Differential systems with Fuchsian linear part: correction and linearization, normal forms and matrix valued orthogonal polynomials

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Lazarus Fuchs

Born on 5 May 1833 in **Mosina**, Grand Duchy of Poznań

(as learned in yesterday's trip)

Motivation

- Equations have been thoroughly *studied near one singularity*, but few results (if any) in regions with two (or more) singularities.
- Start the study with the simplest type of singularity: regular. Then, *irregular* singularities could be studied by *coalescence* (limits when regular singular points tend to coincide).
- Integrability of complex ODEs - existence of independent, single-valued first integrals.
  Equations "known" (i.e. strongly suspected) to be non-integrable reduce to differential systems with Fuchsian linear part (RDC - '96, '97).
- Results with a similar flavor (connected?):
  - Écalle and Vallet showed that resonant systems are linearizable after appropriate correction (1998);
  - Gallavotti showed that there exists appropriate corrections of Hamiltonian systems so that the new system is integrable (1982), convergence proved by Eliasson (1988).

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Differential equation in the complex domain

We consider ordinary differential equations

$$\frac{d\mathbf{u}}{dx} = \mathbf{F}(\mathbf{u}, x), \qquad \mathbf{u} \in \overline{\mathbb{C}}^d, \quad x \in \overline{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$$

where $\mathbf{F}$ is analytic on a domain $D_{\mathbf{F}} \subset \overline{\mathbb{C}}^d \times \overline{\mathbb{C}}$.

Definition

$(\mathbf{u}_0, x_0)$ is a regular point (of the equation) if $\mathbf{F}$ is analytic at $(\mathbf{u}_0, x_0)$.

Definition

$(\mathbf{u}_0, x_0)$ is a singular point (of the equation) if $\mathbf{F}$ is *not* analytic at $(\mathbf{u}_0, x_0)$.

We will consider here $\mathbf{u}_0 = \mathbf{0}$.

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Analytic Equivalence

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Two equations

$$(\mathcal{E}_F): \quad \frac{d\mathbf{u}}{dx} = \mathbf{F}(x, \mathbf{u}); \qquad (\mathcal{E}_G): \quad \frac{d\mathbf{w}}{dz} = \mathbf{G}(z, \mathbf{w})$$

are analytically equivalent in the domains $D_{\mathbf{F}}$, respectively $D_{\mathbf{G}}$ if they are transformed into each other after an analytic change of variables.

(The change of coordiantes $(x, \mathbf{u}) = \mathbf{H}(z, \mathbf{w})$ should be a biholomorphism $\mathbf{H} : D_{\mathbf{G}} \longrightarrow D_{\mathbf{F}}$ -possibly local.)

Question: what is the simplest form an equation can have after an analytic change of variables? Normal form.

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Any two equations are equivalent near one regular point

The Rectification Theorem

Let $(\mathbf{u}_0, x_0)$ be a regular point of $\mathbf{F}$. Then

$$\frac{d\mathbf{u}}{dx} = \mathbf{F}(x, \mathbf{u}), \qquad \frac{d\mathbf{w}}{dz} = \mathbf{0}$$

are analytically equivalent in a **small** neighborhood $D_{\mathbf{F}}$ of $(\mathbf{u}_0, x_0)$, respectively a small neighorhood D_0 of $(0, \mathbf{0})$

So equations can be distinguished by

- looking near singular points
- looking at larger domains

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Near a regular singular point not all equations are equivalent

Consider equations with $\mathbf{u} = \mathbf{0}$ stationary point: $\mathbf{F}(\mathbf{0}, x) \equiv \mathbf{0}$.

$$\frac{d\mathbf{u}}{dx} = \frac{1}{x} L(x)\,\mathbf{u} + \frac{1}{x}\,\mathbf{f}(x, \mathbf{u})$$

with $\mathbf{f}(x, \mathbf{u})$ having a zero of order 2 at $\mathbf{u} = \mathbf{0}$ (so $\mathbf{f}$ is the nonlinear part of the equation). Assume $L, \mathbf{f}$ analytic at $x = 0, \mathbf{u} = \mathbf{0}$.

Note: $x = 0$ is a regular singular (Fuchsian) point of the linear part.

How much can we simplify the equation near $(0, \mathbf{0})$?

