POWER SERIES SOLUTIONS OF NON-LINEAR q-DIFFERENCE EQUATIONS AND THE NEWTON-PUISEUX ALGORITHM

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Abstract. Adapting the Newton-Puiseux Polygon algorithm to $q-$difference equations of any order and degree, we give a bound for the $q-$Gevrey order of their solutions and, for the specific case of order and degree 1, we also bound the rational rank of generalized power series solutions.

1. Introduction

The Newton Polygon construction and its generalization by Puiseux (see [5] for an interesting detailed historical narrative) has been successfully used countless times both in the algebraic [11], [12] [8] and in the differential contexts [6], [9], [3], [4], [17] (this is just a biased and briefest of samples, obviously).

We extend the Newton-Puiseux algorithm its use to non-linear $q-$difference equations. For the linear case it was introduced by Adams [1], and used by Ramis [14] in a more modern context. Since then, it has been extensively used in this setting (see, just to give an example, [16]).

What we attempt is to use it to produce generalized power series solutions of non-linear $q-$difference equations, and to study the asymptotic behaviour of these solutions in terms of those of the original equation, in the same spirit as [2]. The method allows us to bound the $q-$Gevrey order of a formal power series solution in terms of that of the original equation, thus generalizing Zhang's [18] result in the $q-$difference case (Zhang's result is for $q-$difference-differential equations, but analytic). This generalization is analogue to the one given by Cano in the same paper for the Malgrange-Maillet theorem [10].

Let $K = \mathbb{C}[[x]]$ be the ring of formal power series in one variable over the complex field and denote by σ the $K-$automorphism given by $\sigma(x) = qx$ for some $q \in \mathbb{C}$ with $|q| > 1$ (the case $|q| < 1$ is equivalent, but we do not deal with $|q| = 1$).

A $q-$*difference equation* is a "functional equation" of the form

$$P(x, y_0, y_1, \dots, y_n) = 0 \tag{1}$$

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where $P \in K[[x, y_0, \dots, y_n]]$, y_0 is the "unknown", understood as a "function of x" and $y_i(x) = y_{i-1}(qx) = \sigma(y_{i-1}(x))$ for $i > 0$. This equation is plainly analogue to the differential case —whose Polygon was completely studied in [2], correcting the classical "and so on…" mistakes in [9]. We shall also say that P is a $q-$difference equation, obviating the $= 0$.

Unlike the differential case, however, the variables y_i will have the same $x-$weight as y_0 because the order valuation of K is invariant by σ, i.e. $\mathrm{ord}_x(\sigma(f)) = \mathrm{ord}_x(f)$ for $f \in K$.

Let $\mathcal{K} = \mathbb{C}[[x^{\mathbb{R}_{\geq 0}}]]$ be the ring of Hahn power series [7] in x with exponents in $\mathbb{R}_{\geq 0}$, that is, the ring of power series whose exponents are a well-ordered subset of $\mathbb{R}_{\geq 0}$. We shall say that an element

$$g = \sum_{\gamma \in \Gamma} x^\gamma \in \mathcal{K}$$

is a *solution* of (1) if

$$P(x, g(x), \sigma(g(x)), \dots, \sigma^n(g(x))) = P(x, g(x), g(qx), \dots, g(q^n x)) = 0,$$

where we are assumming that a determination of the logarithm has been chosen and fixed in order to compute all the values q^α for $\alpha \in \Gamma \setminus \mathbb{N}$.

2. The Newton Polygon in the general case

Let $\mathcal{F} = \mathbb{C}[[x^{\mathbb{R}_{\geq 0}}]][[y_0, \dots, y_n]]$ denote the ring of formal power series in $n+1$ variables over the ring of Hahn series in one variable over $\mathbb{C}$. As is customary, given an element $P(x, y_0, \dots, y_n) = \sum P_{\gamma,\rho} x^\gamma y_0^{\rho_0} \dots y_n^{\rho_n} \in \mathcal{F}$, its associated *cloud of points* is the set of those $(\gamma, j) \in \mathbb{R}^2$ of indices of nonzero coefficients:

$$\mathcal{C}(P) = \left\{ (\gamma, j) \,\middle|\, \exists \rho \text{ with } |\rho| = j \text{ and } P_{\gamma,\rho} \neq 0 \right\}$$

and given a subset $S \subset \mathbb{R}^2_{\geq 0}$, its associated *Newton Polygon*, $\mathcal{N}(S)$ is the border of the convex hull of

$$S_+ = \left\{ p + \mathbb{R}^2_{\geq 0} \,\middle|\, p \in S \right\}$$

(the set obtained from adjoining at each point of S the first quadrant of the real plane).

Definition. *A $q-$difference equation of order n is $P \in \mathcal{F}$.*

Definition 1. *The* Newton Polygon $\mathcal{N}(E)$ *of P*

$$P = \sum_{\gamma,\rho} P_{\gamma,\rho} x^\gamma y_0^{\rho_0} \dots y_n^{\rho_n} \tag{2}$$

is the Newton Polygon of $\mathcal{C}(P)$.

Notice that, as we mentioned in the introduction, there is no "translation to the left" because the operator $y_k(x) = q^k x$ does not modify the degree in x.

