The Newton-Puisuex Polygon for $q-$difference equations. Applications.

Pedro Fortuny Ayuso

Będlewo, August 2011

Universidad de Oviedo, Spain.

First things first

1.-Thanks to the Scientific Committee

2.-Thanks to the Organizing Committee
(Prof. Walser, Greg, Galina, Sławomir)

3.-Enjoy

Prior Art

- Maillet-Malgrange (1903-1989): Any power series solution of an analytic ODE is of Gevrey type.
- J. Cano (1993): Any power series solution of a Gevrey ODE is Gevrey, explicit computation (Newton Polygon).
- Zhang 1998 (& di Vizio 2007 ultrametric): Any power series solution of an analytic $q-$difference equation is of $q-$Gevrey type.

This work is the natural next step. Techniques like Cano's.

Aims: solve & ($q-$Gevrey) bound

Given a general $q-$difference equation like (no independent term)

$$P(x, y, \sigma(y)) = \sigma(y)y^2 + \sigma(y)^2 + x^2y + x = 0$$

Aims: solve & ($q-$Gevrey) bound

Given a general $q-$difference equation like (no independent term)

$$P(x, y, \sigma(y)) = \sigma(y)y^2 + \sigma(y)^2 + x^2y + x = 0$$

- Compute solutions

Aims: solve & ($q-$Gevrey) bound

Given a general $q-$difference equation like (no independent term)

$$P(x,y,\sigma(y)) = \sigma(y)y^2 + \sigma(y)^2 + x^2y + x = 0$$

- Compute solutions
- Behaviour of solutions

Aims: solve & ($q-$Gevrey) bound

Given a general $q-$difference equation like (no independent term)

$$P(x, y, \sigma(y)) = \sigma(y)y^2 + \sigma(y)^2 + x^2y + x = 0$$

- Compute solutions
- Behaviour of solutions
  - Convergence, $q-$Gevrey asymptotics

Aims: solve & ($q-$Gevrey) bound

Given a general $q-$difference equation like (no independent term)

$$P(x, y, \sigma(y)) = \sigma(y)y^2 + \sigma(y)^2 + x^2y + x = 0$$

- Compute solutions
- Behaviour of solutions
    - Convergence, $q-$Gevrey asymptotics
    - Rational rank, “size” of the exponent’s semigroup

Framework

Framework

Equation: A $q-$difference equation is [we use $|q| > 1$]

$$P(x, y_0, \dots, y_n) \in \mathbb{C}[[x^{\mathbb{R}_{>0}}]][[y_0, \dots, y_n]]$$

(too general, usually $\mathbb{C}[[x, y_0, \dots, y_n]]$), $P(0) = 0$.

Framework

Equation: A $q-$difference equation is [we use $|q|>1$]

$$P(x,y_0,\dots,y_n)\in\mathbb{C}[[x^{\mathbb{R}_{>0}}]][[y_0,\dots,y_n]]$$

(too general, usually $\mathbb{C}[[x,y_0,\dots,y_n]]$), $P(0)=0$.

Solution: A solution of P is a generalized power series

$$f(x)=\sum_{\gamma\in\Gamma}f_\gamma x^\gamma,\ \text{with } \Gamma\subset\mathbb{R}_{>0}$$

where Γ is well ordered, such that

$$P(x,f(x),f(qx),\dots,f(q^nx))=0$$

(a determination of the logarithm is fixed).

Algebraic Curves (Newton)

Solution of [implicit equation]

$$y + xy^3 + y^5 + x^3 = 0$$

Algebraic Curves (Newton)

Solution of [implicit equation]

$$y + xy^3 + y^5 + x^3 = 0$$

Start with an “obvious” term (least degree, $y = -x^3$).

