A notion of boundedness for hyperfunctions and Massera type theorems

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Outline

classsical Massera theorem

J. L. Massera (1950) studied the existence of a periodic solution to a periodic ordinary differential equation of normal form: $\frac{dx}{dt} = X(t, x)$, where $x(t)$ is an $\mathbb{R}^m$-valued unknown function, and $X(t, x)$ is 1-periodic in t.

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J. L. Massera (1950) studied the existence of a periodic solution to a periodic ordinary differential equation of normal form: $\frac{dx}{dt} = X(t,x)$, where $x(t)$ is an $\mathbb{R}^m$-valued unknown function, and $X(t,x)$ is 1-periodic in t.

In the linear case, he gave the following result for the equation

$$\frac{dx}{dt} = A(t)x + f(t),$$

where $A : \mathbb{R} \to \mathbb{R}^{m\times m}$ and $f : \mathbb{R} \to \mathbb{R}^m$ are 1-periodic and continuous.

Theorem (Massera, linear case)

For the equation above, the existence of a bounded solution in the future (i.e., solution defined and bounded on a set $\{t > t_0\}$ with some t_0) implies the existence of a 1-periodic solution.

Note: Since periodic C^1-functions are bounded, we have

$$\exists \text{ a bounded solution in the future.} \iff \exists \text{ a 1-periodic solution.}$$

question

After Massera, many generalizations by many authors have appeared also on linear functional equations.

From now on, let ω be a positive constant, (representing the period).

Problem

Consider an ω-periodic linear functional equation. Does "the existence of a bounded solution in the future" imply "the existence of an ω-periodic solution"?

a few references

- Chow-Hale (1974) studied functional differential equations with retarded type. An example is,

$$\frac{dx}{dt} = A(t)x + \int_0^r B(t,s)x(t-s)ds + f(t).$$

  A, B, (resp. f): square matrices (resp. a vector) of size m, whose entries are continuous and ω-periodic in t,
  $r > 0$: a constant, (representing a "finite delay").
- Hino-Murakami (1989)
- Zubelevich (2006)

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- Chow-Hale (1974)

$$\frac{dx}{dt} = A(t)x + \int_0^r B(t,s)x(t-s)ds + f(t).$$

- Hino-Murakami (1989) studied similar equations with infinite delay.

$$\frac{dx}{dt} = A(t)x + \int_0^\infty B(t,s)x(t-s)ds + f(t).$$

A, B, f: the continuity, the ω-periodicity in t,
B: some condition on the integrability in s,
x: some restriction on the behavior near $-\infty$.

- Zubelevich (2006)

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$$\frac{dx}{dt} = A(t)x + \int_0^\infty B(t,s)x(t-s)ds + f(t).$$

- Zubelevich (2006) studied discrete dynamical systems in reflexive Banach spaces and those in sequentially complete locally convex spaces with Montel property.

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Question

Does this phenomenon appear commonly in periodic linear equations?

Sato's hyperfunction

The notion of hyperfunction was introduced by M. Sato (1959, 1960), and plays impotant roles in the study of analytic ordinary and partial differential equations.

Hyperfunctions admit many good properties, for ex.,

- flabbiness,
- boundary value representations by holomorphic defining functions,
- (comparatively) direct action of linear differential operators.

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Hyperfunctions admit many good properties, for ex.,

- flabbiness,
- boundary value representations by holomorphic defining functions,
- (comparatively) direct action of linear differential operators.

On the other hand, we must suffer from the inconveniences:

- no inequality nor boundedness for hyperfunctions,
- no good topology on the spaces $\mathscr{B}(\Omega)$ of hyperfunctions for $\Omega \overset{\text{open}}{\subset} \mathbb{R}^n$.

univariate hyperfunctions

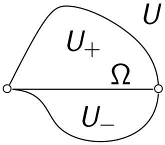
Recall the notion of hyperfunction on $\mathbb{R}$. $\mathscr{O}$ denotes the sheaf of holomorphic functions on $\mathbb{C}$.

