Generalized transseries and global asymptotics of ODEs

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- For linear ODEs we distinguish two types of singularities: *regular*, iff the general solution is given by a convergent Frobenius expansion –a series in z and $\ln z$ of the form $\sum_{\mathbf{j},\mathbf{k},\mathbf{l}} c_{\mathbf{j},k,l} z^{\mathbf{j}\cdot\mathbf{a}+k}(\ln z)^l$, and *irregular*, all the others. Here $|\mathbf{j}| \le 1$, $l \le n$, $k = 0, 1, ...,$.

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- Simplest: regular ones. For a scalar *nth* order ODE $y^{(n)} + a_1(z)y^{(n-1)} + \cdots + a_n(z)y = 0$, $z = 0$ is a regular singularity if for $j = 1, \ldots, n$, a_j has a pole of order at most j.

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- The equation $y'' + (z^{-1} - z^{-2})y' + z^{-3}y = 0$ has an irregular singularity at 0 and the general solution is

$$y = Ae^{-1/z} + Be^{-1/z}\mathrm{Ei}(1, -z^{-1});\ A = 0 \Rightarrow y \sim -B\sum_{k=1}^{\infty}(k-1)!z^k,\ z \to 0$$

- Placing the singularity at infinity, a generic nth order system at an irregular singular point (of rank one) can be brought the form

$$y' = \Lambda y + x^{-1}By + g(1/x)\ y + f_0(1/x); \quad y \in \mathbb{C}^n$$

and a nonlinear one to

$$y' = \Lambda y + x^{-1}By + g(1/x, y) + f_0(1/x);$$

$$\Lambda = \mathrm{diag}\lambda_i, B = \mathrm{diag}\beta_i \text{ const. matr., } f_0 = o(x^{-m})$$

$g = O(1/x^2, y/x^2, y^2)$ bianalytic at 0 and the eigenvalues of Λ are nonresonant.

- General formal solution in the linear case is a transseries

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{0<|\mathbf{k}|\le 1} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} x^{\mathbf{k}\cdot\boldsymbol{\beta}} \tilde{\mathbf{y}}_{\mathbf{k}}$$

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$$= \sum_{k_1,\dots,k_m; m\le n} (C_1 e^{-\lambda_1 x} x^{\beta_1})^{k_1} \cdots (C_m e^{-\lambda_m x} x^{\beta_m})^{k_m} y_{k_1,\dots,k_m}$$

where now C_j must be set to zero if $|x^{\beta_j} e^{-\lambda_j x}| \not\to 0$.

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where now C_j must be set to zero if $|x^{\beta_j} e^{-\lambda_j x}| \not\to 0. \Rightarrow$ generic loss of dimensions.

- In fact, this is a (formal) multiseries in x^{-1} and $\boldsymbol{\xi}$, $\xi_k = C_k e^{-\lambda_k x} x^{\beta_k}$,

$$\tilde{\mathbf{y}}(\boldsymbol{\xi}, x^{-1}) = \sum_{0\leq|\mathbf{k}|,j} c_{\mathbf{k},j} \boldsymbol{\xi}^{\mathbf{k}} x^{-j}$$

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- Answer to first question is yes; extensive literature, Écalle, Balser, Braaksma, Ramis, Malgrange,... and is based on generalized Borel summation $\mathcal{LB}$.
- In general, the answer to second one: in general, no. What lies beyond? Can more general expansions characterize uniquely the general solution?

- Generic nonlinear systems. The small formal solutions are represented by transseries,

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- (O.C., Duke Math. J., 1998) For generic systems of ODEs, the formal series $\tilde{\mathbf{y}}_{\mathbf{k}}(x^{-1})$ are generalized Borel summable in a common domain $|\boldsymbol{\xi}| < \varepsilon$, and $|\mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}}| \le \mathbf{c}^{\mathbf{k}}$ indep of $\mathbf{k}, \varepsilon$. The function series
$$\sum_{0\le|\mathbf{k}|} \boldsymbol{\xi}^{\mathbf{k}}\mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}}$$
converges and gives the general *small* solution of the system.

- Transseries contain the complete description (in sectors near ∞) of the general solution of linear ODEs and small solutions of nonlinear ones: $\tilde{\mathbf{y}} = \sum_{\mathbf{k},j} \boldsymbol{\xi}^{\mathbf{k}} x^{-j} = \tilde{\mathbf{y}}(\boldsymbol{\xi}, x)$, with $\boldsymbol{\xi}, x^{-1}$ small.