Algebraic Curves (Newton)

Solution of [implicit equation]

$$y + xy^3 + y^5 + x^3 = 0$$

Start with an "obvious" term (least degree, $y = -x^3$).
Substitute: $y = y - x^3$ to get

$$y - x^{10} - x^{15} + [\dots] + 10x^9y^2 + [\dots] + y^5 = 0$$

Algebraic Curves (Newton)

Solution of [implicit equation]

$$y + xy^3 + y^5 + x^3 = 0$$

Start with an "obvious" term (least degree, $y = -x^3$).
Substitute: $y = y - x^3$ to get

$$y - x^{10} - x^{15} + [\dots] + 10x^9y^2 + [\dots] + y^5 = 0$$

Following "obvious" term is $y = x^{10}$.

Algebraic Curves (Newton)

Solution of [implicit equation]

$$y + xy^3 + y^5 + x^3 = 0$$

Start with an "obvious" term (least degree, $y = -x^3$).
Substitute: $y = y - x^3$ to get

$$y - x^{10} - x^{15} + [\dots] + 10x^9y^2 + [\dots] + y^5 = 0$$

Following "obvious" term is $y = x^{10}$.
Go on. This one was easy.

Algebraic Curves (Puisuex)

What happens with

$$y^2 - x^3 = 0?$$

(no formal power series solution).

Algebraic Curves (Puisuex)

What happens with

$$y^2 - x^3 = 0?$$

(no formal power series solution).
Obviously

$$y = x^{3/2}$$

is a “solution”.

Algebraic Curves (Puisuex)

What happens with

$$y^2 - x^3 = 0?$$

(no formal power series solution).
Obviously

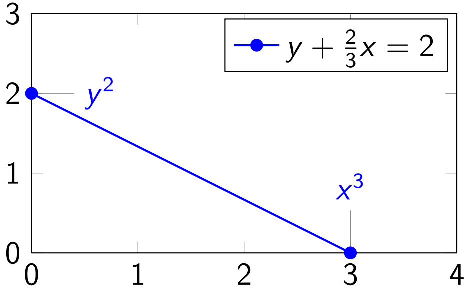
$$y = x^{3/2}$$

is a "solution".
What with ...

$$y^6 + x^5y^5 + xy^4 + x^3y^2 + x^{10} = 0?$$

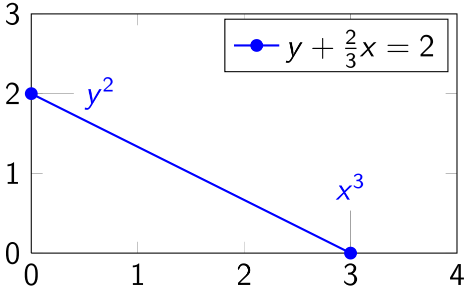
The Polygon

For $y^2 - x^3$, the number $\frac{3}{2}$ corresponds to the side *inclination*



The Polygon

For $y^2 - x^3$, the number $\frac{3}{2}$ corresponds to the side *inclination*

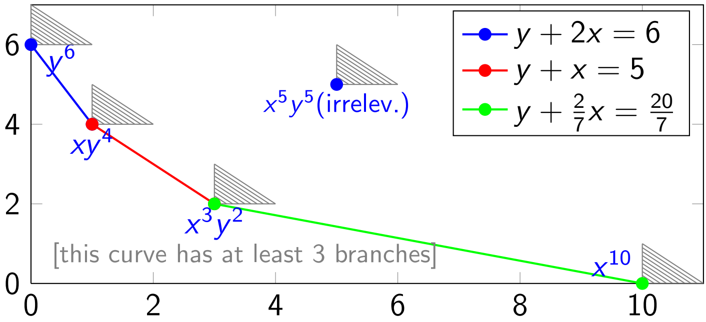


The coefficient $y = 1 \cdot x^{3/2}$ comes from substituting $y = y + cx^{3/2}$, which gives at $y = 0$:

$$x^3 + c^2x^3 = 0 \Rightarrow c = \pm 1.$$

The Polygon: positive convex hull of cloud

For $y^6 + x^5y^5 + xy^4 + x^3y^2 + x^{10} = 0$:



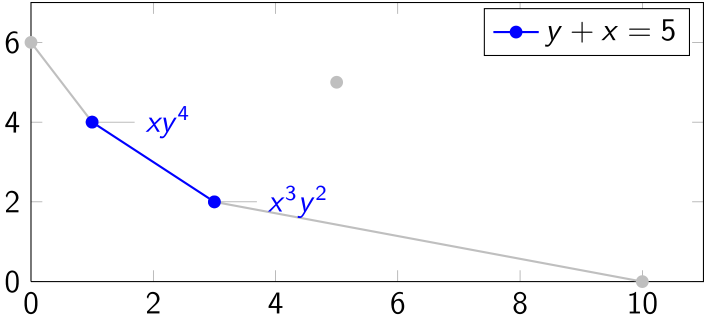
The inner point $(5, 5)$ is irrelevant for starting a solution.
Solutions $y = cx^{\alpha} + \dots$ admit $\alpha = \frac{1}{2}, 1, \frac{7}{2}$, the *inclinations*.

The Polygon: coefficients

If a solution starts with $y = cx^{\alpha}$, α is the inclination, and c?

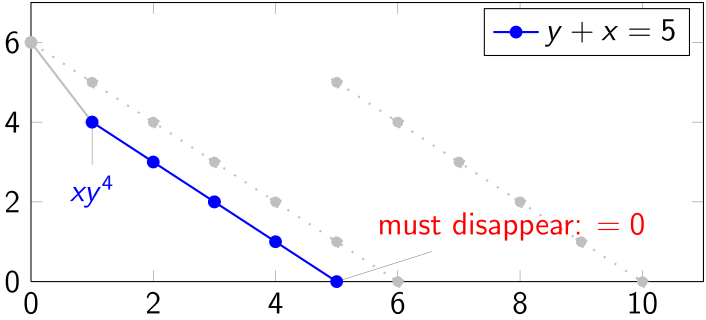
The Polygon: coefficients

If a solution starts with $y = cx^{\alpha}$, α is the inclination, and c?
Substitute: $y = y + cx^{\alpha}$. Example: $y = y + cx$ ($\alpha = 1$):



The Polygon: coefficients

If a solution starts with $y = cx^{\alpha}$, α is the inclination, and c?
Substitute: $y = y + cx^{\alpha}$. Example: $y = y + cx$ ($\alpha = 1$):

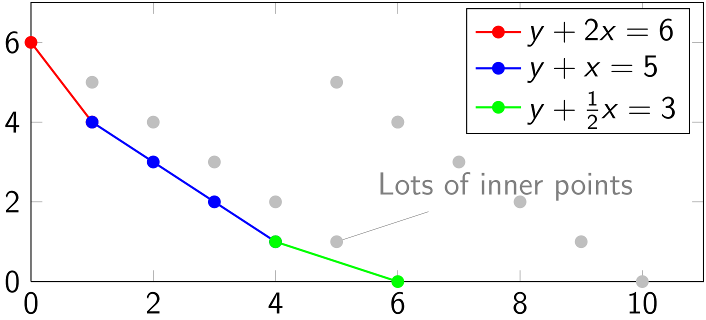


Lowest $x-$term for that side must be 0:

$$P_{\alpha}(C) = x(y + cx)^4 + x^3(y + cx)^2\Big]_{y=0} = 0 \simeq c^4 + c^2 = 0.$$

The Polygon: coefficients

If a solution starts with $y = cx^{\alpha}$, α is the inclination, and c?
Substitute: $y = y + cx^{\alpha}$. Example: $y = y + cx$ ($\alpha = 1$):



Setting $c = i$, a new Newton Polygon appears. The green side is *new*, and has inclination$> \alpha = 1$.
Recurse with new equation and Polygon.

Lexicon

- N-P Polygon: convex envelope of "cloud of points (i, j)."

