+2n\pi)/q})$ for every $n \in \mathbb{N}$.

Remark

Observe that for $1/k_1 = q/k_2$ (resp. $1/k_1 < q/k_2$, $1/k_1 > q/k_2$) the condition $\tilde{\varphi} \in \mathcal{O}^{q\tilde{k}}(\hat{S}_{(d+\arg\lambda+2n\pi)/q})$ for every $n \in \mathbb{N}$ is equivalent to (resp. stronger than, weaker than) k-summability of $\hat{\varphi}$ in directions $(d + \arg\lambda + 2n\pi)/q$ for every $n \in \mathbb{N}$.

Application. Divergent Cauchy data

Idea of proof.

We have that $\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{m_2,z})\varphi(z)}{m_1(n)} t^n$ is $\tilde{k}$-summable in a direction $d \Longleftrightarrow$ $\hat{w}(t,z) := \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{\tilde{m}_2,z})\tilde{\varphi}(z)}{m_1(n)} t^n$ is $\tilde{k}$-summable in a direction d, where $\tilde{m}_2(u) := m(u)m_2(u)$ is a moment function of order $\tilde{k}_2 := (1/k_2 + 1/k)^{-1}$.

Application. Divergent Cauchy data

Idea of proof.

We have that $\hat{u}(t,z)=\sum_{n=0}^{\infty}\frac{\lambda^n(\partial_{m_2,z})\varphi(z)}{m_1(n)}t^n$ is $\tilde{k}$-summable in a direction $d \iff \hat{w}(t,z):=\sum_{n=0}^{\infty}\frac{\lambda^n(\partial_{\tilde{m}_2,z})\tilde{\varphi}(z)}{m_1(n)}t^n$ is $\tilde{k}$-summable in a direction d, where $\tilde{m}_2(u):=m(u)m_2(u)$ is a moment function of order $\tilde{k}_2:=(1/k_2+1/k)^{-1}$.

Observe that $\hat{w}$ is a formal solution of

$$(\partial_{m_1,t}-\lambda(\partial_{\tilde{m}_2,z}))w=0,\qquad w(0,z)=\tilde{\varphi}(z)=\sum_{n=0}^{\infty}\frac{\varphi_n}{m(n)}z^n\in\mathcal{O}(D).$$

Application. Divergent Cauchy data

Idea of proof.

We have that $\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{m_2,z})\varphi(z)}{m_1(n)} t^n$ is $\tilde{k}$-summable in a direction $d \Longleftrightarrow$ $\hat{w}(t,z) := \sum_{n=0}^{\infty} \frac{\lambda^n(\partial_{\tilde{m}_2,z})\tilde{\varphi}(z)}{m_1(n)} t^n$ is $\tilde{k}$-summable in a direction d, where $\tilde{m}_2(u) := m(u)m_2(u)$ is a moment function of order $\tilde{k}_2 := (1/k_2 + 1/k)^{-1}$.

Observe that $\hat{w}$ is a formal solution of

$$(\partial_{m_1,t} - \lambda(\partial_{\tilde{m}_2,z}))w = 0, \qquad w(0,z) = \tilde{\varphi}(z) = \sum_{n=0}^{\infty} \frac{\varphi_n}{m(n)} z^n \in \mathcal{O}(D).$$

Applying the previous theorem we obtain the assertion. □

References

W. Balser
Formal power series and linear systems of meromorphic ordinary differential equations,
Springer-Verlag, New York, 2000.

W. Balser, M. Yoshino
Gevrey order of formal power series solutions of inhomogeneous partial differential equations with constant coefficients,
Funk. Ekvac. 53 (2010), 411–434.

S. Michalik
On the multisummability of divergent solutions of linear partial differential equations with constant coefficients,
J. Differential Equations 249 (2010), 551–570.