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- However, except for first order eq., small solutions form a lower dimensional manifold (because of the cond. $\xi_i \to 0$). More generally solutions that are not small are not covered in any generality by classical or exponential asymptotics.

- Two examples of nonlinear equations:

$$y' = y^3 + z \quad \text{Abel's equation, nonintegrable} \tag{1}$$

$$u'' = u^2 + z \quad \text{Painlevé P1, integrable} \tag{2}$$

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- Their normal forms at infinity:

$$y' + 3y^3 - \frac{1}{9} + \frac{1}{5x}y = 0 \tag{3}$$

$$u'' + \frac{u'}{x} - u - \frac{u^2}{2} - \frac{392}{625x^4} = 0 \ (*) \tag{4}$$

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- First order equations are special –no dimensionality loss in transseries for small solutions:

$$\tilde{y} = \tilde{y}_0 + \sum_{k=1}^{\infty} C^k e^{-kx} x^{k\beta} \tilde{y}_k =: \tilde{y}_0 + \sum_{k=1}^{\infty} \xi^k \tilde{y}_k = y(\xi, 1/x) \quad (1)$$

The transseries is valid if $\xi = Ce^{-x}x^{\beta}$ is small. What happens when ξ is not small? Thinking of the transseries as a double series, we simply do not expand in ξ anymore. Then,

$$y(\xi, 1/x) = \sum_{k=0}^{\infty} F_k(\xi) x^{-k}; \quad (2)$$

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- (2) is valid until $\xi \neq o(x)$.
- A simple way to find the F_ks is by inserting (2) in the eq. and solving order by order the autonomous equations for F_k (by quadratures).

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- (1) is valid for any fixed $\xi \in \mathbb{C}$. This includes the transseries region, $\xi \ll 1$. But it breaks down if $\xi \neq o(x)$.

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- $y \sim \sum_{k=0}^{\infty} x^{-k} F_k(Ce^{-x}x^{\beta})(1)$, $Ce^{-x}x^{\beta} = o(x)$ works for arbitrary but fixed ξ (away from singularities). A better behaved quantity: $C = C(y,x)$: it remains constant whether y is small or not.

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- Solving $F_0(Ce^{-x}x^{\beta}) = y + \cdots$ for C we get $C = e^x x^{-\beta} F_0^{-1}(y) + \cdots$ and taking the log,

$$C = C(y, x) = x - \beta \log x + K_0(y) + x^{-1} K_1(y) + O(1/x^2)(*)$$

an asymptotic constant of motion.

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- **Rigidity:** If C is presented in additive form, then (*), with $-\beta \log x$, etc. is unique up to trivial transformations.
- To obtain K_j: solve $\frac{dC}{dx} := C_x + C_y y' = 0$, order by order (in $1/x$). The eqns. are solved by by quadratures (once more, we are dealing with autonomous systems).

Analysis of general solutions; first order ODEs

Consider first the simplest case, scalar first order equations.

- Typically, they can be brought to the form

$$y' = P_0(y) + Q(y, 1/x) = \sum_{k=0}^{\infty} \frac{P_k(y)}{x^k} \tag{6}$$

Assume for simplicity P_k are polynomials (analytic would be OK; genericity conditions are imposed). Solutions are described by transseries when/if $P_0(y) \ll 1$.

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- Global description: assuming x_0 is regular and $y_0 = y(x_0)$, one calculates, say by local power series, the solution in a compact set, while outside it, asymptotic formulae give accurate description.

$y' = P_0(y) + Q(y, 1/x)$; $C = x - \beta \ln x + K_0(y) + K_1(y)/x + \cdots$.

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Given an IC:

1. When $P_0(y)$ is small y is given by a transseries in a half plane.
2. In the complementary regions $\{(y,x) : |P_0(y)| > \epsilon > 0, |x| > R, \arg(x) = \varphi(|x|)\}$, a constant of motion C describes solutions. The asymptotic expansion of C in $1/x$ can be calculated by order by quadratures.

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3. The different regions overlap and the constants of motion match each other.

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- This description incorporates transseries and describes the *general solution*.
- For (2): $C(y(x), x) = const \Rightarrow C_y y' + C_x = 0 \Rightarrow K_0'(y) P_0(y) + 1 = 0$ etc.