The elements of the Newton Polygon are essentially two: corners and sides. Let $\boldsymbol{x}, \boldsymbol{y}$ denote the standard coordinates on $\mathbb{R}^2$:

Definition 2. *A point $p=(\gamma,j)\in\mathcal{N}(P)$ is a* corner *of $\mathcal{N}(P)$ if there exists a line $\boldsymbol{y}+\mu\boldsymbol{x}=k$ (with $\mu\in\mathbb{R}_{>0}$) whose intersection with $\mathcal{N}(P)$ is the singleton $\{p\}$.*

From the definition and the structure of $\mathcal{F}$ it is clear that there is a finite number of corners in $\mathcal{N}(P)$ (at most one for each height $\boldsymbol{y}$ up to the highest one). We shall always assume they are ordered by their $\boldsymbol{x}$ coordinate (so that there is a *first* corner —the leftmost one— and a *last* one). The *height* of $\mathcal{N}(P)$ is the $\boldsymbol{y}-$coordinate of the leftmost vertex.

Definition 3. *A* side *of $\mathcal{N}(P)$ is either the vertical ray starting at the first corner of the Newton Polygon or the horizontal ray starting at the last one or any segment joining two consecutive corners of $\mathcal{N}(P)$. A compact side s has an associated* slope *$\mu(s)$ or μ_s, and* value *$\nu(s)$, both given by the equation of the only line $\boldsymbol{y}+\mu_s\boldsymbol{x}=\nu(s)$ containing it.*

And the key elements to compute solutions using the Newton Polygon are those of *valuation*, *initial form* and *inicial polynomial* associated to a *slope*:

Definition 4. *Given a slope $\mu\in\mathbb{R}_{>0}$, the* valuation associated to μ *is the map*

$$\nu_\mu:\mathcal{F}\to\mathbb{R}$$

given by

$$\nu_\mu(P)=\nu_\mu\left(\sum P_{\gamma,\rho}x^\gamma y_0^{\rho_0}\dots y_n^{\rho_n}\right)=\min\left\{j+\mu\gamma\mid(\gamma,j)\in\mathcal{C}(P)\right\}.$$

Definition 5. *Given a slope $\mu\in\mathbb{R}_{>0}$, the* initial form *of P with respect to μ is*

$$\Phi_\mu(P)(x,y_0,\dots,y_n)=\sum P_{\gamma,\rho}x^\gamma y_0^{\rho_0}\dots y_n^{\rho_n}$$

where the sum is taken for $|\rho|+\mu\gamma=\nu_\mu(P)$.

Definition 6. *The* initial polyonomial *of P with respect to (or associated to) slope μ (or to the corresponding side) is*

$$Q_\mu^E(T)=\Phi_\mu(P)(1,T,qT,\dots,q^nT).$$

If $Q_\mu^E(T)=0$, μ will be called an exceptional *slope for E.*

Given $g(x)\in\mathcal{F}$, one can consider the substitution

$$y=\bar{y}+g(x)\tag{3}$$

into equation P, which will be denoted $P[\bar{y}+g(x)]$, which gives

$$\begin{aligned}&P[\bar{y}+g(x)]:=\bar{P}(x,\bar{y}_0,\dots,\bar{y}_n)=\\&=\sum P_{\gamma,\rho}x^\gamma\left((\bar{y}_0+g(x))^{\rho_0},\dots,(\bar{y}_n+g(q^nx))^{\rho_n}\right)=\sum\bar{P}_{\gamma,\rho}x^\gamma\bar{y}_0^{\rho_0}\dots\bar{y}_n^{\rho_n},\end{aligned}\tag{4}$$

which is another $q-$difference equation of order n, called the *translation of E by $g(x)$* or *by substitution (3)*.

Definition 7. *We shall say that an element*

$$g = \sum_{\gamma\in\Gamma} c_\gamma x^\gamma \in \mathcal{F}$$

is a solution of (2) *if*

$$P(x, g(x), \dots, g(q^n x)) = 0$$

(using the already fixed determination of the logarithm for computing q^α for any α).

Remark 1. Let $g(x) \in \mathcal{F}$ be any Hahn series and let

$$\bar{P}(x, \bar{y}_0, \dots, \bar{y}_n)$$

be the "translated equation". It is clear that $g(x)$ is a solution of P if and only if $\bar{P}(x, 0, 0) = 0$.

Remark 2. From the definition (and from the basic properties of Hahn series) it follows that if

$$g(x) = \sum_\Gamma g_\gamma x^\gamma$$

is a solution of P and

$$\lfloor g \rfloor_\eta(x) = \sum_{\gamma\leq\eta} g_\gamma x^\gamma$$

is its truncation up to order η and if $\bar{P}$ is the translation of P by $\lfloor g \rfloor_\eta(x)$, then $g(x) - \lfloor g \rfloor_\eta(x)$ is a solution of $\bar{P}$.

The following is essentially what motivates the Newton-Puiseux Polygon construction:

Lemma 1. *Let $c \in \mathbb{C}$ and $\alpha \in \mathbb{R}_{>0}$ be a coefficient and a slope, respectively. If $g = cx^\alpha + \cdots \in \mathcal{K}$ is a solution of P whose least order term is cx^α, then*

$$Q_\alpha^E(c) = 0. \tag{5}$$

(The coefficient of order α is a root of the initial polynomial for α).

