Asymptotic expansions of solutions to the fifth Painlevé equation

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The fifth Painlevé equation

We consider the fifth Painlevé equation (P5):

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) +$$

$$+ \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1},$$

where $\alpha, \beta, \gamma, \delta$ are complex parameters, z and w are the respective independent and dependent variables. The P5 equation has two singular points $z = 0$ и $z = \infty$.

The P5 equation is invariant under the substitution (symmetry)

$$(\tilde{z}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = (z, \frac{1}{w}, -\beta, -\alpha, -\gamma, \delta).$$

The aim of the present work

By means of Power Geometry we are looking for all asymptotic expansions of solutions to the P5 equation as $z \to 0$ and as $z \to \infty$ of the following form:

$$w = c_r(z) z^r + \sum_{s \in \mathbf{K}} c_s(z) z^s,$$

where $c_r(z), c_s(z), r, s \in \mathbb{C}$, $\mathbf{K} \subset \{s \mid \mathrm{Re}\ s > \mathrm{Re}\ r\}$ for the expansions as $z \to 0$ and $\mathbf{K} \subset \{s \mid \mathrm{Re}\ s < \mathrm{Re}\ r\}$ for the expansions as $z \to \infty$; the set $\mathbf{K}$ is countable.

1. *Power expansions*: $c_r(z)$ and $c_s(z)$ are constants.
2. *Power-logarithmic*: $c_r(z)$ is constant, $c_s(z)$ are polinomials in $\log z$.
3. *Complicated*: $c_r(z)$, $c_s(z)$ are series in decreasing powers of $\log z$.
4. *Exotic*: $r, s \in \mathbb{R}$, $c_r(z)$ and $c_s(z)$ are series in z^i, c_r is a sum of countable (*for half-exotic* the sum is just finite) number of terms, the set of power exponents of z^i in c_r is bounded either from above or from below.

Basic notions

1. A differential sum is a sum of differential monomials:
$$a(z,w) = Cz^{q_1}w^{q_2}\prod_{j=1}^{k}\left(\frac{d^{l_j}w}{dz^{l_j}}\right)^{r_j}.$$

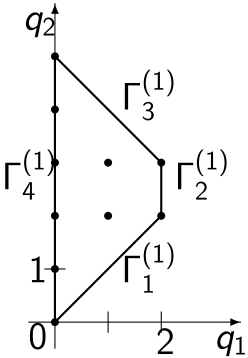
2. The vector degree: $Q(a(z,w)) = (q_1 - \sum_{j=1}^{k} l_j r_j, q_2 + k\sum_{j=1}^{k} r_j)$.

3. To transform the P5 equation to a differential sum we multiply it by $z^2w(w-1)$:

$$-z^2w(w-1)w'' + z^2\left(\frac{3}{2}w - \frac{1}{2}\right)(w')^2 - zw(w-1)w' +$$

$$+(w-1)^3(\alpha w^2 + \beta) + \gamma z w^2(w-1) + \delta z^2 w^2(w+1) = 0.$$

The polygon $\Gamma(\mathbf{f})$ as $\alpha\beta\gamma\delta \neq 0$



Expansions of solutions corresponding to the edges $\Gamma_1^{(1)}$, $\Gamma_2^{(1)}$, $\Gamma_3^{(1)}$ as $\alpha\beta\delta \neq 0$, $z \to \infty$

$\Gamma_1^{(1)}$ (2 power expansions):

$$\mathcal{D}_k : w = (-1)^k \sqrt{\frac{\beta}{\delta}} \frac{1}{z} + \left(-\frac{2\beta}{\delta} + (-1)^k \frac{\gamma}{2\delta} \sqrt{\frac{\beta}{\delta}} \right) \frac{1}{z^2} + \sum_{s=3}^{\infty} \frac{c_{sk}}{z^s}. \tag{1}$$

$\Gamma_2^{(1)}$ (power expansion):

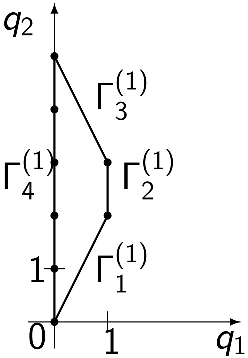
$$\mathcal{E}_1 : \; w = -1 + \frac{2\gamma}{\delta z} + \sum_{s=2}^{\infty} \frac{c_s}{z^s}.$$

$\Gamma_3^{(1)}$ (2 power expansions):

$$\mathcal{F}_k : \; w = (-1)^k \sqrt{-\frac{\delta}{\alpha}} z + 2 + (-1)^k \frac{1}{2} \frac{\gamma}{\sqrt{-\alpha\delta}} + \sum_{s=1}^{\infty} \frac{c_{s,k}}{z^s},$$

where c_s, c_{sk} are uniquely defined constants, $k = 1, 2$.

