Reduction into Hukuhara-Turrittin's canonical form of a singular system of ordinary differential equations

MASATAKE MIYAKE (Nagoya University)

0 Introduction

We are concerned with a system or a matrix of ordinary differential operators which is singular in a neighborhood of the origin in the complex plane,

$$L = (p, A(x)) \qquad L = x^{p+1}\frac{d}{dx}I_N - A(x), \quad A(x) \in M_N(\mathbb{C}\{x\}),\ A(0) \neq O,\ p > 0.$$

By the theory of Hukuhara(1937) - Turrittin(1952), we know that the homogeneous equation $Lu = 0$ has a formal fundamental system of solutions $W(x)$ of the following form,

$$W(x) = P(x^{1/r})x^C \exp[\Lambda(x^{-1/r})],$$

where $P(x^{1/r})$ is an invertible matrix of formal meromorphic series of $x^{1/r}$ ($\exists r \in \mathbb{N}$), C is a constant matrix and $\Lambda(x^{-1/r})$ is a diagonal matrix with each diagonal entry of polynomials of $x^{-1/r}$ without constant term of degree at most p. The diagonal matrix $\Lambda(x^{-1/r})$ is called the **determining factor** which characterizes the irregular singularity of the system L.

In fact, the regular singularity of L at $x = 0$ is characterized by $\Lambda(x^{-1/x}) = O$. In this case, we can take $r = 1$ and $P(x)$ has entries of meromorphic functions.

The expression of $W(x)$ is obtained from the reduced canonical form of L which is obtained by the formal transformation matrix $P(x^{1/r})$. Precisely, by the matrix $P(x^{1/r})$ the system L is reduced into

$$P^{-1}LP = x^{p+1}\frac{d}{dx}I_N - B(x), \quad B(x) = P^{-1}AP - x^{p+1}P^{-1}\frac{d}{dx}P, \tag{0.1}$$

with

$$B(x) = \bigoplus_{j=1}^{k} B_j(x), \quad B_j(x) = p_j(x)I_{n_j} + J_{n_j}, \ \sum_{j=1}^{k} n_j = N, \tag{0.2}$$

where J_{n_j} denotes the Jordan canonical form of a nilpotent matrix of size n_j, and $p_j(x)$ is a polynomial of $x^{1/r}$ of order at most p

These results are commonly known, but unfortunately for me, for a given system $L = (p, A(x))$, we did not know an effective algorithm for the reduction procedure into a canonical form.

The purpose of this talk is to present an algorithm for the reduction into a canonical form by using the new expansion for matrix functions called T**-expansion** instead of the Taylor expansion which may be called the **flat expansion**.

The reduction procedure is done like peeling an onion one piece and one piece. We reveal, in conclusion, that a piece of onion indicates the Newton polygon for each subsystem which appears in the reduction procedure.

At the end of the talk, we shall sho some numerical experiments which were done by Mathematica.

This talk is based on the paper which will appear from Funkcialaj Ekvacioj,

"Newton polygon and Gevrey hierarchy in the index formulas for a singular system of ordinary differential equations"

1 Fundamental Notations and Definitions

Let a singular system $L = (p, A(x))$ be given by

$$(p, A(x)) \qquad L = x^{p+1}\frac{d}{dx} - A(x), \quad A(x) = (a_{i,j}(x))_{i,j=1,\cdots,N} \in M_N(\mathcal{R}[[x]]),$$

where $p \in \mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathcal{R}[[x]]$ denotes the set of formal Laurent series of x.

[0] Order of zeros $O[A]$ $O[A] \in \mathbb{Z}$ denotes the order of zeros of $A(x)$ at $x = 0$,

$$A(x) = \sum\nolimits_{n=O[A]}^{\infty} A_n x^n, \quad A_{O[A]} \neq O.$$

[1] The Volevič weight or the V-weight $V[A] \in \mathbb{Q} \cup \{+\infty\}$

Let $r_{ij} = O(a_{ij}) \in \mathbb{Z} \cup \{+\infty\}$, the order of zeros at $x = 0$. This means that if $a_{ij}(x)$ has pole singular at $x = 0$, $-r_{ij}$ denotes the order of pole. If $a_{ij}(x) \equiv 0$, then $O(a_{ij}) = +\infty$.

The Volevič weight or the V-weight $V[A] \in \mathbb{Q} \cup \{+\infty\}$ is defined by

$$(1.1) \qquad V[A] := \min_{1\leq n\leq N} \min_{1\leq i_1<i_2<\cdots<i_n\leq N} \min_{\sigma\in S_n} \frac{1}{n}\sum_{k=1}^{n} r_{i_k,i_{\sigma(k)}},$$

where $+\infty + r := +\infty$ and $+\infty > r$ for $\forall r \in \mathbb{Q}$, and S_n denotes the permutation group of $\{1, 2, \cdots, n\}$. It is obvious that $V[A] \geq O[A]$.

[2] The V-numbers $T[A] = \{t[i]\}_{i=1}^N$ associated with $V[A] \in \mathbb{Q}$

If $V[A] \in \mathbb{Q}$, then $\exists T[A] = \{t[i]\}_{i=1}^N \subset \mathbb{Q}$, which is called V-numbers, s.t.

$$r_{ij} \geq t[i] - t[j] + V[A], \quad \forall i, j = 1, 2, \cdots, N. \tag{1.2}$$

If $V[A] = m/K \in \mathbb{Z}/K$ $(1 \leq \exists K \leq N)$, then we can take $T[A] = \{t[i]\} \subset \mathbb{Z}/K$.

The proof is done as follows. Let $s_{i,j} = K\{r_{i,j} - V[A]\} \subset \mathbb{Z} \cup \{+\infty\}$. Then we have

$$\sum_{k=1}^{n} s_{i_k, i_{\sigma(k)}} \geq 0, \quad 1 \leq i_1 < \cdots < i_n \leq N,\ \forall \sigma \in S_n,\ 1 \leq \forall n \leq N. \tag{1.3}$$

Then *Volevič's lemma* (Dokl. 1960) says that $\exists \{s[i]\} \subset \mathbb{Z}$ s.t.

$$s_{i,j} \geq s[i] - s[j], \quad \forall i, j. \tag{1.4}$$

Then $T[A] = \{t[i] = s[i]/K\}$ becomes the desired V-numbers.

An actual determination of the numbers $\{s[i]\}$ was obtained by M. Miyake (RIMS. Kyoto Univ., 1979) (cf. Miy-Ichinobe (Funk.Ekvac., 2009)) as follows.