Lexicon

- N-P Polygon: convex envelope of "cloud of points (i, j)."
- Given a slope μ, the valuation $\nu_\mu(P)$ is the minimal value for $j + \mu i$ and the initial form

$$In_\mu(P) = \sum_{i+\mu x=\nu_\mu(P)} p_{ij}x^i y^j$$

Lexicon

- N-P Polygon: convex envelope of "cloud of points (i,j)."
- Given a slope μ, the valuation $\nu_\mu(P)$ is the minimal value for $j+\mu i$ and the initial form

$$In_\mu(P) = \sum_{i+\mu x=\nu_\mu(P)} p_{ij}x^i y^j$$

- Given $In_\mu(P)$, plausible coeffs c are the roots of the initial polynomial

$$Q_\mu(T) \equiv In_\mu(P)(1,T) = 0$$

Lexicon

- N-P Polygon: convex envelope of "cloud of points (i, j)."
- Given a slope μ, the valuation $\nu_\mu(P)$ is the minimal value for $j + \mu i$ and the initial form

$$In_\mu(P) = \sum_{i+\mu x=\nu_\mu(P)} p_{ij} x^i y^j$$

- Given $In_\mu(P)$, plausible coeffs c are the roots of the initial polynomial

$$Q_\mu(T) \equiv In_\mu(P)(1, T) = 0$$

But: in the algebraic case, μ is always rational and any slope and root give rise to a solution.

Newton-Puisuex for q-difference equations

The polygon for $q-$diff equation $P = \sum p_{ij_0\dots j_n} x^i y_0^{j_0} \dots y_n^{j_n}$.

Newton-Puisuex for q-difference equations

The polygon for $q-$diff equation $P = \sum p_{ij_0\dots j_n} x^i y_0^{j_0} \dots y_n^{j_n}$.

- Notice $\deg_x(y_n(x^\alpha)) = \deg_x(y_0(x^\alpha))$ (not so in differential equations).

Newton-Puisuex for q-difference equations

The polygon for $q-$diff equation $P = \sum p_{ij_0\ldots j_n} x^i y_0^{j_0} \ldots y_n^{j_n}$.

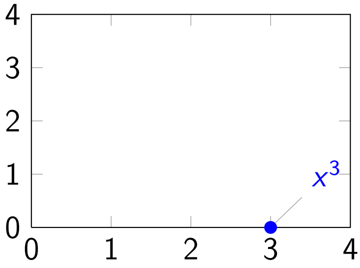
- Notice $\deg_x(y_n(x^\alpha)) = \deg_x(y_0(x^\alpha))$ (not so in differential equations).
- So, one should place $x^i y_0^{j_0} \ldots y_n^{j_n}$ at (i, j) with $j = j_0 + \cdots + j_n$.

Newton-Puisuex for q-difference equations

The polygon for $q-$diff equation $P = \sum p_{ij_0\dots j_n} x^i y_0^{j_0} \dots y_n^{j_n}$.

- Notice $\deg_x(y_n(x^\alpha)) = \deg_x(y_0(x^\alpha))$ (not so in differential equations).
- So, one should place $x^i y_0^{j_0} \dots y_n^{j_n}$ at (i, j) with $j = j_0 + \dots + j_n$.

For the equation $y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0$

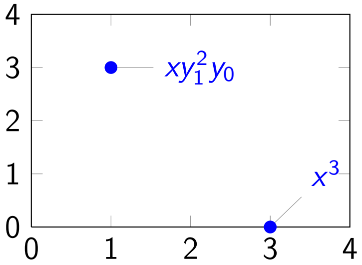


Newton-Puisuex for q-difference equations

The polygon for $q-$diff equation $P = \sum p_{ij_0\dots j_n} x^i y_0^{j_0} \dots y_n^{j_n}$.

- Notice $\deg_x(y_n(x^\alpha)) = \deg_x(y_0(x^\alpha))$ (not so in differential equations).
- So, one should place $x^i y_0^{j_0} \dots y_n^{j_n}$ at (i, j) with $j = j_0 + \dots + j_n$.