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  The normalized form of this equation is obtained by the transformation $u(x) = Bx^{1/5}y(x)$ where $x = At^{5/3}$ with $A = -B^2/5, 15/B^5 = -1/9$ is

$$y' + 3y^3 - \frac{1}{9} + \frac{1}{5x}y = 0 \tag{7}$$

- In regions where $y(x)$ is not too small, the constant is given asymptotically by an expansion of the general form mentioned before ($\beta = 1/5$ here),

$$C(y,x) = x - \frac{1}{5}\log x + K_0(y) + \frac{K_1(y)}{x} + \frac{K_2(y)}{x^2} + O(x^{-3})$$

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- In the PDE we substitute $C = x - \frac{1}{5}\ln x + K_0(y) + x^{-1}K_1(y) + \cdots$, and solve, order by order in x^{-k}. It follows that $(27y^3 - 1)K_0' = 1$ etc. Continuing and solving, we get:

$$K_0(y) = \sqrt{3}\arctan\left(\frac{6y+1}{\sqrt{3}}\right) - \log(3y-1) + \frac{1}{2}\log(9y^2+3y+1)$$

$$K_1(y) = \frac{1}{10}\left(\frac{54y^2}{1-27y^3} - 4\sqrt{3}\arctan\left(\frac{6y+1}{\sqrt{3}}\right)\right) + \frac{1}{25}$$

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- rotating more than $\sim \pi/2$ we enter a narrow region with singularities, three arrays in all. There the solution is given by inverting C.

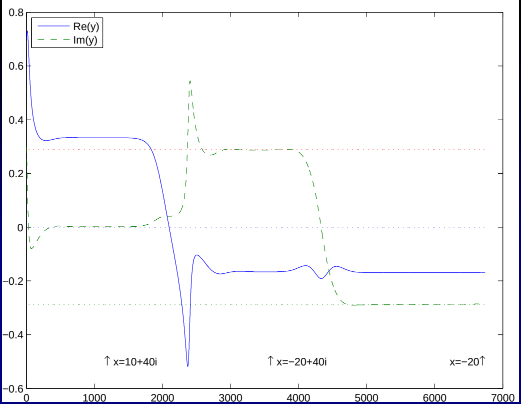


Figure: Monodromy at ∞ of general solution. horizontal axis=arclength. Dotted horizontal lines are the imaginary parts of the three roots. There are two periodic arrays of singularities in every transition region. In integrable systems, connection formulas follow (in nonintegrable equations, the solutions are multivalued).

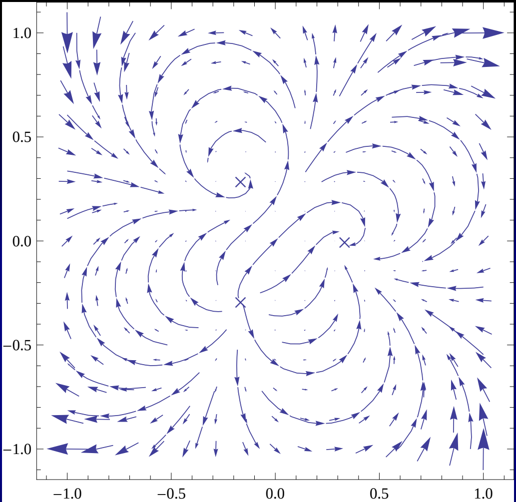


Figure: The attractor function K_0. Phase portrait of $\mathrm{Re}\,(K_0)$ and $\mathrm{Im}\,(K_0)$ for $\theta = \pi/3$. The " $\times$ " marks are the three roots–each corresponding to a transseries.

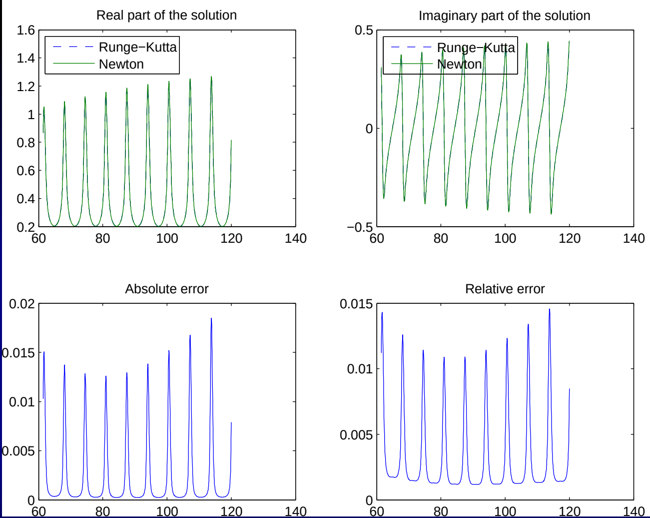


Figure: Comparison of solutions from the Runge-Kutta method and from Newton's method using the formal constant of motion.