Since the numbers $\{s[i]\}$ are free from translation, we take $s[1] = 0$. The others are chosen inductively as follows. Suppose that $s[p]$ are chosen for $1 \leq p < i$. Then $s[i] \in \mathbb{Z}$ is taken in the non trivial interval

$$(1.5)\qquad \max_{1\le p<i}\left\{s[p]-s_{p,i_1}-s_{i_1,i_2}-\cdots-s_{i_k,i}\right\} \le s[i] \le \min_{1\le q<i}\left\{s_{i,j_1}+s_{j_1,j_2}+\cdots+s_{j_\ell,q}+s[q]\right\},$$

where $\{i_1,\cdots,i_k\},\{j_1,\cdots,j_\ell\}\subset\{i,i+1,\cdots,N\}$.

The following estimation is called the span condition which plays a important role when we estimate the pole order of the transformation matrix into its canonical form.

Proposition 1 (span condition) We can choose the V-numbers $T[A]=\{t[i]\}$ associated with $V[A]$ which satisfies

$$(1.6)\qquad \sigma(T[A])=\max_{i,j}|t[i]-t[j]|\le(N-1)(V[A]-O[A]).$$

In the reduction procedure of the system into the canonical form we always take the V-numbers (associated with the V-weight) which satisfies the span condition (1.6).

[3] Singular system of first kind in V-sense, that is, $V[A] \geq p$

When $V[A] \geq p$, we call the system $L = (p, A(x))$ to be a singular system of first kid in V-sense.

In this case, we can find the numbers $T[A] = \{t[i]\} \subset \mathbb{Z}$ instead of the V-numbers associated with $V[A]$ such that

$$r_{ij} \geq t[i] - t[j] + p, \quad \forall i, j = 1, 2, \cdots, N. \tag{1.7}$$

We define a diagonal matrix function $x^T = \mathrm{Diag}\left[x^{t[1]}, \cdots, x^{t[N]}\right]$, and transform the system $L = (p, A(x))$ by x^T,

$$\begin{aligned} x^T : L \quad &\mapsto \quad x^{-T} L T = x^p \left\{ x \frac{d}{dx} - B(x) \right\} = x^p \, (0, B(x)), \\ &B(x) = x^{-p} x^{-T} A(x) x^T - T \in M_N(\mathbb{C}\{x\}). \end{aligned} \tag{1.8}$$

Thus the system $L = (p, A(x))$ is reduced into a singular system of first kind in usual sense $(0, B(x))$.

Hence our interest is in the case $V[A] < p$.

[4] (Principal matrix and principal matrix function)

For a matrix function $A(x) = (a_{ij}(x))$, let $V[A] < p$ and the associated V-numbers be $T[A] = \{t[i]\}$. Then,

$$a_{ij}(x) = \left\{ \overset{\circ}{a}_{ij} + o(1) \right\} x^{t[i]-t[j]+V[A]}, \qquad \overset{\circ}{a}_{ij} = 0 \text{ if } t[i] - t[j] + V[A] \notin \mathbb{N}. \tag{1.9}$$

The following matrices are called the principal matrix and the principal matrix function, respectively.

$$\overset{\circ}{A} = \left(\overset{\circ}{a}_{ij} \right), \qquad \overset{\circ}{A}(x) = \left(\overset{\circ}{a}_{ij} x^{t[i]-t[j]+V[A]} \right) \in M_N(\mathbb{C}[z]). \tag{1.10}$$

They depend on $T[A]$ which is not uniquely determined, but the eigenvalues do not. There are some definitions.

- **(full rank system)** If $\det(\overset{\circ}{A}) \neq 0$, the system $L = (p, A(x))$ is called of **full rank** of irregular singular type. In this case, we define a number $\rho(L)$ by

  $$\rho(L) = p - V[A] \; (> 0), \tag{1.11}$$

  which we call the **irregularity** of the system $L = (p, A(x))$. Exactly saying, the **number irregularity** indicates the possible **maximal exponential growth order** of solutions $u(x)$ of $Lu = 0$ as $x \to 0$.

Let $\{\alpha_j\}_{j=1}^{N}$ be the eigenvalues of the principal matrix $\overset{\circ}{A}$. Then the leading term of the determining factor $\Lambda(x)$ of the fundamental system of solutions $W(x)$ of $Lu = 0$ is given by

$$\text{Diag}\left[-\frac{\alpha_1}{\rho(L)}x^{-\rho(L)}, \cdots, -\frac{\alpha_N}{\rho(L)}x^{-\rho(L)}\right]. \tag{1.12}$$

- **(Non degenerate system)** If $\overset{\circ}{A}$ is **non nilpotent**, L is called a non-degenerate system of irregular singular type. In this case, we have $\rho(L) = p - V[A]$ for the irregularity.

- **(Degenerate system)** If $\overset{\circ}{A}$ is **nilpotent**, L is called a degenerate system. In this case we know that

$$\rho(L) \leq p - V[A] - 1/N(N-1). \tag{1.13}$$

- **(Irreducibility)** We call the system $L = (p, A(x))$ to be irreducible if it is a **full rank** system of irregular singular type **or** a **first kind** system in V-sense, that is, $V[A] \geq p$.

Remark. The irregularity $\rho(L)$ was characterized in various form in the paper with K. Ichinobe (Funkcialaj Ekvacioj, 2009).

2 First Step of Reduction; Irreducible Decomposition

The reduction procedure is done to prove the following decomposition theorem.

Theorem 1 (Irreducible decomposition) *Let a system $L = (p, A(x))$ with $A(x) \in M_N(\mathbb{C}\{x\})$ be given. Then by a formally invertible matrix $P_\delta(x) \in GL_N(\mathcal{R}[[x]])$ the system L is decomposed by irreducible subsystems,*

$$(2.1)\quad P_\delta(x) : L \;\mapsto\; L_{P_\delta} := P_\delta^{-1}\, L\, P_\delta = \bigoplus_{i=1}^{k+1} L_i, \qquad L_i = (p, A_i(x)),\;\; A_i(x) \in M_{N_i}(\mathcal{R}[[x]])$$

with $V[A_1] < V[A_2] < \cdots < V[A_k] < p \leq V[A_{k+1}]$ and $\sum_i^{k+1} N_i = N$.

Moreover, the pole order of $P_\delta(x)$ and $A_i(x)$ are estimated by

$$(2.2)\quad \begin{cases} O(A_i) \geq -\displaystyle\prod_{j=1}^{i}(M_j - 1)\{p - O(A)\} + p, \quad (1 \leq i \leq k), \\ O[P_\delta] \geq -\displaystyle\sum_{j=1}^{k}\prod_{j=1}^{i}(M_j - 1)\{p - O[A]\}, \end{cases}$$

$O[A_{k+1}] \geq O[A_k]$, where $M_1 = N$ and $M_j = N - (N_1 + \cdots + N_{j-1})$ $(j \geq 2)$.

Theorem 1 is proved after establishing several proposition.

The first step is the following Proposition.