Proof. One needs only perform the substitution $\bar{y} = y + cx^\alpha$ and verify on one hand that $\Phi_\alpha(P)(x, 0, 0) = Q_\alpha^P(c)$ and on the other that the term of least degree of $P(x, y_0 + cx^\alpha, \dots, y_n + cq^{n\alpha}x^\alpha)$ is precisely $\Phi_\alpha(P)(x, 0, \dots, 0) = Q_\alpha^P(c)$. ∎

Finally, a substitution of slope α *only modifies the Polygon* for slopes less than or equal to α, that is, it only modifies the sides *to the right* of the side of slope α, or the corner met by the line $\boldsymbol{y} + \alpha\boldsymbol{x} = \nu_\alpha(P)$. Moreover, the cloud of points of the transformed equation belongs to the semigroup generated by the previous cloud of points "and α":

Lemma 2. *Let $\mathcal{N}(P)$ be the Newton Polygon of P and $\mathcal{N}(\bar{P})$ be the one of the translated equation by the substitution $y = \bar{y} + cx^\alpha$. Then*

(1) *The cloud of points $\mathcal{C}(\bar{P})$ —and hence $\mathcal{N}(\bar{P})$ is included in the semi-group generated by $\mathcal{C}(P)$ and $(\alpha,-1)$.*
(2) *The Newton Polygons $\mathcal{N}(P)$ and $\mathcal{N}(\bar{P})$ have the same sides of slope greater than α (that is, to the left of slope α).*
(3) *For any $\beta > \alpha$, $\Phi_\beta(P) = \Phi_\beta(\bar{P})$ and consequently $Q_\beta^P(T) = Q_\beta^{\bar{P}}(T)$.*
(4) *If $A_{\gamma,\rho}$ is such that $(\gamma,|\rho|)$ is the leftmost corner of $\mathcal{N}(P)$ with value $\nu_\alpha(P)$, then $\bar{A}_{\gamma,\rho} = A_{\gamma,\rho}$ and $(\gamma,|\rho|)$ is also the leftmost corner of $\mathcal{N}(\bar{P})$ with value $\nu_\alpha(P)$.*

Remark 3. As a matter of fact, *the coefficients corresponding to sides of slope greater than α do not change*, but this result will not be used.

The proof is an straightforward computation. However, one has a specific and relevant behaviour in the case of exceptional slopes corresponding to corners when the equation has order and degree 1:

Lemma 3. *Assume P is of order and degree 1. Let μ be an exceptional slope of P corresponding to a corner of $\mathcal{N}(P)$ and let c be any nonzero coefficient. Then the Newton Polygon of the translation $\bar{P}$ of P by cx^μ has a corner at height 1.*

Proof. Let $\Phi_\mu(P)$ be the initial form of P for the slope μ. We only need to show that $\Phi_\mu(\bar{P})$ has a point at height 1 and no point at height 0. But, since μ meets $\mathcal{N}(P)$ at a single point,

$$\Phi_\mu(P)(x,y,\sigma(y)) = A_{\gamma,j,0}x^\gamma y_0^j + A_{\gamma,j-1,1}x^\gamma y_0^{j-1}y_1,$$

for just one (γ,j) (the coordinates of the point). The translation $y=\bar{y}+cx^\mu$ works as follows

$$\Phi_\mu(P)(x,y,\sigma(y)) = (\ldots) = Q_\mu^P(c)x^{\nu_\mu(P)} + \bar{y}_1 A_{\gamma,j-1,1}c^{j-1}x^{\gamma+\mu(j-1)} + \ldots$$

On one hand, $Q_\mu^P = 0$ by hypothesis and on the other, $A_{\gamma,j-1,1} \neq 0$ (because otherwise Q_μ^P cannot be 0, μ corresponding to a single point of $\mathcal{N}(P)$). Thus, the translation gives rise to the point $(\nu_\mu(P)-1,1) = (\gamma+\mu(j-1),1)$ in $\mathcal{C}(\bar{P})$ and (since $Q_\mu^P = 0$) the point $(\nu_\mu(P),0)$ does not belong to $\mathcal{C}(\bar{P})$, which gives the result. □

Lemma 4. *Let $g(x) = \sum_\Gamma c_\gamma x^\gamma$ be a solution of P. For $\eta \in \Gamma$, let P_η be the Newton Polygon of the translation of P by the substitution*

$$y = \bar{y} + \sum_{\gamma \leq \eta} c_\gamma x^\gamma.$$

(The truncation up to order η). Then either $\lfloor g \rfloor_\eta(x)$ is a solution of P or $\mathcal{N}(P_\eta)$ has at least a compact side of slope less than η.

Proof. This is a straightforward consequence of Remark 2 and Lemma 2. □

The following corollary, which follows from the well-ordering of $\mathbb{N}$ serves also as a definition:

Corollary 1 (Definition of pivot point)**.** *Let $g(x) \in \mathcal{F}$ be a Hahn series with $g(0) = 0$*

$$g(x) = \sum_{\Gamma \subset \mathbb{R}_{>0}} c_\gamma x^\gamma.$$

For each $\gamma \in \Gamma$, let P_γ denote the equation

$$P_\gamma = P[y + \lfloor g \rfloor_\gamma] = P(x, y_0 + g_\gamma(x), \dots, y_n + g_\gamma(q^n x))$$

where $g_\gamma(x)$ is the truncation of $g(x)$ up to order γ. Assume Γ is not finite. Then there is $\gamma_0 \in \Gamma$ and $j \in \mathbb{Z}_{>0}$ such that the corner (γ_0, j) belongs to $\mathcal{N}(P_\gamma)$ for any $\gamma \geq \gamma_0$ and γ_0 is maximal with this property. This corner is called the pivot point *of P with respect to $g(x)$ —or for any of the corresponding translations. Moreover, γ_0 is in the first ω_0-component of Γ (that is, there is a finite number of terms of $g(x)$ before γ_0).*