For the equation $y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0$

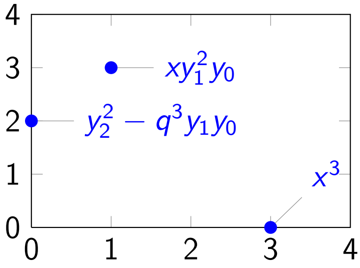


Newton-Puisuex for q-difference equations

The polygon for $q-$diff equation $P = \sum p_{ij_0\dots j_n} x^i y_0^{j_0} \dots y_n^{j_n}$.

- Notice $\deg_x(y_n(x^\alpha)) = \deg_x(y_0(x^\alpha))$ (not so in differential equations).
- So, one should place $x^i y_0^{j_0} \dots y_n^{j_n}$ at (i, j) with $j = j_0 + \dots + j_n$.

For the equation $y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0$

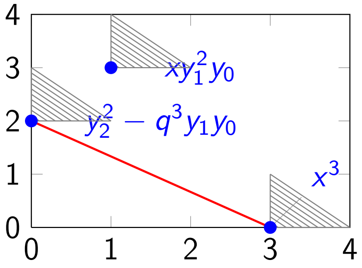


Newton-Puisuex for q-difference equations

The polygon for $q-$diff equation $P = \sum p_{ij_0\ldots j_n} x^i y_0^{j_0} \ldots y_n^{j_n}$.

- Notice $\deg_x(y_n(x^\alpha)) = \deg_x(y_0(x^\alpha))$ (not so in differential equations).
- So, one should place $x^i y_0^{j_0} \ldots y_n^{j_n}$ at (i, j) with $j = j_0 + \cdots + j_n$.

For the equation $y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0$



Pathologies

Sample equation: $y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0$

Pathologies

Sample equation: $y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0$

Null Q: For $\mu > \frac{2}{3} \longrightarrow In_\mu = y_2^2 - q^3 y_1 y_0$, so:

$$Q_\mu(T) = (q^2 T)^2 - q^3 (qT) T \equiv 0.$$

Any c will make $Q_\mu(c) = 0$: free slopes.

Pathologies

Sample equation: $y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0$

Null Q: For $\mu > \frac{2}{3} \longrightarrow In_\mu = y_2^2 - q^3 y_1 y_0$, so:

$$Q_\mu(T) = (q^2 T)^2 - q^3 (qT) T \equiv 0.$$

Any c will make $Q_\mu(c) = 0$: free slopes.

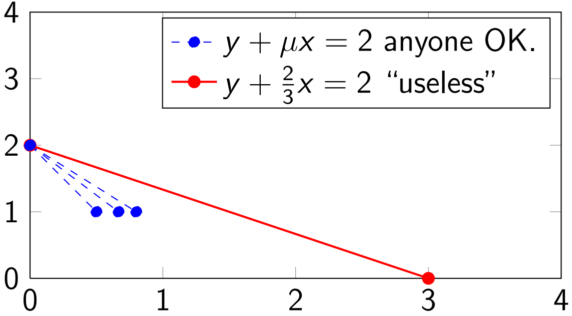
Constant Q: For $\mu = \frac{2}{3} \longrightarrow In_\mu = y_2^2 - q^3 y_1 y_0 + x^3$, so

$$Q_\mu(T) = (q^2 T)^2 - q^3 (qT) T + 1 \equiv 1$$

No c will make $Q_\mu(c) = 0$. Useless slope.