[1] Reduction into non degenerate system [Miy-Ich, Funk. Evac., 2009]

For every system $L = (p, A(x))$, we can find $Q(x) \in GL_N(\mathbb{C}[x])$ such that which reduces L into either a first kind system in V-sense or a non degenerate system of irregular singular type, that is, for

$$Q(x):\ L = (p, A(x)) \ \mapsto\ L_Q = (p, B(x)), \tag{2.3}$$

we have either $V[B] \geq p$ or in case $V[B] < p$ with property that $\overset{\circ}{B}$ is not nilpotent. In latter case, $\overset{\circ}{B}$ has a form

$$\overset{\circ}{B} = \begin{bmatrix} B^{11} & B^{12} \\ O & B^{22} \end{bmatrix}, \quad \det B^{11} \neq 0, \quad B^{22} = \begin{bmatrix} 0 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{2.4}$$

The proof is done as follows. It is enough to study the case where $V[A] < p$ and $\det \overset{\circ}{A} = 0$. We assume that $t[1] \leq t[2] \leq \cdots \leq t[N]$ for the V-numbers $T[A]$. Then we can take a left null vector $\vec{\ell} = (\ell_1, \cdots, \ell_{k-1}, 1, 0, \cdots 0)$ of $\overset{\circ}{A}$ such that

$$\vec{\ell}(x) = (\ell_1 x^{t[k]-t[1]}, \cdots, \ell_{k-1} x^{t[k]-t[k-1]}, 1, 0, \cdots, 0) \quad \text{(polynomial components)}$$

becomes a left null vector of $\overset{\circ}{A}(x)$. This means $\ell_i = 0$ if $t[k] - t[i] \notin \mathbb{N}$. Now we define

$$R^{-1}(x) = \begin{array}{r} \\ \\ \\ k > \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ & \cdots & \vec{\ell}(x) & \cdots & \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in GL_N(\mathbb{C}[x]).$$

Let $L_R = (p, B(x))$. If $V[B] \geq p$, the proof ends. If $V[B] = V[A] < p$, then $\overset{\circ}{B} = R^{-1}(1)\overset{\circ}{A}R(1)$ shows that the k-th row vector vanishes. We exchange the k-row and the N-th row by an equivalent transformation matrix, and we repeat the above procedure to the principal minor matrix of size $N-1$ which is obtained by removing the N-th row and column.

By continuing such operations, we get the desired reduced system. $\square$

[2] Decomposition into two blocks $\Longrightarrow$ Completion of Proof

Let $L_Q = (p, B(x))$ *be the system* (2.3) *with the property* (2.4). *Then* $\exists R(x) \in GL_N(\mathcal{R}[[x]])$ *s.t.*

$$(2.5) \qquad R(x): \ L_Q = (p, B(x)) \ \mapsto \ L_{QR} = (p, C(x)), \quad C(x) = C^{11}(x) \oplus C^{22}(x)$$

with the following properties; $V[B] = V[C]$, *the* V*-numbers* $T[B]$ *for* $B(x)$ *becomes the* V*-numbers for* $C(x)$ *again, and they have a common principal matrix, i.e.,* $\overset{\circ}{C} = B^{11} \oplus B^{22}$, *where* $\det B^{11} \neq 0$ *and* B^{22} *is an upper triangular matrix in strict sense.*

This shows that $L_1 := (p, C^{11}(x))$ is a full rank system of irregular singular type with $V[C^{11}] = V[B](< p)$. On the other hand, for $\tilde{L}_2 = (p, C^{22}(x))$, since the principal matrix for $C^{22}(x)$ is upper triangular matrix in strict sense, we have

$$V[C^{22}| > V[B^{11}].$$

Remark. 1) We continue the above procedures for the subsystem $\tilde{L}_2 = (p, C^{22}(x))$ and so on, we finally obtain the irreducible decomposition (Theorem 1).

2) The pole order the the transformation matrix $Q(x)R(x)$ and the growth of pole order of $C(x)$ in the reduced system is estimated by the span condition, $\sigma(T[B]) \leq (N-1)(V[B]-O[B]) \leq (N-1)(p - O[A])$ as follows.

$$
(2.6)\qquad \begin{cases} O[QR] \geq -\sigma(T[B]) \geq -(N-1)(p-O[A]), \\ O[C] \geq -\sigma(T[B]) + V[B] \geq -(N-1)(V[B]-O[B]) + V[B] \\ \qquad\qquad\qquad\qquad \geq -(N-2)(p-O[A]) + O[A] \quad (\because O[A] \leq O[B]) \end{cases}
$$

Idea of proof of [2]

We reduce the system $L_Q = (p, B(x))$ into an **upper triangular system** with the same principal matrix $\overset{\circ}{B}$ associated with the V-numbers $T[B]$ for $B(x)$.

This is done by killing the $(2,1)$ block $B^{21}(x)$ of $B(x)$ by a transformation matrix $R(x) \in GL_N(\mathcal{R}[[x]])$ of the form

$$
(2.7)\qquad R^{-1}(x) = \begin{bmatrix} I & O \\ R^{21}(x) & I \end{bmatrix}, \quad V[R] = 0 \text{ with the same V}-\text{numbers } T[B].
$$

To do the requested task we introduce the T**-expansion** associated with the V-numbers $T[B]$ as follows.

Let $V[B] = m/K \in \mathbb{Z}/K$. Then $T[B] = \{t[i] = s[i]/K\} \subset \mathbb{Z}/K$. Then

$$b_{ij}(x) = \sum_{k=0}^{\infty} b_k^{ij} x^{(s[i]-s[j]+m+k)/K}, \quad B(x) = (b_{ij}(x))_{i,j=1,2,\cdots,N}.$$

Note that $b_k^{ij} = 0$ is $(s[i] - s[j] + m + k)/K \notin \mathbb{Z}$.

Then the following expansion is called the T**-expansion** associated with $T[B]$.

$$B(x) = \sum_{k=0}^{\infty} B_k(x), \quad B_k(x) = \left(b_k^{ij} x^{(s[i]-s[j]+m+k)/K} \right), \; B_0(x) = \overset{\circ}{B}(x). \tag{2.8}$$

The usual Taylor expansion may be called the **flat-expansion**, i.e., $\{t[i] = 0\}$.

The reduction is done by killing the (2,1)-block $B^{21}(x)$ in the order of T-expansion $\{B_k^{21}(x)\}_{k=1}^{\infty}$ ($B_0^{21}(x) \equiv O$) by using $B_0^{11}(x)$. Actually, the transformation matrix is obtained by repeated operations of the following form,

$$\begin{bmatrix} I & O \\ B_k^{21}(x)(B_0^{11})^{-1}(x) & I \end{bmatrix}, \quad k = 1, 2, 3, \cdots \tag{2.9}$$

This is only symbolical, and we need to recalculate the coefficient matrices at each change of system.