The rational rank of any Hahn series solution of a $q-$difference equation is finite. The proof of the following result can be translated from Proposition 1 of [3]:

Theorem 1. *Let $g(x) = \sum_\Gamma c_\gamma x^\gamma \in \mathcal{F}$ be a solution of P. Then Γ is a finitely generated semigroup over $\mathbb{Z}_{>0}$. As a consequence, Γ has no accumulation points in $\mathbb{R}$.*

Finally, "a vertex at height one is always s pivot point":

Remark 4. Notice that if $\mathcal{C}(P_\gamma)$ above has a point $p = (\eta, 1)$ with $\eta = \nu_\gamma(P) - 1$ —that is, if the slope γ "falls to height 1"—, then p is automatically the pivot point of P for $g(x)$.

2.1. **Relative Pivot points.** All the above constructions can be made *relative to a monomial $\underline{\rho} = (\rho_0, \dots, \rho_n)$*, by taking from the start

Definition. *The cloud of points of P relative to $\underline{\rho}$, $\mathcal{C}_\rho(P)$ is the set of points $(\alpha, j) \in \mathbb{R}^2$ for which there is $\underline{\gamma}$ such that*

$$P_{\alpha, \underline{\gamma}} \neq 0 \text{ and } \gamma_i \geq \rho_i \text{ for } i = 0, \dots, n.$$

One can construct the Newton Polygon relative to $\underline{\rho}$, $\mathcal{N}_{\underline{\rho}}(P)$ and, what is most important:

Definition 8. *The* pivot point of P relative to $\underline{\rho}$ with respect to $g(x) \in \mathcal{F}$ *is (γ, j) if and only if (γ, j) is a vertex of $\mathcal{N}_{\underline{\rho}}(P_\gamma)$ for all $\gamma >> 0$ and j is minimal. It will be denoted $Q_{\underline{\rho}}(P, f)$.*

Notice that the Pivot point of P with respect to g is the pivot point of P relative to $(0, \dots, 0)$.

With this notation, one has

Proposition 1. *Let $g(x) \in \mathcal{F}$ be a solutionof P and $p = (a, b)$ be the pivot point of P relative to $\underline{\rho}$ with respect to $g(x)$. Then $b > |\underline{\rho}|$ if and only if $g(x)$*

is a solution of

$$\frac{\partial^{|\underline{\rho}|}P}{\partial y_0^{\rho_0}\dots\partial y_n^{\rho_n}}=0.$$

Which is a consequence of the following easy lemma:

Lemma 5. *Let $c\in\mathbb{C}$, $\alpha\in\mathbb{R}_{>n}$ and $j\in\{0,\dots,n\}$. Then*

(1) *One has $\frac{\partial P}{\partial y_j}[y+cx^\alpha]=\frac{\partial}{\partial y_j}(P[y+cx^\alpha])$.*
(2) *Let $\underline{\rho}\in\mathbb{N}^{n+1}$ be such that $\rho_j\geq 1$ and let $\underline{e}_j=(0,\dots,1,\dots,0)$. Then*

$$Q_{\underline{\rho}-\underline{e}_j}\left(\frac{\partial P}{\partial y_j},g\right)=Q_{\underline{\rho}}(P,g)-(0,1). \tag{6}$$

Corollary 2. *The power series $g(x)\in\mathcal{F}$ is a solutionof $\frac{\partial P}{\partial y_j}$ if and only if $Q_{\underline{e}_j}(P,f)$ has ordinate greater than 1. If, on the other hand, $Q_{\underline{e}_j}(P,f)=(a,1)$, then $\mathrm{ord}_x\left(\frac{\partial P}{\partial y_j}\right)[f]=a$.*

3. Recursive Formulæfor the coefficients

We are working now in the case $\Gamma=\mathbb{Z}$, so that the equation P is in $\mathbb{C}[[x,y_0,\dots,y_n]]$ and solutions lie in $\mathbb{C}[[x]]$. Notice that this case includes that of $\Gamma\subset\mathbb{Q}$, because, as we have proved, after a ramification we can obtain a new semigroup included in $\mathbb{Z}$.

Let $g(x)=\sum c_ix^i$ be a power series solution of $P=0$ and denote, $P_1=\dots=P_p=P$, $P_i=P_{i-1}[y+c_ix^i]$ for $i\geq p$. By Lemma 1, c_i satisfies

$$Q_{1/i}^{P_{i-1}}(c_i)=0.$$

This equation has a unique solution (it is actually a linear equation) for i large enough (whenever the pivot point has been reached).

Assume from now until further notice that the pivot point $p=p(P,g)$ is $(a,1)$ for some $a\in\mathbb{N}$. Let

$$(P_{i_0})(p)=a_0x^ay_0+\dots+a_rx^ay_r,\ a_r\neq 0, 0\leq r\leq n.$$

(the terms of P at the pivot point).

The above equation is equivalent to

$$c_i\Psi(i)=-\mathrm{Coeff}_{x^{a+i}}(P_{i-1}),\ i\geq i_0 \tag{7}$$

where $\Psi(i)=a_0+a_1q+\dots+a_rq^r$.