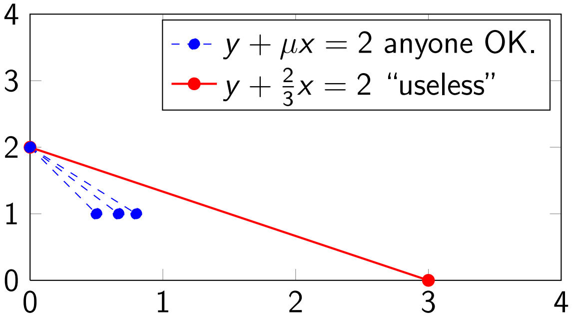
Exceptional behaviour

Recall: $y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0$



Exceptional behaviour

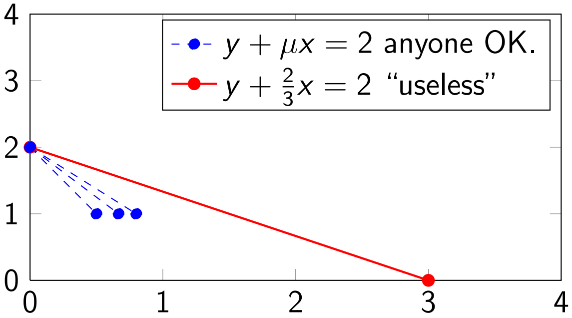
Recall: $y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0$



- $Q_\mu(T) = 0$ for $\mu > 2/3$: $y = cx^{1/\mu} + \ldots$ possible (may have $\mu \notin \mathbb{Q}$: not so in algebraic curves).

Exceptional behaviour

Recall: $y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0$



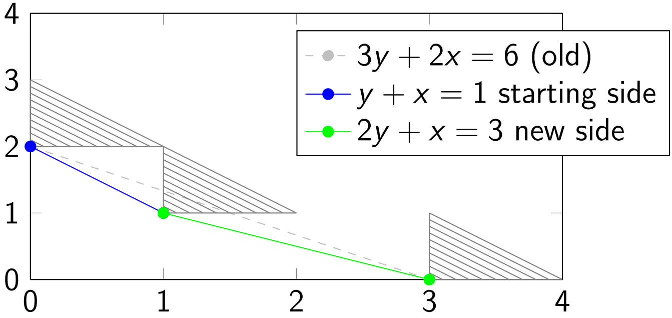
- $Q_\mu(T) = 0$ for $\mu > 2/3$: $y = cx^{1/\mu} + \dots$ possible (may have $\mu \notin \mathbb{Q}$: not so in algebraic curves).
- $Q_{2/3}(T) = 1$: no solution starting with $cx^{3/2}$.

Substitution step

After substituting

$$y = y + cx^{\alpha}$$

new Polygon, continue with greater inclination (to the right).
Say $y = y + x$ in $y_2^2 - q^3 y_1 y_0 + x y_1^2 y_0 + x^3 = 0$,



Results: Solution construction

Lemma

Generically, the Newton-Puiseux algorithm gives rise to a solution. And solutions correspond to the algorithm.

Results: Solution construction

Lemma

Generically, the Newton-Puiseux algorithm gives rise to a solution. And solutions correspond to the algorithm.

Theorem

The semigroup generated by Γ (exponents) is finitely generated (like for ODE). [Hence $\sum x^{(2-\frac{1}{n})}$ is forbidden].

Results: Solution construction

Lemma

Generically, the Newton-Puiseux algorithm gives rise to a solution. And solutions correspond to the algorithm.

Theorem

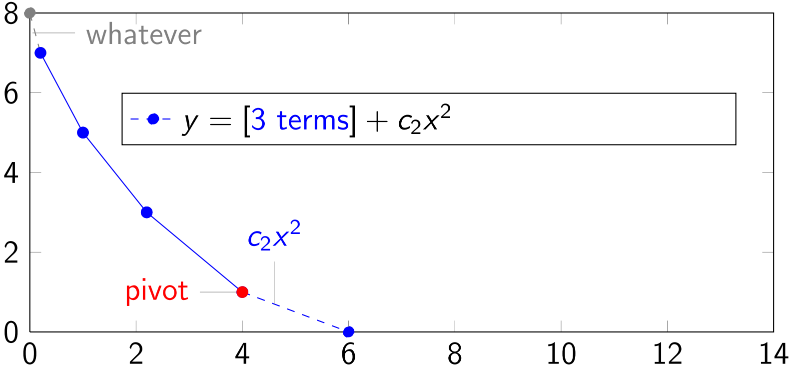
The semigroup generated by Γ (exponents) is finitely generated (like for ODE). [Hence $\sum x^{(2-\frac{1}{n})}$ is forbidden].