After completing the blocked upper triangular system, we continue the reduction to kill the (1,2) block. It is done similarly. $\square$

Remark

1) (Newton polygon and index formula) In case $L = (p, A(x))$ with $A(x) \in M_N(\mathbb{C}\{x\})$, it is interesting to know how we define the Newton polygon $\mathrm{N}(L)$ which is a basic notion in he study of summability problem. It is actually done by using Theorem 1 as follows; $\mathrm{N}(L) = N(\det L_{P_\delta}(x, d/dx))$, where $\det L_{P_\delta}(x; d/dx)$ denotes a single operator by taking a symbolical determinant of $L_{P_\delta}(x; d/dx)$. Then the index formula of the operator $L = (p, A(x))$ on formal Gevrey spaces are obtained from the coordinates of vertexes of $\mathrm{N}(L)$ (cf. J.-P. Ramis).

2) (multi-summability) The multi-summability of the transformation matrix $P_\delta(x)$ is understand by the following observation. In the transformation matrix $R(x)$ in (2.5) in the statement [2], the (2,1) block $R^{21}(x)$ of $R^{-1}(x)$ (cf. (2.7)) satisfies the following singular nonlinear equation,

$$x^{p+1}\frac{d}{dx}R^{21} = B^{21}(x) + B^{22}(x)R^{21} - R^{21}B^{11}(x) - R^{21}B^{12}(x)R^{21}, \tag{2.10}$$

where $B^{ij}(x)$ are holomorphic at $x = 0$, B_0^{22} is an upper triangular matrix in strict sense and $\det B_0^{11} \neq 0$. This equation is reduced into the usual one by making the expansion flat, for which the theory of Braaksma is applicable.

3 Second Step of Reduction; for Each Subsystem

Let suppose that the system $L = (p, A(x))$ $(A(x) \in M_N(\mathbb{C}\{x\})$ is decomposed by irreducible subsystems by $P_\delta(x) \in GL_N(\mathcal{R}[[x]])$, which is multi-summable,

$$
\text{(3.1)} \qquad P_\delta(x) : \; L = (p, A(x)) \;\mapsto\; L_{P_\delta} = \bigoplus_{i=1}^{k+1} L_i = (p, A_i(x)), \quad A_i(x) \in M_{N_i}(\mathcal{R}[[x]]
$$

$$
V[A_1] < V[A_2] < \cdots < V[A_k] < p \leq V[A_{k+1}].
$$

As we have seen, this decomposition gives the Newton polygon $\mathrm{N}(L)$. Our nest task is to deal each irreducible factor $L_i = (p, A_i(x))$ $A_i(x) \in N_i(\mathcal{R}[[x]])$ in order to reduce into the study in the first step. This task is like peeling an onion one peace and one piece. This means that the Newton polygon $\mathrm{N}(L)$ is like one piece of onion's peel.

3.1 Each subsystem

[1] (L_{k+1} ; first kind system in V-sense)

Let the associated V-numbers be $T_{k+1} = \{t_\ell^{(k+1)}\}_{\ell=1}^{N_{k+1}} \subset \mathbb{Z}$. Then

$$
\text{(3.2)} \qquad x^{T_{k+1}} : L_{k+1} \;\mapsto\; \tilde{L}_{k+1} = x^p(0, B_{k+1}(x)), \quad B_{k+1}(x) \in M_{N_{k+1}}(\mathbb{C}[[x]]),
$$

which is a singular system of first kind in the usual sense.

The reduction procedure ends here to this factor.

[2] $L_i = (p, A_i(x))$ $(1 \leq 1 \leq k)$ **; full rank system of irregular singular type**

- **(flatten the T-expansion by fractional power)**

Take $V[A_i] = m_i/N_i \in \mathbb{Z}/N_i$. The associated V-numbers becomes $T_i = \{t^i[j] = s^i[j]/N_i\}_{j=1}^{N_i} \subset \mathbb{Z}/N_i$. Then

$$x^{T_i} : L_i \mapsto \tilde{L}_i = x^{p+1} D - x^{V[A_i]} C_i(x^{1/N_i}), \quad C_i(t) \in M_{N_i}(\mathbb{C}[[t]]), \tag{3.3}$$

with $C_i(0) = (\overset{\circ}{A}_i)_{T_i}$ for the principal matrix and $\det C_i(0) \neq 0$. The Poincaré rank is reduced to $p - V[A_i]$.

- **(introduction of fractional power $\Longrightarrow$ formal power series)**

A change of variable $t = x^a$ is denoted by $\big|_{t=x^a}$. Then

$$\big|_{t=x^{1/N_i}} : \tilde{L}_i(z, D_z) \mapsto \tilde{L}_i(t, D_t) = \frac{1}{N_i} t^{pN_i+1} D_t - t^{m_i} C_i(t), \quad pN_i > m_i, \tag{3.4}$$

with $\det C_i(0) \neq 0$.

Remark. The number of denominator N_i may not be best possible. Exactly saying, $V[A_i] \in \mathbb{Z}/K_i$ by some K_i which is a divisor of N_i. If one wants to take a common integer K for all subsystems L_i $(1 \leq i \leq k)$, it is enough we take $K = \text{L.C.M.}\{K_i\}_{1\leq i\leq k}$.

- **Constant matrix transformation matrix and blocked diagonalization**

Let $\{\alpha_\ell^{(i)}\}_{\ell=1}^{n_i}$ be the distinct eigenvalues of $C_i(0)$. Let a constant matrix $D_i \in GL_{N_i}(\mathbb{C})$ transform the matrix $C_i(0)$ into its Jordan canonical form J_i,

$$D_i:\ C_i(0) \ \mapsto\ J_i = \mathrm{Diag}[J(\alpha_1^{(i)}), \cdots, J(\alpha_{n_i}^{(i)})],$$

where $J(\alpha_\ell^{(i)}) \in M_{N_\ell^{(i)}}(\mathbb{C})$ is the collection of the Jordan blocks for $\alpha_\ell^{(i)}$.

By the well known reduction procedure (cf. the book by W. Wasaw), $(\tilde{L}_i)_{D_i}(t, D_t)$ is blocked diagonalizable by an invertible matrix $Q_i(t) \in GL_{N_i}(\mathbb{C}[[t]])$,

$$D_iQ_i(t)\ :\ \tilde{L}_i(t, D_t)\ \mapsto\ \mathcal{L}_i = \bigoplus_{\ell=1}^{n_i} \mathcal{L}_\ell^{(i)}(t, D_t), \tag{3.5}$$

with

$$\mathcal{L}_\ell^{(i)} = \frac{1}{N_i} t^{pN_i+1} D_t - t^{m_i} B_\ell^{(i)}(t),\ B_\ell^{(i)}(0) = J(\alpha_\ell^{(i)}). \tag{3.6}$$

• **Exponential transformation** Let $\Lambda_\ell^{(i)}(t) = \frac{N_i \alpha_\ell^{(i)}}{m_i - pN_i} t^{-(pN_i - m_i)} \cdot I_{N_\ell^{(i)}}$. We define

$$\exp[\Lambda_i(t)] = \exp\left[\mathrm{Diag}\left[\Lambda_1^{(i)}(t), \cdots, \Lambda_{n_i}^{(i)}(t)\right]\right]. \tag{3.7}$$

Then

$$\exp[\Lambda_i(t)] : \bigoplus_{\ell=1}^{n_i} \mathcal{L}_\ell^{(i)}(t, D_t) \ \mapsto \ \bigoplus_{\ell=1}^{n_i} \tilde{\mathcal{L}}_\ell^{(i)}(t, D_t), \tag{3.8}$$

where

$$\tilde{\mathcal{L}}_\ell^{(i)}(t, D_t) = \frac{1}{N_i} t^{pN_i+1} D_t - t^{m_i} \mathcal{A}_i(t), \quad \mathcal{A}_\ell^{(i)}(t) = B_\ell^{(i)}(t) - \alpha_\ell^{(i)} I_{N_\ell^{(i)}}. \tag{3.9}$$

This ends to peel one piece of onion.