The polygon $\Gamma(\mathbf{f})$ as $\alpha\beta\gamma \neq \mathbf{0},\ \delta = \mathbf{0}$



Expansions of solutions corresponding to the edges $\alpha\beta\gamma \neq \mathbf{0}$, $\delta = 0$, $z \longrightarrow \infty$

$\Gamma_1^{(1)}$ (2 power expansions):

$$\mathcal{D}_k: \ w = (-1)^k \sqrt{-\frac{\beta}{\gamma}} \frac{1}{\sqrt{z}} + \frac{\beta}{\gamma}\frac{1}{z} + \sum_{s=3}^{\infty} \frac{c_{s,k}}{z^{s/2}}.$$

$\Gamma_2^{(1)}$ (power expansion): $\mathcal{E}_2: \ w = 1.$
$\Gamma_3^{(1)}$ (2 power expansions):

$$\mathcal{F}_k: \quad w = (-1)^k \sqrt{-\frac{\gamma}{\alpha}} \sqrt{z} + 1 + \sum_{s=1}^{\infty} \frac{c_{s,k}}{z^{s/2}},$$

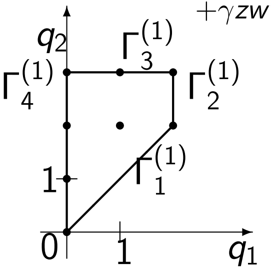
where c_s, c_{sk} are uniquely defined complex constants, $k = 3, 4$.

$\alpha = 0, \ \delta \neq 0$

If $\alpha = 0$ P5 has the following form:

$$-z^2w(w-1)w'' + z^2\left(\frac{3}{2}w - \frac{1}{2}\right)(w')^2 - zw(w-1)w' + \beta(w-1)^3 +$$

$$+\gamma zw^2(w-1) + \delta z^2w^2(w+1) = 0.$$



$\alpha = 0,\ z \to \infty$

If $\alpha = 0,\ \delta \neq 0,\ z \to \infty$ we obtain the following asymptotics

$$w_\sigma = Cz^{\left(1 - \frac{\sigma\gamma}{\sqrt{-2\delta}}\right)} \exp\{\sigma\sqrt{-2\delta}z + \sum_{s=2}^{\infty} c_s \frac{z^{-s+1}}{-s+1}\}, \sigma = \pm 1,$$

where C is an arbitrary constant, the expansion exists when $\mathrm{Re}(\sigma\sqrt{-\delta}z) > 0$.
The case $\alpha = 0,\ \delta = 0,\ z \to \infty$ we obtain the asymptotics

$$w_\sigma = C\sqrt{z}\exp\{2\sigma\sqrt{-2\gamma z} + \sum_{s=3}^{\infty} c_s \frac{z^{-s/2+1}}{-s/2+1}\}, \sigma = \pm 1,$$

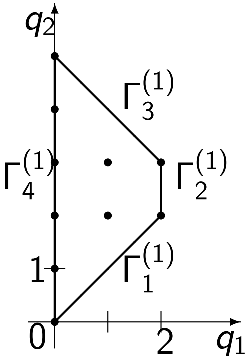
the expansion exists when $\mathrm{Re}(\sigma\sqrt{-\gamma z}) > 0$.

Expansions obtained in the case $z \to \infty$

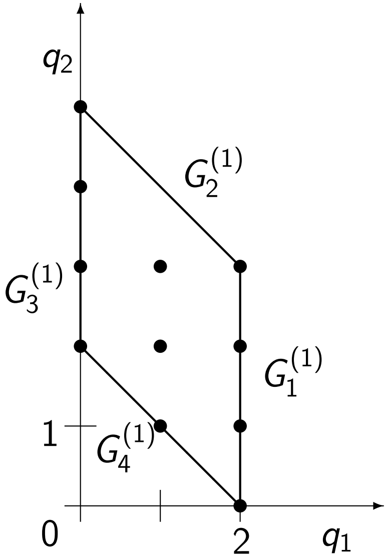
If $\alpha \neq 0$ there exist 10 power expansions. Six of them (in integer degrees of z) have been known before [1, §37], [2]), four of them (in half-integer degrees of z) are new.
If $\alpha = 0$ there exist 4 one-parameter families of expansions of solutions $w(z)$ (they are new).
If $\alpha = \beta = 0$ does not give any new solutions.
The expansions in case $\beta = 0$ are obtained from the expansions in case $\alpha = 0$ with the help of the symmetry:

$$(\tilde{z}, \tilde{w}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = \left(z, \frac{1}{w}, -\beta, -\alpha, -\gamma, \delta\right).$$

The polygon $\Gamma(\mathbf{f})$ as $\alpha\beta\gamma\delta \neq 0$



The polygon after the substitution $w = 1 + y$



Expansions obtained in the case $z \to 0$ corresponding to the edge $G_4^{(1)}$

Let us denote $a = \left(\operatorname{sgn}\operatorname{Re}\sqrt{-\frac{\gamma^2}{2\delta}}\right)\sqrt{-\frac{\gamma^2}{2\delta}}$.

We have obtained the following results

-if $\gamma\delta \neq 0,\ \frac{\gamma^2}{2\delta} = -s^2,\ s \in \mathbb{R} \setminus \mathbb{Z}$ there exists the family of power expansions

$$\mathcal{H}_1 :\ w = 1 - \frac{2\delta}{\gamma} z + \sum_{s \in \mathbf{K}} c_s z^s,$$

where $\mathbf{K} = \{s : s = l + m + ma,\ l, m \in \mathbb{Z},\ l, m \geq 0,\ l + m > 0\}$, c_{a+1} is an arbitrary constant, other coefficients c_s are uniquely defined. The expansion can be found in [1]. If the set $\mathbf{K}$ is a subset of $\mathbb{Z}$ or even $\mathbb{Q}$, this expansion converges according to Theorem 1.7.2 [3];

Expansions obtained in the case $z \to 0$ corresponding to the edge $G_4^{(1)}$

-if $\gamma\delta \neq 0$, $\dfrac{\gamma^2}{2\delta} = s^2$, $s \in \mathbb{R}$ there exist new families of half-exotic expansions

$$\mathcal{H}_1^\tau:\ w = 1 + \left(-\frac{2\delta}{\gamma} + C z^{i\tau \frac{\gamma}{\sqrt{2\delta}}} \right) z + \sum_{\mathrm{Re}\ s>1} c_s z^s,\ c_r \in \mathbb{C},\ \gamma^2/(2\delta) \in \mathbb{R};$$

-if $|\gamma| + |\delta| \neq 0$ there exists a new family of half-exotic expansions

$$\mathcal{H}_4:\ w = 1 + \left(c_r z^{ir} - \frac{\gamma}{r^2} + \frac{\gamma^2 - 2\delta r^2}{4 c_r r^4} z^{-ir} \right) z + \sum_{\mathrm{Re}\ s>1} c_s z^s,\ r \in \mathbb{R}\backslash\{0\},$$

$c_r \in \mathbb{C}$.

Expansions obtained in the case $z \to 0$ corresponding to the edge $G_4^{(1)}$

-if $\gamma\delta \neq 0,\ \dfrac{\gamma^2}{2\delta} = -n^2,\ n \in \mathbb{N}$ there exists a new family of power-logarithmic expansions

$$\mathcal{H}_2 : \ w = 1 - \frac{2\delta}{\gamma} z + \sum_{s=1}^{\infty} c_s z^s,$$

where $c_s, 1 \leq s \leq a$ are constant, $c_s, s \geq a+2$ are polynomials in $\log z$ with uniquely defined coefficients, c_{a+1} is a polynomial in $\log z$ which contains one arbitrary constant.

For example, if $\dfrac{\gamma^2}{2\delta} = -1$, then

$w = 1 - \dfrac{2\delta}{\gamma} z + \left(\mathcal{C} + \dfrac{(\alpha+\beta)\gamma^2}{2} \ln z\right) z^2 + \displaystyle\sum_{s=3}^{\infty} c_s z^s$, where c_s are polynomials in $\log z$ with uniquely defined coefficients, $\mathcal{C}$ is an arbitrary constant.