Definition

The space $\mathscr{B}(\Omega)$ of hyperfunctions on $\Omega \overset{\text{open}}{\subset} \mathbb{R}$ is defined by

$$\mathscr{B}(\Omega) := \varinjlim_{U} \frac{\mathscr{O}(U \setminus \Omega)}{\mathscr{O}(U)}.$$

Here, U runs through complex neighborhoods of Ω.



For $\Omega \subset \mathbb{R}$, a complex neighborhood of Ω is an open set in $\mathbb{C}$ including Ω as a closed subset.
For ex., $U = U_+ \cup \Omega \cup U_-$ ($U_\pm = \{x + iy \in \mathbb{C} \mid x \in \Omega, 0 < \pm y < d_\pm(x)\}$) is a complex neighborhood when $d_\pm : \Omega \to \mathbb{R}$ are positive and upper semicontinuous.

My interest

Interest (of the speaker)

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Is there any counterpart to the Massera type phenomenon in the framework of hyperfunctions?

There are some obstacles.

1. How to give a meaning to "a bounded solution in the future" in hyperfunctions? (Note. No notion of bdd'ness for hyperfunctions).
2. There are many periodic hyperfunctions satisfying periodic equations, which, at a glance, do not seem bounded. Can we treat them? For ex., $u(t) := \tan(t + i0)$ is a π-periodic hyperfunction solution to a π-periodic differential equation $(\cos^2 t)\frac{du}{dt} = 1$.
3. If we want to consider equations with infinity delay, (for ex., equations containing a term like $\int_0^\infty k(t,s)u(t-s)ds$), what kind of restriction we should impose on the behavior of $u(t)$ near $t = -\infty$?

Let us construct a new class of “bounded hyperfunctions”!

boundedness for hyperfunctions

We will introduce the sheaf $\mathscr{B}_{L^\infty}$ of *bounded hyperfunctions at infinity* in one variable, in a similar manner as Sato defined the sheaf $\mathscr{B}$ of hyperfunctions and the sheaf $\mathscr{Q}$ of Fourier hyperfunctions in one variable.

In fact, we define the sheaf $\mathscr{B}_{L^\infty}$ on a compactification $\mathbb{D}^1 := [-\infty, +\infty] = \mathbb{R} \sqcup \{\pm\infty\}$ of $\mathbb{R}$, using the sheaf $\mathscr{O}_{L^\infty}$ of bounded holomorphic functions on $\mathbb{D}^1 + i\mathbb{R}$.

$$\begin{array}{ccc} \mathbb{C} = \mathbb{R} + i\mathbb{R} & \subset & \mathbb{D}^1 + i\mathbb{R} \\ \cup & & \cup \\ \mathbb{R} =]-\infty, +\infty[& \subset & \mathbb{D}^1 = [-\infty, +\infty] \end{array}$$

We take coordinates $t \in \mathbb{R}$ and $w = t + i\tau \in \mathbb{C} = \mathbb{R} + i\mathbb{R}$.

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$$\begin{array}{ccccccc} \mathscr{O} & \cdots & \mathbb{C} = \mathbb{R} + i\mathbb{R} & \subset & \mathbb{D}^1 + i\mathbb{R} & \cdots & \mathscr{O}_{L^\infty} \\ & & \cup & & \cup & & \\ \mathscr{B} & \cdots & \mathbb{R} =]-\infty, +\infty[& \subset & \mathbb{D}^1 = [-\infty, +\infty] & \cdots & \mathscr{B}_{L^\infty} \end{array}$$

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bounded holomorphic functions

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Definition ($\mathscr{O}_{L^\infty}$)

$\mathscr{O}_{L^\infty}$: the sheaf of bounded holomorphic functions on $\mathbb{D}^1 + i\mathbb{R}$, is defined as the sheaf associated with the presheaf

$$\mathbb{D}^1 + i\mathbb{R} \supset U \mapsto \mathscr{O}(U \cap \mathbb{C}) \cap L^\infty(U \cap \mathbb{C}).$$