[3] Turn back to the First Step of Reduction

Let us consider the sub-subsystem $\tilde{\mathcal{L}}_\ell^{(i)}(t, D_t)$ in (3.9).

The leading term $\mathcal{A}_\ell^{(i)}(0) = J(\alpha_\ell^{(i)}) - \alpha_\ell^{(i)} I_{N_\ell^{(i)}}$ $(\alpha_\ell^{(i)} \neq 0)$ is a collection of Jordan canonical forms of nilpotent matrices, that is, an upper triangular matrix in strict sense. This means that

$$O[t^{m_i} B_\ell^{(i)}(t)] = V[t^{m_i} B_\ell^{(i)}] = t^{m_i} < V[t^{m_i} \mathcal{A}_\ell^{(i)}]. \tag{3.10}$$

Now we repeat the reduction procedures from the the first step for each sub-subsystems $\{\tilde{\mathcal{L}}_\ell^{(i)}(t, D_t)\}_{i,\ell}$, and we continue so on.

SUMMARY

For a singular system $L = (p, A(x))$ $(A(x) \in M_N(\mathbb{C}\{x\}))$, at each stair in the reduction procedures, the transformation matrix consists of the following product of matrices,

$$P_\delta(x)\, x^{\{T_i\}}\, \big|_{t=x^{\{1/N_i\}}}\, D\, Q(t)\, \exp[\Lambda(t)], \tag{3.11}$$

with $P_\delta(x)$ which reduces into irreducible decomposition,

$$\begin{cases} x^{\{T_i\}} = \bigoplus_{i=1}^{k+1} x^{T_i}, \\ \big|_{t=x^{\{1/N_i\}}} = \bigoplus_{i=1}^{k+1} \big|_{t=x^{1/N_i}}, \qquad \big|_{t=x^{1/N_{k+1}}} := I_{N_{k+1}}, \\ D = \bigoplus_{i=1}^{k+1} D_i, \qquad D_{k+1} := I_{N_{k+1}}, \\ Q(t) = \bigoplus_{i=1}^{k+1} Q_i(t), \qquad Q_{k+1}(t) := I_{N_{k+1}}, \\ \Lambda(t) = \bigoplus_{\ell,i} \Lambda_\ell^{(i)}(t), \qquad \Lambda_*^{(k+1)}(t) := O_{N_{k+1}}. \end{cases} \tag{3.12}$$

4 Some numerical experiments

4.1 Degenerate system = Nilpotent principal matrix

The system $L = x^{p+1}\frac{d}{dx} - A(x) = (p, A(x))$ we consider is

$$L = (3, A(x)), \quad A(x) = \begin{bmatrix} x^2 & 3 & x \\ -x^4 & -2x^2 & x^4 \\ x^4 & x & x^2 \end{bmatrix}. \tag{4.1}$$

The V-weight and V-numbers; $V[A] = 2,\ T[A] = \{t[1] = 0,\ t[2] = 2,\ t[3] = 1\}$.

The principal matrix and its eigenvalues ;

$$\overset{\circ}{A} = \begin{bmatrix} 1 & 3 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad \{0,\ 0,\ 0\} \text{ (nilpotent)}.$$

Reduce the system into a non degenerate system.

First, exchange rows and columns in the order of V-numbers, i.e, the second and the third rows and columns are exchanged.

The system and the principal matrix functions become ;

$$L \sim \left(3, \begin{bmatrix} x^2 & x & 3 \\ x^4 & x^2 & x \\ -x^4 & x^4 & -2x^2 \end{bmatrix}\right), \qquad \begin{bmatrix} x^2 & x & 3 \\ 0 & x^2 & x \\ -x^4 & 0 & -2x^2 \end{bmatrix}.$$

A left null vector of the principal matrix function and a transformation matrix ;

$$(x^2,\ -x,\ 1), \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x^2 & x & 1 \end{bmatrix}.$$

The reduced system becomes ;

$$L \ \sim \ \left(3, \begin{bmatrix} -2x^2 & 4x & 3 \\ -x^3 + x^4 & 2x^2 & x \\ x^5 & 0 & 0 \end{bmatrix}\right).$$

To this system the principal matrix function and its principal minor ;

$$\begin{bmatrix} -2x^2 & 4x & 3 \\ -x^3 & 2x^2 & x \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} -2x^2 & 4x \\ -x^3 & 2x^2 \end{bmatrix}.$$

A left null vector of the principal minor and a transformation matrix ;

$$\left(-\frac{x}{2},\ 1\right), \qquad \begin{bmatrix} 1 & 0 & 0 \\ x/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The reduced system becomes ;

$$L \sim \left(3, \begin{bmatrix} 0 & 4x & 3 \\ x^4/2 & 0 & -x/2 \\ x^5 & 0 & 0 \end{bmatrix}\right) \underset{\text{put}}{=} (3, B(x)).$$

To this reduced system, the V-weight and V-numbers are

$$V[B] = \frac{7}{3}, \quad T[B] = \left\{t[1] = 0,\ t[2] = \frac{4}{3},\ t[3] = \frac{8}{3}\right\}$$

Then the principal matrix and its eigenvalues are

$$\begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & -1/2 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \left\{-2^{1/3},\ (-2)^{1/3},\ -(-1)^{2/3}2^{1/3}\right\}$$

Conclusion

- The reduced system $(3, B(x))$ is a full rank system of irregular type with a principal matrix of distinct eigenvalues. Therefore it is reduced into a diagonal system by a transformed matrix of formal power series of $x^{1/3}$.

- The leading term of the determining factor is

$$\begin{bmatrix} -\frac{3\alpha}{2}x^{-2/3} & 0 & 0 \\ 0 & -\frac{3\beta}{2}x^{-2/3} & 0 \\ 0 & 0 & -\frac{3\gamma}{2}x^{-2/3} \end{bmatrix},$$

  where $\{\alpha, \beta, \gamma\}$ are the above eigenvalues.