Expansions obtained in the case $z \to 0$ corresponding to the edge $G_4^{(1)}$

- if $\gamma \neq 0$ there exists a family of complicated expansions

$$\mathcal{H}_3 : \; w = 1 + \left(-\frac{\gamma}{2}\ln^2 z + \mathcal{C}\ln z\right) z + \sum_{p=2}^{\infty} \varphi_p z^p .$$

It can be obtained from the expansion of solution to the P3 equation [4], [5];
-if $\gamma = 0, \; \delta \neq 0$ there exist two families of complicated expansions

$$\mathcal{H}_j^{(1)} : \; w = 1 + \left((-1)^j \sqrt{-2\delta}\ln z + \mathcal{C}\right) z + \sum_{p=2}^{\infty} \varphi_p z^p ,$$

where $j = 5, 6$, φ_p are series in decreasing powers of $\log z$ with uniquely defined coefficients.

We have also obtained more than 20 expansions from the corresponding expansions of solutions to the P6 equation.
We do not consider the case $\gamma = \delta = 0$ in this work as for these values of parameters the equation can be solved directly (see [3]).

Expansions obtained in the neighbourhood of the nonsingular point

To explore the expansions near the nonsingular point $z = z_0,\ z_0 \neq 0,\ z_0 \neq \infty$ of the equation we perform the transformation $z = t + z_0$ which permits us to apply the algorithms of Power Geometry described above to the transformed equation. In the neighbourhood of the nonsingular point $z = z_0$ of the P5 equation there exist 10 families of asymptotic expansions of its solutions:

$$\mathcal{O}_{1,2}:\ \ w = (-1)^j \frac{\sqrt{-2\beta}}{z_0}(z - z_0) + \sum_{s=2}^{\infty} c_{sj}(z - z_0)^s,\ j = 1, 2,$$

where c_{2j} are arbitrary constants. The expansions exist when $\beta \neq 0$;

$$\mathcal{O}_{3,4}:\ \ w = (-1)^j \frac{z_0}{\sqrt{2\alpha}(z - z_0)} + \sum_{s=0}^{\infty} c_{sj}(z - z_0)^s,\ j = 3, 4,$$

where c_{0j} are arbitrary constants. The expansions exist when $\alpha \neq 0$;

Expansions obtained in the neighbourhood of the nonsingular point

$$\mathcal{O}_5: \;\; w = \sum_{s=0}^{\infty} c_s (z - z_0)^s,$$

where $c_0, c_1 \in \mathbb{C}$ are arbitrary constants, $c_0 \neq 0$, $c_0 \neq 1$. The expansions exist for all values of parameters;

$$\mathcal{O}_{6,7}: \;\; w = 1 + (-1)^j \sqrt{-2\delta}(z - z_0) + \sum_{s=2}^{\infty} c_{sj} (z - z_0)^s, \;\; j = 6, 7,$$

where c_{3j} are arbitrary constants. The expansions exist when $\delta \neq 0$;

$$\mathbf{O_8}: \;\; \mathbf{w} = \mathbf{1} - \frac{\gamma}{\mathbf{2z_0}}(\mathbf{z} - \mathbf{z_0})^{\mathbf{2}} + \sum_{\mathbf{s}=\mathbf{4}}^{\infty} \mathbf{c_s}(\mathbf{z} - \mathbf{z_0})^{\mathbf{s}},$$

where c_4 is an arbitrary constant. The expansions exist when $\gamma \neq \mathbf{0}$, $\delta = \mathbf{0}$;

Expansions obtained in the neighbourhood of the nonsingular point

$$\mathcal{O}_9: \ w = \sum_{s=-2}^{\infty} c_s (z - z_0)^s,$$

expansions exist when $\alpha = 0$;

$$\mathcal{O}_{10}: \ w = \sum_{s=2}^{\infty} c_s (z - z_0)^s,$$

where c_2 is an arbitrary constant. The expansions exist when $\beta = 0$.
The constants $c_{ij}, \ c_j$ about which it has not been mentioned that they are arbitrary are uniquely defined.
The family $\mathcal{O}_5$ is two-parametric, the rest families are one-parametric.
The expansions $\mathcal{O}_j, \ j = 1, 2, 5, 6, 7, 8, 10$ (they are Taylor series) converge in the neighbourhood of $z = z_0$, the expansions $\mathcal{O}_3, \ \mathcal{O}_4, \ \mathcal{O}_9$ (they are Laurent series) converge in the deleted neighbourhood of $z = z_0$.

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