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Fact

- $\mathscr{O}_{L^\infty}(U) = \{f \in \mathscr{O}(U \cap \mathbb{C}); \forall K \Subset U, \|f\|_K < +\infty\}$, *where* $\|f\|_K := \sup_{w \in K \cap \mathbb{C}} |f(w)|$.
- $\mathscr{O}_{L^\infty}(U)$ *is a Fréchet space, and* $\mathscr{O}_{L^\infty}|_{\mathbb{C}} = \mathscr{O}$.

bounded hyperfunctions at infinity

$$\begin{array}{ccccccc}
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Definition ($\mathscr{B}_{L^\infty}$)

$\mathscr{B}_{L^\infty}$: the sheaf of *bounded hyperfunctions at infinity* on $\mathbb{D}^1$, is defined as the sheaf associated with the presheaf

$$\mathbb{D}^1 \overset{\text{open}}{\supset} \Omega \mapsto \varinjlim_{U} \frac{\mathscr{O}_{L^\infty}(U \setminus \Omega)}{\mathscr{O}_{L^\infty}(U)}.$$

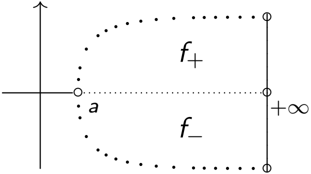
Here U runs through complex neighborhoods of Ω.

Cf. hyperfunctions: $\mathscr{B}(\Omega) = \varinjlim_U \frac{\mathscr{O}(U \setminus \Omega)}{\mathscr{O}(U)}$.

properties of $\mathscr{B}_{L^\infty}$

- $\mathscr{B}_{L^\infty}$ is an extension of $\mathscr{B}$ to $\mathbb{D}^1$. That is, $\mathscr{B}_{L^\infty}|_{\mathbb{R}} = \mathscr{B}$.
- $\mathscr{B}_{L^\infty}$ is a flabby sheaf.
- $u \in \mathscr{B}_{L^\infty}(]a, +\infty])$ admits a boundary value representation:

$$u(t) = f_+(t + i0) - f_-(t - i0).$$



- There exists a natural embedding $L^\infty(]a, +\infty[) \hookrightarrow \mathscr{B}_{L^\infty}(]a, +\infty])$.

Moreover, for E: a sequentially complete locally convex space over $\mathbb{C}$, we can also define vector-valued variants ${}^E\mathscr{O}_{L^\infty}$ on $\mathbb{D}^1 + i\mathbb{R}$ and ${}^E\mathscr{B}_{L^\infty}$ on $\mathbb{D}^1$ in a similar manner.

operators of type K

Let $K = [a, b]$ be a closed interval in $\mathbb{R}$, or $K = [-\infty, b] \subset \mathbb{D}^1$.

Now we introduce classes of operators for bounded hyperfunctions, what we call "*operators of type* K *for* $\mathscr{O}_{L^\infty}$". But, before giving the definition, we give some properties and some examples.

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Let $K = [a, b]$ be a closed interval in $\mathbb{R}$, or $K = [-\infty, b] \subset \mathbb{D}^1$.

Now we introduce classes of operators for bounded hyperfunctions, what we call "*operators of type K for $\mathscr{O}_{L^\infty}$*". But, before giving the definition, we give some properties and some examples.

An operator P of type K for $\mathscr{O}_{L^\infty}$ induces, for any open $\Omega \subset \mathbb{D}^1$, a linear map

$$P_\Omega : \mathscr{B}_{L^\infty}(\Omega + K) \to \mathscr{B}_{L^\infty}(\Omega),$$

and we have, for any pair $\Omega_2 \subset \Omega_1$,

$$(P_{\Omega_1} u)|_{\Omega_2} = P_{\Omega_2}(u|_{\Omega_1 + K}), \quad \forall u \in \mathscr{B}_{L^\infty}(\Omega_2 + K).$$

typical examples of our operators

An operator P of type K induces $P_\Omega : \mathscr{B}_{L^\infty}(\Omega + K) \to \mathscr{B}_{L^\infty}(\Omega)$.