Theorem: invariants

Any analytic change of coordinates preserves the spectrum of $L(0)$.

So we can at most hope to bring the equation to the form

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Generic systems are indeed linearizable **locally**:

Theorem

If $\sigma(L(0))$ is *not 'too close' to resonance* then $\exists\ \mathbf{u} = \mathbf{h}(x, \mathbf{w})$ analytic for $|x| < \epsilon,\ |\mathbf{u}| < \epsilon_1$

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Consequence

The study of the local analytic properties of nonlinear systems reduces to the study of linear equations with (almost) constant coefficients.

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Can equations be **simultaneously** linearized near two singularities?

E.g., instead of $\dfrac{d\mathbf{u}}{dx} = \dfrac{1}{x}L(x)\,\mathbf{u} + \dfrac{1}{x}\,\mathbf{f}(x,\mathbf{u})$ for x near 0, consider

$$\frac{d\mathbf{u}}{dx} = \left(\frac{1}{x}A + \frac{1}{x-1}B\right)\mathbf{u} + \frac{1}{x(x-1)}\,\mathbf{f}(x,\mathbf{u})$$

for x in a domain containing both singularities $x = 1$ and $x = 0$.

A change of variables analytic at both $x = 0$ and $x = 1$ preserves both $\sigma(A)$ and $\sigma(B)$.

Question: Perhaps for $x \in D \ni 0, 1$ the equation is equivalent to

$$\frac{d\mathbf{w}}{dx} = \left(\frac{1}{x}A + \frac{1}{x-1}B\right)\mathbf{w}$$

In other words, is the equation linearizable simultaneous at $x = 0$ and $x = 1$?

This type of semi-local questions received very little attention.

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Linearizability and integrability are equivalent

Linearizability and integrability are equivalent - at least in the generic case, in the scalar case:

Theorem (RDC, M.D. Kruskal, *Nonlin.*'03)

If the equation $\dfrac{du}{dx} = \left(\dfrac{a_0}{x} + \dfrac{a_1}{x-1}\right) u + \dfrac{1}{x(x-1)} f(x,u)$
is *not* analytically linearizable
then for generic a_0, a_1 (precise conditions given) no single-valued integrals exist for x in a domain encircling both singularities.

Question: when is the equation linearizable near both singularities?

Answer to the simultaneous linearization question

In the general multi-dimensional setting:

$$(nl)\ \frac{d\mathbf{u}}{dx} = \left(\frac{1}{x}A + \frac{1}{x-1}B\right)\mathbf{u} + \frac{\mathbf{f}(x,\mathbf{u})}{x(x-1)} \quad (l)\ \frac{d\mathbf{w}}{dx} = \left(\frac{1}{x}A + \frac{1}{x-1}B\right)\mathbf{w}$$

Question: Are the equations equivalent for x in a domain containing both singularities $x = 1$ and $x = 0$?

Answer: No.

$\exists!\ \mathbf{u} = \mathbf{h}_0(x, \mathbf{w})$ analytic at $x = 0$ s.t. $(nl) \Leftrightarrow (l)$.

and $\exists!\ \mathbf{u} = \mathbf{h}_1(x, \mathbf{w})$ analytic at $x = 1$ s.t. $(nl) \Leftrightarrow (l)$.

But $\mathbf{h}_0 \neq \mathbf{h}_1$. ($\mathbf{h}_0$ is ramified at $x = 1$ and $\mathbf{h}_1$ is ramified at $x = 0$.)

If we do not take NO for an answer, then...

Question 2: Which systems (nl) are linearizable?

Question 3: What are the normal forms of equations (nl)?

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Question: Are the equations equivalent for x in a domain containing both singularities $x=1$ and $x=0$?

Answer: No.

$\exists!\ \mathbf{u} = \mathbf{h}_0(x,\mathbf{w})$ analytic at $x=0$ s.t. $(nl) \Leftrightarrow (l)$.

and $\exists!\ \mathbf{u} = \mathbf{h}_1(x,\mathbf{w})$ analytic at $x=1$ s.t. $(nl) \Leftrightarrow (l)$.

But $\mathbf{h}_0 \neq \mathbf{h}_1$. ($\mathbf{h}_0$ is ramified at $x=1$ and $\mathbf{h}_1$ is ramified at $x=0$.)

If we do not take NO for an answer, then...