- The Newton polygon $N(L)$ is obtained from $q(x, \lambda) = \det(x^4 \lambda I - \overset{\circ}{B}(x)) = x^{12}\lambda^3 + 2x^7$. Hence, $N(L)$ has two vertexes $(0, 7)$ and $(3, 9)$ with a side of slope $2/3$.

4.2 Non degenerate system, but not of full rank

The system $L = (p, A(x))$ we consider is

$$(4.2) \qquad L = (4, A(x)), \quad A(x) = \begin{bmatrix} x^2 & 1 & x^2 \\ x^5 & x^2 & -x^3 \\ 2x^3 & x & x^2 \end{bmatrix}.$$

The V-weight and V-numbers ; $\quad V[A] = 2,\ T[A] = \{t[1] = 0,\ t[2] = 2,\ t[3] = 1\}$.
The principal matrix and its eigenvalues ;

$$\overset{\circ}{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}, \quad \left\{ \frac{3 \pm i\sqrt{7}}{2},\ 0 \right\}.$$

We exchange the row and columns in the order of V-numbers ; Then the system and the principal matrix function become

$$L \ \sim \left(4, \begin{bmatrix} x^2 & x^2 & 1 \\ 2x^3 & x^2 & x \\ x^5 & -x^3 & x^2 \end{bmatrix} \right), \quad \begin{bmatrix} x^2 & 0 & 1 \\ 2x^3 & x^2 & x \\ 0 & -x^3 & x^2 \end{bmatrix}$$

The left null vector of the principal matrix function and a transformation matrix ;

$$(-2x^2, x, 1), \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2x^2 & -x & 1 \end{bmatrix}$$

The reduced system and the principal matrix function of reduced form ;

$$L \sim x^5 \frac{d}{dx} - \begin{bmatrix} 3x^2 & -x + x^2 & 1 \\ 4x^3 & 0 & x \\ x^5 - 4x^6 & -2x^4 + x^5 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 3x^2 & -x & 1 \\ 4x^3 & 0 & x \\ 0 & 0 & 0 \end{bmatrix}$$

We kill $\{(3,1),(3,2)\}$ block. First, we consider the following matrix ;

$$\begin{bmatrix} 3x^2 & -x \\ 4x^3 & 0 \\ x^5 & -2x^4 \end{bmatrix}.$$

The left null vector of this matrix and a transformation matrix ;

$$(-2x^3,\ 5x^2/4,\ 1), \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2x^3 & -5x^2/4 & 1 \end{bmatrix}$$

The reduced system ;

$$\left(4, \begin{bmatrix} 3x^2+2x^3 & -x-x^2/4 & 1 \\ 4x^3+2x^4 & -5x^3/4 & x \\ -\frac{11}{2}x^6-6x^7 & -\frac{1}{16}x^5+5x^6/2 & -3x^3/4 \end{bmatrix}\right).$$

We consider the following matrix ;

$$\begin{bmatrix} 3x^2 & -x \\ 4x^3 & 0 \\ -\frac{11}{2}x^6 & -\frac{1}{16}x^5 \end{bmatrix}.$$

The left null vector and a transformation matrix ;

$$\left(-\frac{1}{16}x^4,\ \frac{91}{64}x^3,\ 1\right), \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \frac{1}{16}x^4 & -\frac{91}{64}x^3 & 1 \end{bmatrix}$$

The reduced system ;

$$\left(4, \begin{bmatrix} 3x^2+2x^3+\frac{1}{16}x^4 & -x-\frac{1}{4}x^2-\frac{91}{64}x^3 & 1 \\ 4x^3+2x^4+\frac{x^5}{16} & -\frac{5}{4}x^3-\frac{91}{64}x^4 & x \\ -\frac{213}{64}x^7+O[x]^8 & \frac{231}{128}x^6+O[x]^7 & -\frac{3}{4}x^3+\frac{87}{64}x^4 \end{bmatrix}\right)$$

Next, we kill $\{(1,3),(2,3)\}$ block $\begin{pmatrix} 1 \\ x \end{pmatrix}$

We consider the following matrix;

$$\begin{bmatrix} 3x^2 & -x & 1 \\ 4x^3 & 0 & x \end{bmatrix}$$

The right null vector of this matrix and a transformation matrix ;

$$\begin{bmatrix} -\frac{1}{4x^2} \\ \frac{1}{4x} \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & -\frac{1}{4x^2} \\ 0 & 1 & \frac{1}{4x} \\ 0 & 0 & 1 \end{bmatrix}$$

The reduced system is

$$\left(4, \begin{bmatrix} 3x^2 + 2x^3 + O[x]^4 & -x - \frac{1}{4}x^2 + O[x]^3 & -\frac{3}{4}x + O[x]^2 \\ 4x^3 + 2x^4 + O[x]^5 & -\frac{5}{4}x^3 + O[x]^4 & -\frac{5}{8}x^2 + O[x]^3 \\ O[x]^7 & O[x]^6 & -\frac{3}{4}x^3 + \frac{87}{64}x^4 + O[x]^5 \end{bmatrix} \right)$$

We make a further reduction. We consider the matrix

$$\begin{bmatrix} 3x^2 & -x & -\frac{3}{4}x \\ 4x^3 & 0 & -\frac{5}{8}x^2 \end{bmatrix}.$$

The right null vector of this matrix and a transformation matrix ;

$$\begin{bmatrix} \frac{5}{32x} \\ -\frac{9}{32} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & \frac{5}{32x} \\ 0 & 1 & -\frac{9}{32} \\ 0 & 0 & 1 \end{bmatrix}.$$

Then the reduced system is

$$\left(4, \begin{bmatrix} 3x^2 + O[x]^3 & -x + O[x]^2 & -\frac{1}{32}x^2 + O[x]^3 \\ 4x^3 + O[x]^4 & O[x]^3 & -\frac{1}{128}x^3 + O[x]^4 \\ O[x]^7 & O[x]^6 & -\frac{3}{4}x^3 + O[x]^4 \end{bmatrix} \right).$$

We make a reduction once more. We consider the following matrix

$$\begin{bmatrix} 3x^2 & -x & -\frac{1}{32}x^2 \\ 4x^3 & 0 & -\frac{1}{128}x^3 \end{bmatrix}.$$

The right null vector and a transformation matrix are

$$\begin{bmatrix} \frac{1}{512} \\ -\frac{13}{512}x \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & \frac{1}{512} \\ 0 & 1 & -\frac{13}{512}x \\ 0 & 0 & 1 \end{bmatrix}.$$

The reduced system is

$$\left(4,\ \begin{bmatrix} 3x^2+O[x]^3 & -x+O[x]^2 & O[x]^3 \\ 4x^3+O[x]^4 & O[x]^3 & O[x]^4 \\ O[x]^7 & O[x]^6 & -\frac{3}{4}x^3+O[x]^4 \end{bmatrix}\right).$$