Theorem

If $P = A(x, y_0) + B(x, y_0)y_1$ ("order and degree 1"), then rational rank$(\Gamma) \leq 2$. ("Only one irrational exponent").

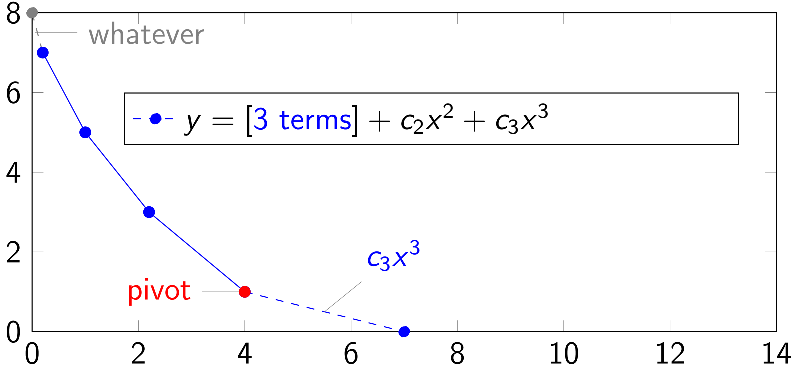
Pivot point

In a finite number of steps, the topmost vertex of the *interesting* side is fixed (usually at height 1).



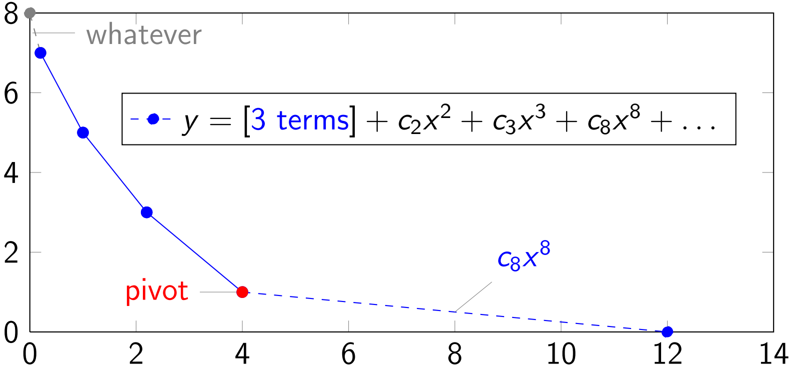
Pivot point

In a finite number of steps, the topmost vertex of the *interesting* side is fixed (usually at height 1).



Pivot point

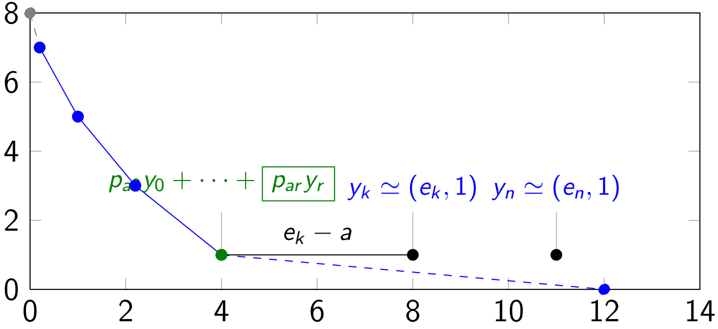
In a finite number of steps, the topmost vertex of the *interesting* side is fixed (usually at height 1).