Let $U^\circ := \mathbb{R}^1 + i]-d, d[$ be a strip domain, and ω, $r > 0$.

differential operator $\sum_{j=0}^{m} a_j(t)\partial_t^j$, an operator of type $K = \{0\}$.
$a_j(w)$ are bounded and holomorphic on U°.

translation operator $T_\omega : u(t) \mapsto u(t+\omega)$, $K = \{\omega\}$.

difference operator $T_\omega - 1 : u(t) \mapsto u(t+\omega) - u(t)$, $K = [0, \omega]$.

integral operator $u(t) \mapsto \int_0^r k(t,s)u(t-s)ds$, $K = [-r, 0]$.
$k(w,s) \in (C \cap L^\infty)(U^\circ \times [0, r])$, and is holomorphic in w.

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$\int_0^\infty \sup_{w \in U^\circ} |k(w,s)| ds < +\infty$.

definition of our operators

Let $U \subset \mathbb{D}^1 + i\mathbb{R}$ be an open set, K a closed interval $[a, b] \subset \mathbb{R}$ or $[-\infty, b]$.

Definition (Operators of type K)

Let $P = \{P_V : \mathscr{O}_{L^\infty}(V + K) \to \mathscr{O}_{L^\infty}(V)\}_{V \subset U}$ be a family of linear continuous maps. P is said to be *an operator of type K* for $\mathscr{O}_{L^\infty}$ on U, if the diagram below commutes for any pair $V_1 \supset V_2$ in U.

$$\begin{array}{ccc}
\mathscr{O}_{L^\infty}(V_1 + K) & \xrightarrow{P_{V_1}} & \mathscr{O}_{L^\infty}(V_1) \\
\downarrow & & \downarrow \\
\mathscr{O}_{L^\infty}(V_2 + K) & \xrightarrow{P_{V_2}} & \mathscr{O}_{L^\infty}(V_2)
\end{array}$$

Here the vertical arrows are the restriction maps.

As is seen in the definition, we require the "*continuity*" and the "*commutativity with restrictions*" in terms of defining functions. Note that the spaces $\mathscr{B}_{L^\infty}(\Omega)$ ($\Omega \subset \mathbb{D}^1$) have no good topology.

periodicity for bounded hyperfunctions

We fix a period $\omega > 0$, and denote by T_ω the ω-translation operator $u(t) \mapsto u(t+\omega)$.

We introduce the notion of ω-periodicity
for bounded hyperfunction u by the equation $(T_\omega - 1)u = 0$, and
for operators P of type K by the commutativity $P \circ T_\omega = T_\omega \circ P$.

As for periodic bounded hyperfunction, we can prove the followings:

- Every ω-periodic hyperfuntion $f \in \mathscr{B}(\mathbb{R})$ has the unique ω-periodic extension $\hat{f} \in \mathscr{B}_{L^\infty}(\mathbb{D}^1)$.
- Every ω-periodic bounded hyperfunction $f \in \mathscr{B}_{L^\infty}(\mathbb{D}^1)$ admits an ω-periodic boundary value representation.

Note that an ω-periodic operator preserves the ω-periodicity of its operands.

Massera type theorem for the case $K \subset \mathbb{R}$

$K = [a, b] \subset \mathbb{R}$, $\omega > 0$, and $U = \mathbb{D}^1 + i]-d, d[$.
P: an ω-periodic operator of type K for $\mathscr{O}_{L^\infty}$ on U,
f: an ω-periodic (bounded) hyperfunction,
$(\mathscr{B}_{L^\infty})_{+\infty} = \varinjlim_R \mathscr{B}_{L^\infty}(]R, +\infty])$: the stalk of $\mathscr{B}_{L^\infty}$ at $+\infty$.

Theorem (O. 2008)

$Pu = f$ has an ω-periodic $\mathscr{B}(\mathbb{R})$-solution if and only if it has an $(\mathscr{B}_{L^\infty})_{+\infty}$-solution.