Thus, if we can compute $\mathrm{Coeff}_{x^{a+i}}(P_{i-1})$ in terms of $c_p,\dots,c_{i-1}$, we shall get a family of recursive formulæ for c_i which may allow us to bound for them.

Consider the formal series (analogue to the one in [3])

$$
\begin{aligned}
&H_i(T_{(\alpha,\rho)},C_{0,p},\dots,C_{0,i-1},\dots,C_{n,p},\dots,C_{n,i-1},x,y_0,\dots,y_n) = \\
&\sum_{(\alpha,\rho)} T_{(\alpha,\rho)}x^{\alpha}\Big[(C_{0,p}x^p+\cdots+C_{0,i-1}x^{i-1}+y_0)^{\rho_0}\dots \\
&\qquad (C_{n,p}q^{np}x^p+\cdots+C_{n,i-1}q^{n(i-1)}x^{i-1}+y_n)^{\rho_n}\Big] = \\
&\qquad = \sum_{(\beta,\underline{\gamma})\in\mathcal{C}(P)} L^i_{(\beta,\underline{\gamma})}(T_{(\alpha,\rho)},\underline{C}^i_0,\dots,\underline{C}^i_n)x^{\alpha}y_0^{\gamma_0}\dots y_n^{\gamma_n}
\end{aligned}
$$

where $\underline{C}^i_k=(q^{kp}C_{k,p},\dots,q^{k(i-1)}C_{k,i-1})$ and $L^i_{(\beta,\underline{\gamma})}$ is a polynomial with non-negative coefficients.

The polynomial $L^i=L^i_{(a+i,0)}=\mathrm{Coeff}_{x^{a+i}}(H_i)$ has the following form

(8)
$$L^i_{(a+i,0)}=\sum_{(\beta,\underline{\gamma},\underline{d})\in\mathcal{F}_i} B^i_{(\beta,\underline{\gamma},\underline{d})}T_{(\beta,\underline{\gamma})}q^{\eta(\underline{d})}C_{0,p}^{d_{0,p}}\dots C_{0,i-1}^{d_{0,i-1}}\dots C_{n,p}^{d_{n,p}}\dots C_{n,i-1}^{d_{n,i-1}},$$

where $B^i(\beta,\underline{\gamma},\underline{d})\in\mathbb{N}$ and

$$
\begin{aligned}
&(\beta,\underline{\gamma},\underline{d})=(\beta,\gamma_0,\dots,\gamma_n,d_{0,p},\dots,d_{0,i-1},\dots,d_{n,p},\dots,d_{n,i-1})\\
&\eta(\underline{d})=\sum_{\substack{k=0,\dots,n\\ l=p,\dots,i-1}} kld_{k,l}
\end{aligned}
$$

and the summation set $\mathcal{F}_i$ in (8) comprises those $(\beta,\underline{\gamma},\underline{d})$ for which the following formula holds:

$$
\begin{aligned}
a+i=\beta+pd_{0,p}+\cdots+(i-1)d_{0,i-1}+\cdots+pd_{n,p}+\cdots+(i-1)d_{n,i-1}= \\
=pd_p+\cdots+(i-1)d_{i-1}.
\end{aligned}
$$

If we write $\underline{c}^i_j=\left(q^{jp}c_p,q^{j(p+1)}c_{p+1},\dots,q^{j(i-1)}c_{i-1}\right)$, then

$$H_i(A_{(\alpha,\rho)},\underline{c}^i_0,\dots,\underline{c}^i_n,x,y_0,\dots,y_n)=P_{i-1},\ i>p$$

and hence

$$\mathrm{Coeff}_{x^{a+i}}(P_{i-1})=L^i(A_{(\alpha,\rho)},\underline{c}^i_0,\dots,\underline{c}^i_n).$$

Corollary 3. *If $g(x)$ is a solution of P and the pivot point of P with respect to g has ordinate 1, then*

$$c_i\Psi(i)=-L^i(A_{(\alpha,\rho)},\underline{c}^i_0,\dots,\underline{c}^i_n),\quad \text{for } i>p. \tag{9}$$

4. The Gevrey bound

Let P be the power series

$$P=\sum_{(a,\rho)\in\mathbb{N}\times\mathbb{N}^{n+1}} A_{(\alpha,\rho)}x^{\alpha}y_0^{\rho_0}\dots y_n^{\rho_n}$$

We say that P has $q-$Gevrey order $t+1$ (see [14] and [15]) if

$$\beta_t(P) = \sum \frac{A_{(\alpha,\rho)}}{|q|^{t(\alpha+|\rho|)(\alpha+|\rho|+1)/2}} x^\alpha y_0^{\rho_0} \dots y_n^{\rho_n} \in \mathbb{C}\{x, y_0, \dots, y_n\}$$

(the exponent on the $|q|$ is t times the sum $1+\dots+(\alpha+|\rho|)$, which appears quite naturally in this context).