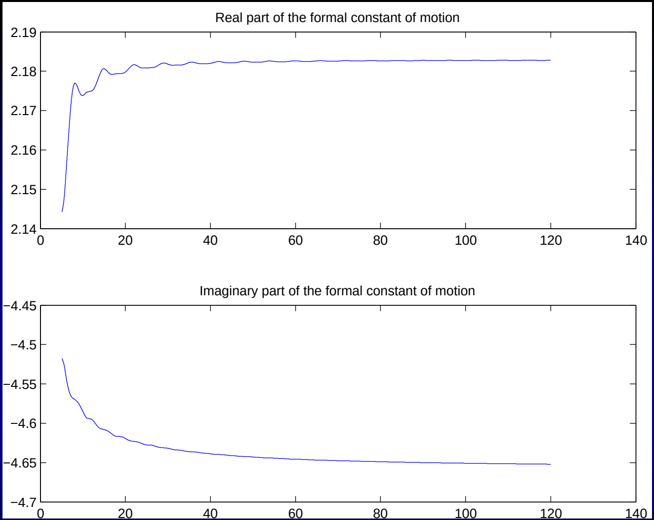
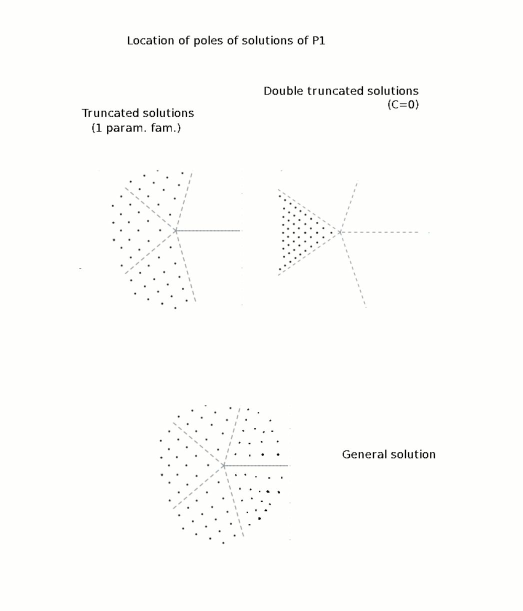


Figure: Formal constant of motion with K_0 and K_1.

The Painlevé equation P1, $y'' = y^2 + z$



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- (Modified) Boutroux normalized equation: $u''+\frac{u'}{x}-u-\frac{u^2}{2}-\frac{392}{625x^4}=0$ $(*)$.
- The first constant of motion can be obtained as before, from the transseries. A second one can be obtained by reduction of order.

1. However, alternative approach: note that when not large, $u = u(Ce^{-x}x^{\beta})$. Because of e^{-x} it is periodic, of period $2\pi i$. It acts like a cyclic variable.
2. As in Hamiltonian dynamics, it is useful to pass to action- angle variables. u is an angle-like variable, which we naturally take as an independent variable. Dependent variables: x, s, $s = {u'}^2 - u^2 - u^3/3$.

Action angle variables: radial directions

First (Boutroux): if $x = re^{it}$, $r \to +\infty$, then $s(u) \to D_\infty$ and $u(x)$ asympt. traverses a cycle –closed loop surrounding zeros of cubic–indefinitely.

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- Thus, we let u be the indep. variable, evolving on the limiting cycle and s, x become the dependent ones. We have

$$\frac{ds}{du} = -\frac{2\sqrt{u^3/3 + u^2 + s(u)}}{x(u)}$$
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- A similar equation is satisfied by $\xi := x - L(u, D_\infty)$ (quite a bit of algebra here...); the constants are obtained rigorously by inverting asymptotically for the IC as functions of x, s, u.