Example: periodic integro-differential equation with finite delay.

$$\partial_t u = a(t)u + \int_0^r k(t, s)u(t - s)ds + f,$$

where $a(t)$: an ω-periodic real-analytic function,
$k(w, s) \in (C \cap L^\infty)(U^\circ \times [0, r])$: holomorphic and ω-periodic in w.

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This theorem can be extended to E-valued case, when E admits the Montel property, or E is a reflexive Banach space.

Definition (Montel property)

(M) Any bounded sequence in E has a convergent subsequence.

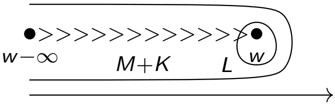
"continuity" of our operators in case $K = [-\infty, b]$

The notion of operators of type K is defined by "continuity" and "commutativity with restrictions". Under the commutativity, "continuity" reads:

$$\forall L \Subset \forall M \Subset U, \forall p \in \mathcal{N}(E), \exists q = q_{L,M,p} \in \mathcal{N}(E), \exists C = C_{L,M,p} > 0,$$
$$\forall V\,(M \Subset V \subset U), \forall f \in {}^{E}\mathscr{O}_{L^\infty}(V+K), \|P_V(f)\|_{L,p} \le C\|f\|_{M+K,q}.$$

Here, $\mathcal{N}(E)$ denotes the system of continuous seminorms of E, and $\|f\|_{L,p}$ denotes the seminorm on ${}^{E}\mathscr{O}_{L^\infty}(V)$ defined by $\|f\|_{L,p} := \sup_{w \in L} p(f(w))$. (In the scalar case, take $\|f\|_L := \sup_{w\in L}|f(w)|$).

In case $K = [-\infty, b]$, $M + K$ always contains points at $-\infty$.
If the contribution of f at $w - \infty$ to Pf at w is not small, we can not expect a Massera type theorem for P.



fading memory condition for operators of type $[-\infty, b]$

We pose a further assumption on P.

Definition (fading memory condition)

Let $P = \{P_V\}_{V \subset U}$ be an operator of type $K := [-\infty, b]$ for ${}^E\mathscr{O}_{L^\infty}$ on $U \subset \mathbb{D}^1 + i\mathbb{R}$. P is said to satisfy the condition (FM), if

$$\forall L \Subset \forall M \Subset \forall V \subset U, \forall p \in \mathcal{N}(E), \exists q = q_{L,M,p} \in \mathcal{N}(E),$$
$$\forall \varepsilon > 0, \exists K^0 = K^0_{L,M,p,\varepsilon} \Subset K \cap \mathbb{R}, \exists C = C_{L,M,p,\varepsilon} > 0,$$
$$\forall f \in {}^E\mathscr{O}_{L^\infty}(V + K), \|P_V(f)\|_{L,p} \leq C\|f\|_{M+K^0,q} + \varepsilon\|f\|_{M+K,q}.$$

This condition (FM) seems to have relation with the notion of (uniform) fading memory space, studied in Hino-Murakami-Naito (1991).

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Fact

Volterra integral operators satisfy the condition (FM).

Massera type theorem under (FM)

$\omega > 0$, and $U = \mathbb{D}^1 + i]-d, d[$: as before.
$f \in {}^{E}\mathscr{B}_{L^\infty}(\mathbb{D}^1)$: ω-periodic.

Theorem (O.)

Let P be an ω-periodic operator of type $K = [-\infty, b]$ for ${}^{E}\mathscr{O}_{L^\infty}$ on $\mathbb{D}^1 + i]-d, d[$. Assume that E admits the Montel property and that P satisfies (FM). Then $Pu = f$ has an ω-periodic ${}^{E}\mathscr{B}(\mathbb{R})$-solution if and only if it has an $({}^{E}\mathscr{B}_{L^\infty})_{+\infty}$-solution.