To make the expansion to flat, we apply a transformation matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x^2 \end{bmatrix}$$

Then we get the following equivalent system ;

$$\left(4,\ \begin{bmatrix} 3x^2+O[x]^3 & -x^2+O[x]^3 & O[x]^5 \\ 4x^2+O[x]^3 & O[x]^3 & O[x]^5 \\ O[x]^5 & O[x]^5 & -\frac{3}{4}x^3+O[x]^4 \end{bmatrix}\right).$$

Conclusion:

- The principal parts of irreducible factors in canonical form ;

$$\left(4,\ \begin{bmatrix} 3x^2 & -x \\ 4x^3 & 0 \end{bmatrix}\right) \quad \left(\text{or} \quad \left(4,\ \begin{bmatrix} 3x^2 & -x^2 \\ 4x^2 & 0 \end{bmatrix}\right)\right), \qquad \left(4, -\frac{3x^3}{4}\right).$$

- Leading term of the determining factor of fundamental system of solutions;

$$\begin{bmatrix} -\frac{\alpha}{2}x^{-2} & 0 & 0 \\ 0 & -\frac{\beta}{2}x^{-2} & 0 \\ 0 & 0 & \frac{3}{4}x^{-1} \end{bmatrix},$$

  where $\{\alpha, \beta\} = \{(3 \pm i\sqrt{7})/2\}$.

  The system can be decomposed into single operators by a transformation of formal power series of x.

- The Newton polygon $N(L)$ has three vertexes $\{(0,7),\ (1,8),\ (3,12)\}$ and two non trivial sides of slopes 1 and 2.

4.3 Full rank system when the principal matrix has only one eigenvalue

The system we consider is

$$(4, A(x)), \quad A(x) = \begin{bmatrix} x + x^2 & x & 1 \\ x^2 & x - x^2 & x^3 \\ x^4 & x^3 & x + x^3 \end{bmatrix} \tag{4.3}$$

The V-weight and V-numbers ;

$$V[A] = 1, \qquad T[A] = \{t[1] = 0,\ t[2] = 0,\ t[3] = 1\}$$

The principal matrix $\overset{\circ}{A}$ and it eigenvalues are

$$\overset{\circ}{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \{1,\ 1,\ 1\}$$

We take an exponential transformation matrix

$$\begin{bmatrix} e^{-1/(3x^3)} & 0 & 0 \\ 0 & e^{-1/(3x^3)} & 0 \\ 0 & 0 & e^{-1/(3x^3)} \end{bmatrix},$$

The reduced system ;

$$\left(4,\ \begin{bmatrix} x^2 & x & 1 \\ x^2 & -x^2 & x^3 \\ x^4 & x^3 & x^3 \end{bmatrix}\right) = (4, B(x)). \tag{4.4}$$

Then $\quad V[B] = \frac{3}{2},\ T[B] = \left\{t[1] = 0,\ t[2] = \frac{1}{2},\ t[3] = \frac{3}{2}\right\}.$

The principal matrix $\overset{\circ}{B}$ and its eigenvalues ;

$$\overset{\circ}{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \{-1,\ 1,\ 0\}$$

The principal matrix function and its principal minor ;

$$\begin{bmatrix} 0 & x & 1 \\ x^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & x \\ x^2 & 0 \end{bmatrix}$$

We, first, kill $\{(3,1),\ (3,2)\}$ block $(x^4,\ x^3)$

We consider the matrix

$$\begin{bmatrix} 0 & x \\ x^2 & 0 \\ 0 & x^3 \end{bmatrix}$$

The left null vector of this matrix and a transformation matrix ;

$$(-x^2,\ 0,\ 1), \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x^2 & 0 & 1 \end{bmatrix}$$

The reduced system ;

$$\left(4,\ \begin{bmatrix} 2x^2 & x & 1 \\ x^2 + x^5 & -x^2 & x^3 \\ -x^4 + x^5 - 2x^6 & 0 & -x^2 + x^3 \end{bmatrix} \right)$$

Next we consider the matrix

$$\begin{bmatrix} 0 & x \\ x^2 & 0 \\ -x^4 & 0 \end{bmatrix}.$$

The left null vector and a transformation matrix ;

$$(0,\ x^2,\ 1), \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -x^2 & 1 \end{bmatrix}$$

The reduced system ;

$$\left(4,\ \begin{bmatrix} 2x^2 & x - x^2 & 1 \\ x^2 + x^5 & -x^2 - x^5 & x^3 \\ x^5 - 2x^6 + x^7 & -x^5 + 2x^6 - x^7 & -x^2 + x^3 + x^5 \end{bmatrix} \right)$$

Next we try to kill $\{(1,\ 3),\ (2,\ 3)\}$ block $\begin{pmatrix} 1 \\ x^3 \end{pmatrix}$.

We consider the matrix

$$\begin{bmatrix} 0 & x & 1 \\ x^2 & 0 & 0 \end{bmatrix}.$$

The right null vector and a transformation matrix ;

$$\begin{bmatrix} 0 \\ -1/x \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/x \\ 0 & 0 & 1 \end{bmatrix},$$

The reduced system ;

$$\left(4, \begin{bmatrix} 2x^2 & x - x^2 & x \\ x^2 + x^4 + x^5 + x^6 & -x^2 - x^4 + x^5 - x^6 & x^2 + x^3 + x^5 \\ x^5 - 2x^6 + x^7 & -x^5 + 2x^6 - x^7 & -x^2 + x^3 + x^4 - x^5 + x^6 \end{bmatrix} \right).$$

Next, we consider the matrix

$$\begin{bmatrix} 0 & x & 0 \\ x^2 & 0 & x^2 \end{bmatrix}.$$

The right null vector and a transformation matrix ;

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

The reduced system ;

$$\left(4,\ \begin{bmatrix} 2x^2+x^5+O[x]^6 & x-x^2-O[x]^5 & x-3x^2+O[x]^3 \\ x^2+x^4++O[x]^5 & -x^2+O[x]^4 & x^3-x^4+O[x]^5 \\ x^5+O[x]^6 & -x^5+O[x]^6 & -x^2+x^3+O[x]^4 \end{bmatrix}\right)$$

We consider the matrix

$$\begin{bmatrix} 0 & x & x \\ x^2 & 0 & 0 \end{bmatrix}.$$

The right null vector and a transformation matrix ;

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The reduced system ;

$$\left(4,\ \begin{bmatrix} 2x^2+O[x]^5 & x-x^2+O[x]^5 & -2x^2+O[x]^3 \\ x^2+x^4+O[x]^6 & -x^2+O[x]^4 & 2x^3+O[x]^4 \\ O[x]^5 & O[x]^5 & -x^2+x^3+O[x]^4 \end{bmatrix}\right)$$

In order to make the V-numbers flatten, we employ a transformation matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & x^{1/2} & 0 \\ 0 & 0 & x^{3/2} \end{bmatrix}.$$

The reduced system ;

$$\left(4,\ \begin{bmatrix} O[x]^2 & x^{3/2}+O[x]^{5/2} & O[x]^{7/2} \\ x^{3/2}+O[x]^{7/2} & O[x]^2 & O[x]^4 \\ O[x]^{7/2} & O[x]^4 & -x^2+O[x]^3 \end{bmatrix}\right)$$

Conclusion

- The principal parts of irreducible factors in canonical form;

$$\left(4,\ \begin{bmatrix} 0 & x \\ x^2 & 0 \end{bmatrix}\right) \quad \left(\text{or } \left(4,\ \begin{bmatrix} 0 & x^{3/2} \\ x^{3/2} & 0 \end{bmatrix}\right)\right), \qquad (4, -x^2)$$

The system is decomposed into single operators by a transformation of formal power series of $x^{1/2}$.