As in [13] and [3], let

$$s = \max\left\{ \left\{ \frac{k-r}{e_k - a} \,\middle|\, l_k = 1, r < k \leq n \right\} \cup \{0\} \right\}, \text{ where } Q_{\underline{e}_k} = (e_k, l_k)$$

Theorem 2. *If P has $q-$Gevrey order $t+1$, then any solution g of P whose pivot point has height 1, has $q-$Gevrey order $s+t+1$.*

Proof. (compare with [3])
Consider the algebraic equation (for some $K, k_2 \in \mathbb{N}$ later to be specified)

$$x^a w = \frac{|q|^{k_2 p}}{|q|^{s\frac{p(p+1)}{2}}} |c_p| x^{a+p} + \dots + \sum_{P_{(\alpha,\rho)} \in \mathcal{C}'(P)} K|q|^{k_2 i} |A_{(\alpha,\rho)}| x^\alpha w^{\rho_0} \dots w^{\rho_n}$$

where $\mathcal{C}'(P)$ is the whole cloud of points of P without $(a, 1)$ and $(a+p, 0)$. This equation has obviously a unique solution $w(x)$

$$w(x) = \frac{|q|^{k_2 p}}{|q|^{s\frac{p(p+1)}{2}}} |c_p| x^p + \sum_{i=p+1}^{\infty} c'_i x^i,$$

whose coefficients c'_i satisfy the recursive formula

$$c'_i = L^i(K|q|^{k_2 i}|A_{(\alpha,\rho)}|, c'_p, \dots, c'_{i-1}, \dots, c'_p, \dots, c'_{i-1}) \text{ for } i > p,$$
$$c'_p = \frac{|q|^{k_2 p}}{|q|^{s\frac{p(p+1)}{2}}}.$$

Assume that

$$|c_j| \leq \frac{|q|^{s\frac{j(j+1)}{2}}}{|q|^{k_2 i}} c'_j, \text{ for } j = p, \dots, i-1. \tag{10}$$

By Corollary 3, we have

(11)
$$\begin{aligned}|c_i| \leq& \frac{1}{|\Psi(i)|} |L^i(A_{(\alpha,\rho)}, \underline{c}_0^i, \dots, \underline{c}_n^i)| \leq \\ \leq& \frac{1}{|\Psi(i)|} \sum_{(\beta,\underline{\gamma},\underline{d}) \in \mathcal{F}_i} B^i_{(\beta,\underline{\gamma},\underline{d})} |A_{(\beta,\underline{\gamma},\underline{d})}| \frac{K|q|^{k_2\beta}}{K|q|^{k_2\beta}} |c'_p|^{d_p} \dots |c'_{i-1}|^{d_{i-1}} \cdot r_i(\beta,\underline{\gamma},\underline{d}),\end{aligned}$$

where $d_l = d_{0,l} + \cdots + d_{n,l}$ and

$$r_i(\beta, \underline{\gamma}, \underline{d}) = \frac{|q|^{s\frac{p(p+1)}{2}d_p} \ldots |q|^{s\frac{(i-1)(i)}{2}d_{i-1}}}{|q|^{k_2 p d_p} \ldots |q|^{k_2(i-1)d_{i-1}}} |q|^{\sum k l d_{k,l}} =$$

$$\frac{|q|^{\sum_{l=p}^{i-1} s\frac{l(l+1)}{2}}}{|q|^{k_2(a+i-\beta)}} |q|^{\sum k l d_{k,l}}$$

where the last exponent is

$$\sum_{\substack{k=1,\ldots,n \\ l=p,\ldots,i-1}} k l d_{k,l} = p d_{1,p} + \cdots + (i-1) d_{1,i-1} + \cdots + n p d_{n,p} + \cdots + n(i-1) d_{n,i-1}.$$

If for all $(\beta, \underline{\gamma}, \underline{d})$ in $\mathcal{F}_i$,

$$R_i(\beta, \underline{\gamma}, \underline{d}) = \frac{1}{K|\Psi(i)||q|^{k_2\beta}} r_i(\beta, \underline{\gamma}, \underline{d}) \leq \frac{|q|^{s\frac{i(i+1)}{2}}}{|q|^{k_2 i}}, \tag{12}$$

then one has that

$$|c_i| \leq L^i(A_{(\alpha,\rho)}, \underline{c}_0^i, \ldots, \underline{c}_n^i) \cdot \frac{|q|^{s\frac{i(i+1)}{2}}}{|q|^{k_2 i}} \leq c_i' \frac{|q|^{s\frac{i(i+1)}{2}}}{|q|^{k_2 i}}$$

which implies inequality (10) for all $j \geq p$ and we are done.

Thus, we only need to prove the bound in (12). There are two completely different cases: $s = 0$ and $s \neq 0$.

Case $s = 0$**:** we are done if we show that for some $K, k_2 >> 0$ the following inequality holds:

$$\frac{1}{K|\Psi(i)||q|^{k_2\beta}|q|^{k_2(a+i-\beta)}} |q|^{\sum k l d_{k,l}} \leq \frac{1}{|q|^{k_2 i}}.$$

We know that $|\Psi(i)| \simeq |q|^{ri} = |q|^{ni}$ (because $r = n$ in this case, as $s = 0$). By definition,

$$a + i = \beta + p d_p + \cdots + (i-1) d_{i-1},$$

hence $\sum k l d_{k,l} \leq n \sum l d_{kl} = n(a + i - \beta)$. Thus,

$$\begin{aligned} r_i(\beta, \underline{\gamma}, \underline{d}) \leq & \frac{1}{K|\Psi(i)||q|^{k_2(a+i)}} |q|^{n(a+i-\beta)} \leq \\ \leq & \frac{1}{|q|^{ni}|q|^{k_2 a}|q|^{k_2 i}} |q^{na}||q|^{ni} \leq \frac{1}{|q|^{k_2 i}} \end{aligned}$$

if $K >> 0$ and $k_2 > n$ (actually $K|\Psi(i)| > |q|^{ni}$).