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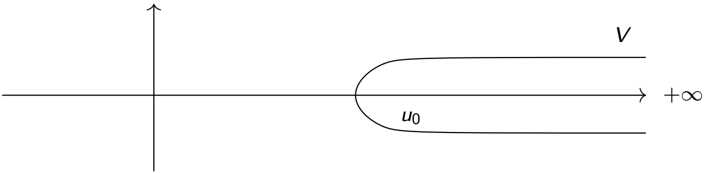
idea of the proof

We give the idea of the proof of "$\exists$ a bdd. sol. $u_0 \Rightarrow \exists\ \omega$-periodic sol. u", for the simplest case: scalar-valued ($E = \mathbb{C}$), P is of type $K = \{0\}$ (a local operator).
Moreover, assume that our bounded solution u_0 belongs to $(\mathscr{O}_{L^\infty})_{+\infty}$. (Therefore, the data f belongs necessarily to $\mathscr{O}_{L^\infty}$). We can find a neighborhood $V \subset \mathbb{D}^1 + i\mathbb{R}$ of $+\infty$, such that $u_0 \in \mathscr{O}_{L^\infty}(V)$ and that $Pu_0 = f$ on V.

We define $v_k := \frac{1}{k}\sum_{j=0}^{k-1} T_\omega^j u_0$, and choose a "convergent" subsequence from $\{v_k\}_k$. Then, its "limit" becomes an ω-periodic solution.



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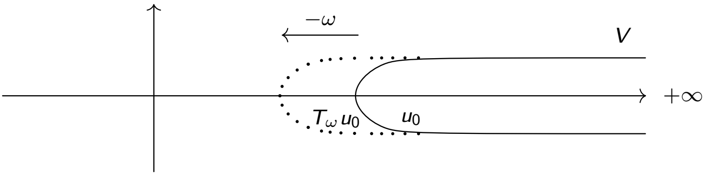

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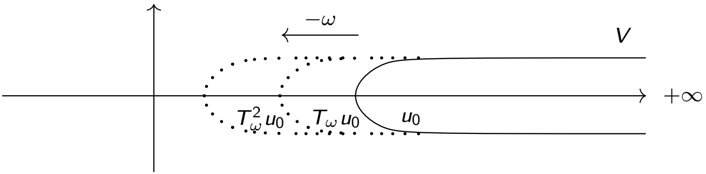

But we omit the details.

idea of the proof

We give the idea of the proof of "$\exists$ a bdd. sol. $u_0 \Rightarrow \exists$ ω-periodic sol. u", for the simplest case: scalar-valued ($E = \mathbb{C}$), P is of type $K = \{0\}$ (a local operator).

Moreover, assume that our bounded solution u_0 belongs to $(\mathscr{O}_{L^\infty})_{+\infty}$. (Therefore, the data f belongs necessarily to $\mathscr{O}_{L^\infty}$). We can find a neighborhood $V \subset \mathbb{D}^1 + i\mathbb{R}$ of $+\infty$, such that $u_0 \in \mathscr{O}_{L^\infty}(V)$ and that $Pu_0 = f$ on V.

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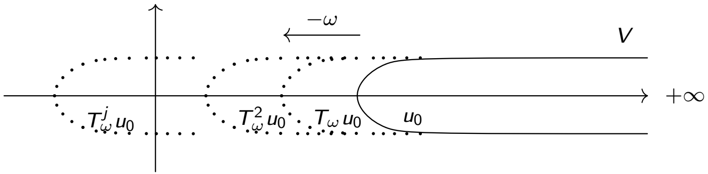

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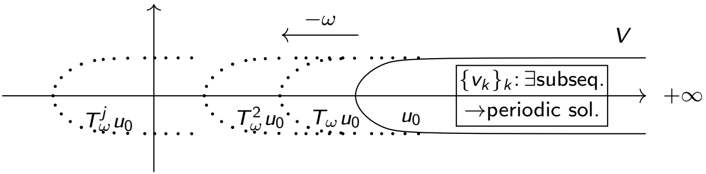

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Thank you for your attention.