Results: $q-$Gevrey bounds (intro)

$$t+1-\text{ Gevrey } \Leftrightarrow \sum \frac{A_n}{|q|^{tn(n+1)/2}} x^n \text{ convergent}$$

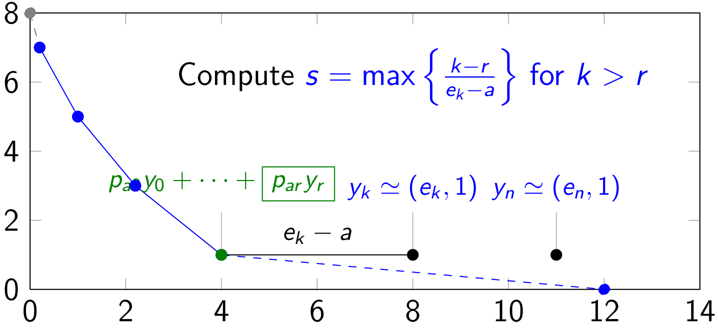
Assume Pivot Point $(a,1)$ contributions up to $\boxed{r}$, $\mathrm{ord}(P)=n$.
Worst point: $(e_k,1)$, high order "bad", far from pivot "good".



Results: $q-$Gevrey bounds (intro)

$$t + 1- \text{ Gevrey } \Leftrightarrow \sum \frac{A_n}{|q|^{tn(n+1)/2}} x^n \text{ convergent}$$

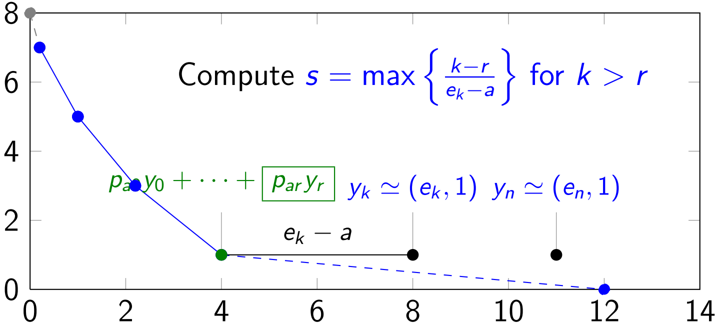
Assume Pivot Point $(a, 1)$ contributions up to $\boxed{r}$, $\mathrm{ord}(P) = n$.
Worst point: $(e_k, 1)$, high order "bad", far from pivot "good".



Results: $q-$Gevrey bounds (intro)

$$t+1-\text{ Gevrey } \Leftrightarrow \sum \frac{A_n}{|q|^{tn(n+1)/2}} x^n \text{ convergent}$$

Assume Pivot Point $(a,1)$ contributions up to $\boxed{r}$, $\operatorname{ord}(P)=n$.
Worst point: $(e_k,1)$, high order "bad", far from pivot "good".



For pivot at (a,b), *analogous* computation.

Results: $q-$Gevrey bounds (I)

Formal power series solution $f(x) \in \mathbb{C}[[x]]$ (can be generalized to rational exponents easily). Recall $s = \frac{\Delta \text{ord}}{\text{distance}}$.

Theorem ($q-$Gevrey Malgrange-Maillet)

If P has $q-$Gevrey order $t+1$, then $f(x)$ has $q-$Gevrey order $\leq s+t+1$.

Results: $q-$Gevrey bounds (I)

Formal power series solution $f(x) \in \mathbb{C}[[x]]$ (can be generalized to rational exponents easily). Recall $s = \frac{\Delta\text{ord}}{\text{distance}}$.

Theorem ($q-$Gevrey Malgrange-Maillet)

If P has $q-$Gevrey order $t+1$, then $f(x)$ has $q-$Gevrey order $\leq s+t+1$.

Corollary

If the pivot point of $f(x)$ contains order n, then $f(x)$ has the same $q-$Gevrey order as P.

Results: closed-eyed bound

Corollary

If $P(x, y_0, \ldots, y_n)$ has $q-$Gevrey order $t+1$, then any solution has at most $q-$Gevrey order $n+t+1$.

No information required on the solution, whereas the results above assume the pivot point is known.

Thanks, etc

Questions?

And many thanks