Case $s = \frac{s1}{s2} \neq 0$**,** with $s_1, s_2 \in \mathbb{N}$. We distinguish two new cases: $|\underline{d}| = 1$ and $|\underline{d}| > 1$, where $|\underline{d}| = d_p + \cdots + d_{i-1}$.

Subcase a: $|\underline{d}| = 1$. Let (k, l) be the only pair such that $d_{k,l} = 1$ (hence $d_l = 1$ as well). We are obviously done if we show that for $i >> 0$,

$$s_1 \frac{l(l+1)}{2} + s_2 k l \leq s_1 \frac{i(i+1)}{2} + s_2 r i \tag{13}$$

independently of (k,l) (recall that in any case, $l \leq i-1$). If $k \leq r$ the bound is obvious. If, on the other hand, $k > r$, let $e_k' = e_k - a$. We know that $a+i = \beta + l$ (and by definition, $\beta \geq e_k$), so that $l \leq i - (e_k - a) = i - e_k'$. Hence, (13) holds if the following inequality does:

$$s_1 \frac{(i-e_k')(i-e_k'+1)}{2} + s_2 k(i-e_k') \leq s_2 ri + s_1 \frac{i(i+1)}{2}, \tag{14}$$

which is equivalent to

$$s_2 ki \leq s_1 i e_k' + s_2 ri.$$

This last equation can be written

$$s_2(k-r) \leq s_1(e_k - a),$$

which is true by definition of $s = s_1/s_2$.

Subcase b: $|\underline{d}| > 1$. The inequality we have to prove is, for $i >> 0$ (or, what amounts to the same, for $p >> 0$):

$$s_1 \sum_{l=p}^{i-1} \frac{l(l+1)}{2} d_l + s_2 \sum k l d_{k,l} \leq s_1 \frac{i(i+1)}{2} + ris_2 + k_2 s_2 (a+i). \tag{15}$$

Let us enumerate $p \leq l_0 \leq \cdots \leq l_h$ those indices for which $d_l \neq 0$. Recall that $a + i = \beta + l_0 d_{l_0} + \cdots + l_h d_{l_h}$.

Again we distinguish two possibilities: $l_h \leq \frac{a+i}{2}$ and $l_h > \frac{a+i}{2}$.

If all the indices are less than or equal to $\frac{a+i}{2}$, the LHS of (15) is bounded by

$$(1 + \cdots + l_0) d_{l_0} + \cdots + (1 + \cdots + l_h) d_{l_h} + s_2 ni,$$

which is easily seen to be bounded, as well, by

$$s_1 \cdot 2 \cdot \left(1 + \cdots + \left\lfloor \frac{a+i}{2} \right\rfloor\right) + s_2 ni = s_1 \frac{i^2}{4} + O(i)$$

whereas the RHS of (15) is a polynomial of degree 2 in i starting with $s_1 \frac{i^2}{2}$, and we are done in this case (taking $p >> 0$).

On the other hand, if $l_h > \frac{a+i}{2}$, then $d_{l_h} = 1$ and the LHS of (13) is bounded by

$$\begin{aligned}(1 + \cdots + l_0) d_{l_0} + \cdots + (1 + \cdots + l_h) d_{l_h} + s_2 ni \leq \\ \leq (1 + \cdots + p) + (1 + \cdots + i - p + a) + s_2 ni,\end{aligned}$$

which is

$$\frac{i^2 + i(2a - 2p + 1 + 2s_2 n) + a^2 + a + p^2 - 2ap - p}{2},$$

while the RHS is greater than

$$\frac{i^2+i}{2},$$

so that if $p >> 0$, the bound holds. $\square$

For the general case, we need to improve the definition of s

Definition 9. *Let $g(x)=\sum_{i=n+1}^{\infty} c_i x^i$ be a solution of P and $Q(P,g)$ be the pivot point of P with respect to f. Consider the set*

$$E(P,g)=\left\{\underline{\rho}\in\mathbb{N}^{n+1}\;\middle|\; Q_{\underline{\rho}}(P,g)=(a,b)\ \textit{with}\ |\underline{\rho}|=b\right\}.$$

For $\underline{\rho}\in E(P,g)$, if $\rho_r>0$, let $\underline{\gamma}_r=\rho-\underline{e}_r=(\rho_0,\ldots,\rho_r-1,\ldots,\rho_n)$. For all $j>r$, write $Q_{\underline{\gamma}_r+\underline{e}_j}(P,g)=(e_{r,j},k_{r,j})$ and consider

$$s_r(\underline{\rho})=\max\left\{\left\{\frac{j-r}{e_{r,j}-a}\;\middle|\; j>r\ \textit{and}\ k_{r,j}=b\right\}\cup\{0\}\right\},$$
$$s(\underline{\rho})=\min\left\{s_r(\underline{\rho})\,|\rho_r>0\right\},$$

and define

$$s(P,g)=\min\left\{s(\underline{\rho})\;\middle|\;\underline{\rho}\in E(P,g)\right\}. \tag{16}$$

Then we have

Theorem 3. *Let g be a solution of P. If P has $q-$Gevrey index $t+1$, then g has $q-$Gevrey index $s(P,f)+t+1$.*