- First two terms of the determining factor ;

$$\begin{bmatrix} -\frac{1}{3x^3}+\frac{2}{5x^{5/2}} & 0 & 0 \\ 0 & -\frac{1}{3x^3}-\frac{2}{5x^{5/2}} & 0 \\ 0 & 0 & -\frac{1}{3x^3}+\frac{1}{2x^2} \end{bmatrix}$$

- Newton polygon $N(L)$ has two vertexes $\{(0,3),\ (3,12)\}$ with a side of slope 3, which is determined from the polynomial $p(x,\lambda)=(x^5\lambda-x)^3$.

4.4 Full rank system with the principal matrix has distinct eigenvalues

The system we consider is

$$L=(3,A(x)),\quad A(x)=\begin{bmatrix} x+4x^2 & 1+4x \\ 3x^2-4x^3 & -x-3x^3 \end{bmatrix}. \tag{4.5}$$

The V-weight and V-numbers are

$$V[A]=1,\quad T[A]=\{t[1]=0,\ t[2]=1\}.$$

The principal matrix and its eigenvalues ;

$$\overset{\circ}{A}=\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix},\quad \{2,-2\}$$

A transformation matrix to make the V-numbers flatten and the reduced system ;

$$\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}, \qquad \left(3,\ \begin{bmatrix} x+4x^2 & x+4x^2 \\ 3x-4x^2 & -x-4x^3 \end{bmatrix}\right)$$

A transformation matrix for diagonalization of the leading matrix, and the reduced system ;

$$\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}, \qquad \left(3,\ \begin{bmatrix} 2x+5x^2-x^3 & -7x^2+3x^3 \\ 3x^2+x^3 & -2x-x^2-3x^3 \end{bmatrix}\right) \underset{\text{put}}{=} (3, B(x)).$$

We try to diagonalize the system by Proposition 0.2. Let the transformation matrix be $\begin{pmatrix} 1 & p^{12}(x) \\ p^{21}(x) & 1 \end{pmatrix}$ with $p^{ij}(0) = 0$. Then $p^{ij}(x)$ are uniquely determined from the following equations,

$$\begin{cases} b^{11}(x)p^{12}(x) - p^{12}(x)b^{22}(x) = -b^{12}(x) + x^{3+1}\dfrac{d}{dx}p^{12}(x) + p^{12}(x)b^{21}(x)p^{12}(x) \\ b^{22}(x)p^{21}(x) - p^{21}(x)b^{11}(x) = -b^{21}(x) + x^{3+1}\dfrac{d}{dx}p^{21}(x) + p^{21}(x)b^{12}(x)p^{21}(x) \end{cases}$$

Here $b^{ij}(x)$ denotes the (i,j) entry of the coefficient matrix $B(x)$

Let $p^{21}(x) = ax + bx^2 + cx^3$ be an approximate solution for the firs equation. Then we get the following equations for $\{a, b, c\}$.

$$3 - 4a = 0,\ 1 - 6a - 4b = 0, -3a + 7a^2 - 6b - 4c = 0,$$

which are solved by

$$a = \frac{3}{4},\ b = -\frac{7}{8},\ c = \frac{111}{64}.$$

That is,

$$p^{21}(x) = \frac{3}{4}x - \frac{7}{8}x^2 + \frac{111}{64}x^3 + O[x]^4.$$

Similarly we have ;

$$p^{12}(x) = \frac{7}{4}x - \frac{27}{8}x^2 + \frac{443}{64}x^3 + O[x]^4.$$

Now the reduced system is

$$\left(3,\ \begin{bmatrix} 2x + 5x^2 - \frac{25}{4}x^3 + O[x]^4 & O[x]^5 \\ O[x]^5 & -2x - x^2 + \frac{9}{4}x^3 + O[x]^4 \end{bmatrix}\right).$$

Conclusion

- The determining factor is given by

$$\mathrm{Diag}\left[-x^{-2} - 5x^{-1},\ x^{-2} + x^{-1}\right]$$

- Newton polygon $N(L)$ has two vertexes $\{(0, 2),\ (2, 6)\}$ with a side of slope 2.

Reference

[Bal 1] W. Balser, *From Divergent Series to Analytic Functions*, Springer Lecture Notes, **1582**, Springer-Verlag, 1994.

[Bal 2] ———, *Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations*, Springer-Verlag, New York, 2000.

[Hs-Si] P-F. Hisieh and Y. Sibuya, *Basic theory of ordinary differential equations*, Springer-Verlag, New York, 1999.

[Mal] B. Malgrange, Sur les points singuliers des équations différentialles linéaires, Enseign. Math. **20** (1970), 146 - 176.

[Math] Wolfram Mathematica, 7

[Miy 1] M. Miyake, On Cauchy-Kowalevki's theorem for general systems, Publ.PIMS Kyoto Univ., **15**(1979), 315 - 337.

[Miy 2] ———, Newton polygon and index formula on formal Gevrey space for singular system of ordinary differential equations, in preparation.

[Miy-Ich] M. Miyake and K. Ichinobe, Irregularity for singular system of ordinary differential operators in complex domain, Funk. Ekvac., **52**(2009), 53 - 82.

[Miy-Yos]] M. Miyake and M. Yoshino, Toeplitz operators and an index theorem for differential operators on Gevrey spaces, Funk. Ekvac., **38** (1995), 329 - 342.

[Ram 1] J.-P. Ramis, Dévissage Gevrey, Astérisque **59/60** (1978), 173 - 204.

[Ram 2] ———, Théorèmes d'indices Gevrey pour les équations différentielles ordinaires, Mer. Amer. Math. Soc., **48**, No.296 (1984).

[Vol] L.R. Volevič, On general system of differential equations, Dokl. Acad. Nauk SSSR —bf 132 (1960), 20-23 = Soviet Math. Dokl., **1** (1960), 458 - 461.

[Was] W. Wasow, Asymptotic Expansions of ordinary differential equations, John Wiley, New York, 1965.