$s = s(u, u') = {u'}^2 - u^2 - \frac{u^3}{3}$. Order by order from contraction:

Theorem (Constants of motion, radial direction)

For any solution, there exist C_1 and C_2 so that

$$x - L(s, u) + \frac{K_1(u, s)}{x} = C_1 + O(x^{-2})$$

$$xJ(s) - J(s)L(u, s) + L(s)J(u, s) + \frac{H_1(u, s)}{x} + \frac{H_2(u, s)}{x^2} = C_2 + O(1/x^3)$$

$J(s) := \oint \sqrt{u^3/3 + u^2 + s}\,du; \quad L(s) := \oint (u^3/3 + u^2 + s)^{-1/2} du$ are complete elliptic integrals. (K_i, H_i are explicit combinations of elliptic integrals.)

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1. Asympt. exp. of $C_{1,2}$ *depend continuously on the direction*: J and L depend on $D_\infty(\theta)$. Analog of Stokes matrices change continuously.
2. Thus, we need a second pair of asymptotic formulas on arc-circles (the curves $\perp \{x : \arg x = \theta\}$)

The lateral connection

$$\begin{aligned}\frac{ds}{du} &= -\frac{2\sqrt{u^3/3+u^2+s(u)}}{x(u)}\\ \frac{dx}{du} &= \frac{1}{\sqrt{u^3/3+u^2+s(u)}}\end{aligned} \tag{9}$$

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- One can show that, for large enough starting point x_0, we have, with Q_0 a number:

Eq. for the Poincaré map and its solution

$$s_{n+1} = s_n - 2\frac{J_n}{x_n} + \frac{J_n L_n}{x_n^2} + O\left(\frac{1}{x_n^3}\right) \tag{10}$$

$$x_{n+1} - x_n = L_n + \frac{Q_0 J_n + \frac{1}{2}\rho(s_n) J_n^2}{x_n} + O(1/x_n^2) \tag{11}$$

$$(J(s) := \oint \sqrt{u^3/3 + u^2 + s}du; \quad L(s) := \oint (u^3/3 + u^2 + s)^{-1/2} du)$$

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- Thus, to leading order $s_{n+1} - s_n \approx \frac{ds}{dn} \cdot 1, x_{n+1} - x_n \approx \frac{dx}{dn}$ (think Euler-Maclaurin). By substitution from (10),(11) and division, we get (note that $J' = \frac{1}{2}L$)

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a first, discrete, constant of motion to which corrections can be calculated similarly. (For a rigorous proof, we estimate $J_{n+1}x_{n+1} - J_n x_n$.)

Let $k=\dfrac{r_3-r_1}{r_2-r_1}$, $(\dfrac{r_i^3}{3}+r_i^2+s=0)$, $\psi(k)=k^3\frac{{}_2F_1(-\frac{1}{2},\frac{3}{2};3;k^2)}{{}_2F_1(-\frac{1}{2},\frac{3}{2};3;1/k^2)}$

Theorem (Constants of motion along arcs of circle)

Let N be the number of arrays of poles traversed. We have

$$N\psi(k(s_N))=\frac{x_0}{2\pi i}(1+o(1));\quad \frac{x_N}{x_0}J_3(s_N)=-\frac{24}{5}+o(1) \qquad (12)$$

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Successive corrections can be calculated as well. The second const. to two orders is given by $Jx + B$ where

$$B = C_1 \left(3s^{1/3} {}_2F_1\left(\frac{1}{6}, \frac{7}{6}; \frac{1}{3}, \frac{-4}{3s}\right) + \frac{7}{2} {}_3F_2\left(\frac{2}{3}, \frac{7}{6}, \frac{13}{6}; \frac{4}{3}, \frac{5}{3} \frac{-4}{3s}\right)\right) - 3s^{-1/3} C_2 \; {}_3F_2\left(\frac{1}{3}, \frac{5}{6}, \frac{11}{6}; \frac{4}{3}, \frac{5}{3} \frac{-4}{3s}\right) \tag{13}$$

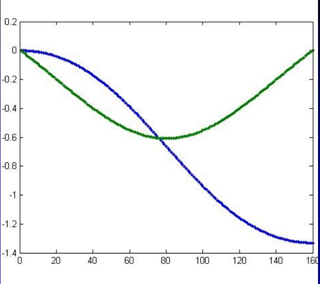

Figure: Evolution of the "energy" s ($J = \oint \sqrt{u^3/3 + u^2 + s}du$) across the region of poles, for the tritronquée solution. The solution goes to a different transseries, corresponding to the fixed point $u = -2$. This is explained by a form of the Stokes phenomenon: the leading order asymptotics of P_1, $y = \sqrt{-x/6}$: y is single valued while $\sqrt{-x}$ is not.

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