Proof. If $Q(P,g)=(a,1)$ then the result is Theorem 2. Assume hence that $s(P,g)=s_r(\underline{\rho})$ for $\underline{\rho}\in E(P,g)$. This means that $Q_{\underline{\gamma}_r}(P,g)=Q(P,g)=(a,b)$ and $|\underline{\gamma}|_r=b-1$. By Proposition 1, the series g is a solution of P', being

$$P'=\frac{\partial P}{\partial y_0^{\gamma_{r,0}}\ldots\partial y_n^{\gamma_{r,n}}}$$

where $\underline{\gamma}=(\gamma_{r,0},\ldots,\gamma_{r,n})$. By Lemma 5, $Q(P',g)=(a',1)$ and the corresponding s in Theorem 2 for P' is exactly $s_r(\underline{\rho})=s(P,g)$. □

5. Rational rank of solutions for order and degree 1

5.1. **Order and degree 1.** In this specific case, the above can be greatly improved, using the following lemmas. Assume in what follows that P is of order and degree 1 (that is, P is of the form $A(x,y_0)+B(x,y_0)y_1$.

Let $g(x)\in\mathcal{F}$ be a solution of P and denote, as before, $P_\gamma=P[y+\lfloor g(x)\rfloor_\gamma]$ for $\gamma\in\Gamma$, the set of exponents of Γ. Fix $\gamma\in\Gamma$ and let γ_+ be the next element of Γ.

Lemma 6. *If γ_+ is an exceptional slope for P_γ corresponding to a corner of $\mathcal{N}(P_\gamma)$ and c is any nonzero coefficient, then the Newton polygon of $P_{\gamma'}$ has a corner at height 1 and $P_{\gamma'}$ has no exceptional slopes greater than γ_+.*

Proof. Let $\Phi_{\gamma'}(P)$ be the initial form of P for slope γ'. We only need to show that $\Phi_{\gamma'}(P_{\gamma'})$ has a point at height 1 and no point at height 0. But, since γ' meets $\mathcal{N}(P_\gamma)$ at a single point,

$$\Phi_{\gamma'}(P_\gamma)(x,y,\sigma(y))=A_{\alpha,j,0}x^\alpha y_0^j+B_{\alpha,j-1,1}x^\alpha y_0^{j-1}y_1,$$

for just one (α, j) (the coordinates of the point). The translation $y = \bar{y} + cx^{\gamma'}$ works as follows

$$\Phi_{\gamma'}(P)(x, y, \sigma(y)) = (...) = Q^P_{\gamma'}(c)x^{\nu_{\gamma'}(P)} + \sigma(\bar{y}_0)B_{\gamma,j-1,1}c^{j-1}x^{\alpha+\gamma'(j-1)}.$$

On one hand, $Q^{P_\gamma}_{\gamma'} = 0$ by hypothesis and on the other, $B_{\gamma,j-1,1} \neq 0$ (because otherwise $Q^{P_\gamma}_{\gamma'}$ would not be 0, γ' corresponding to a single point of $\mathcal{N}(P)$). Thus, the translation gives rise to the point $(\nu_{\gamma'}(P)-1, 1) = (\gamma+\gamma'(j-1), 1)$ in $\mathcal{C}(\bar{P})$ and (since $P^P_{\gamma'} = 0$) the point $(\nu_{\gamma'}(P), 0)$ does not belong to $\mathcal{C}(\bar{P})$, which makes it a corner of $\mathcal{N}(\bar{P})$.

The second part is an easy computation □

Lemma 7. *Let $g(x) \in \mathcal{F}$ be a solution of P*

$$g(x) = \sum_{\gamma\in\Gamma} g_\gamma x^\gamma.$$

Then there is at most one $\gamma \in \Gamma$ which is an exceptional slope of a corner (of the corresponding Netwon polygon).

Proof. Without loss of generality, we may assume the first exponent γ_0 in $g(x)$ corresponds to an exceptional slope at a corner (a, b) of $\mathcal{N}(P)$. Assume there are at least two exceptional slopes. Let $\Phi_{\gamma_0}(P) = Ax^a y_0^b + Bx^a y_0^{b-1} y_1$ be the initial form at (a, b). From the defition of exceptional slope,

$$q^{\gamma_0} = -A/B.$$

A straightforward computation gives

$$\begin{aligned}\Phi_{\gamma_0}(P)(x, y_0 + cx^{\gamma_0}, y_1 + cq^{\gamma_0}x^{\gamma_0}) = (\dots)+ \\ + x^{\alpha+\gamma_0(b-1)}(bAc^{b-1} + (b-1)Bc^{b-1}q^{\gamma_0})y_0 + x^{\alpha+\gamma_0(b-1)}Bc^{b-1}\end{aligned}$$

(where we show only the terms at the pivot point of $\bar{P}$), so that any other exceptional slope η should give

$$q^\eta = -\frac{bAc^{b-1} + (b-1)Bc^{b-1}q^{\gamma_0}}{Bc^{b-1}} = q^{\gamma_0},$$

and we are done.

□

From these two lemmas we deduce

Corollary 4. *Any Hahn series $g(x) \in \mathcal{F}$ solution of a formal $q-$difference equation $P = A(x, y) + B(x, y)\sigma(y)$ of order and degree 1 has at most rational rank 2.*

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