Question 2: Which systems (nl) are linearizable?

Question 3: What are the normal forms of equations (nl)?

Linearization after Correction; Normal Form.

Theorem - Linearization after correction (RDC, Nonlin. 2008)

For any $\mathbf{f}$ there exists a unique correction $\phi(\mathbf{u})$ (formal series) so that

$$(\mathcal{E}_{\mathbf{f}-\phi}) \qquad \frac{d\mathbf{u}}{dx} = \left(\frac{1}{x}A + \frac{1}{x-1}B\right)\mathbf{u} + \frac{\mathbf{f}(x,\mathbf{u}) - \phi(\mathbf{u})}{x(x-1)}$$

is (formally) linearizable (assuming A, B non-resonant).

Convergence of the correction $\phi(\mathbf{u})$ was proved on a subclass.

Theorem - Normal Form (RDC, Nonlin. 2008)

Assume A, B non-resonant. For any $\mathbf{f}$ there exists a unique series $\psi(\mathbf{u})$ so that $(\mathcal{E}_{\mathbf{f}})$ is formally equivalent to

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For many singularities: Correction and Linearization

$$\frac{d\mathbf{u}}{dx} = A(x)\,\mathbf{u} + \frac{\mathbf{f}(x,\mathbf{u})}{Q(x)}$$

$$\text{with} \quad A(x) = \sum_{j=0}^{S+1} \frac{1}{x-p_j} A_j \text{ (Fuchsian matrix)}, \qquad Q(x) = \prod_{j=0}^{S+1}(x-p_j)$$

Theorem (RDC, Nonlin. 2008, J.Diff.Eq. 2009)

Assume $A_0, \ldots, A_{S+1}, A_\infty = \sum A_j$ are nonresonant.

Then $\exists$ **unique** *correction* $\phi(x,\mathbf{u}) = \sum_{\mathbf{m}\in\mathbb{N}^d,\ |\mathbf{m}|\geq 2} \phi_{\mathbf{m}}(x)\mathbf{u}^{\mathbf{m}}$ (formal)

where $\phi_{\mathbf{m}}(x)$ are **polynomials in x of deg.** $\leq S$, s.t. the corrected system $\frac{d\mathbf{u}}{dx} = A(x)\,\mathbf{u} + \frac{\mathbf{f}(x,\mathbf{u}) - \phi(x,\mathbf{u})}{Q(x)}$ is (formally) linearizable.

Note. Equation (*) is linearizable iff $\phi(x,\mathbf{u}) \equiv 0$, so the unique correction ϕ is the obstruction to linearizability.

Normal forms

Since equations are not necessarily linearizable, then they are not all equivalent either. Classification of these equations by specifying formal normal forms:

Theorem (RDC, J.Diff.Eq. 2009)

Assume non-resonance. For any $\mathbf{f}(x,\mathbf{w})$ analytic on $D \times \{|\mathbf{w}| < r\}$ there exists a unique formal series $\mathbf{p}(x,\mathbf{w}) = \sum_{\mathbf{m}\in\mathbb{N}^d,\ |\mathbf{m}|\geq 2} \mathbf{p}_{\mathbf{m}}(x)\mathbf{w}^{\mathbf{m}}$

where $\mathbf{p}_{\mathbf{m}}(x)$ are **polynomials in x of degree at most S**, such that

$\dfrac{d\mathbf{u}}{dx} = A(x)\,\mathbf{u} + \dfrac{\mathbf{f}(x,\mathbf{u})}{Q(x)} \Longleftrightarrow$

$$\frac{d\mathbf{w}}{dx} = A(x)\,\mathbf{w} + \frac{\mathbf{p}(x,\mathbf{w})}{Q(x)}$$

through $\mathbf{u} = \mathbf{h}(x,\mathbf{w}) = \mathbf{w} + \sum \mathbf{h}_{\mathbf{m}}(x)\mathbf{w}^{\mathbf{m}}$ with $\mathbf{h}_{\mathbf{m}}(x)$ analytic on D.

Normal forms in regions with regular singular points (generic cases)

In a region containing one sing. point $(x, \mathbf{u}) = (0, \mathbf{0})$:

$$x\frac{d\mathbf{u}}{dx} = L_0\mathbf{u} + \mathbf{f}(x, \mathbf{u}) \Leftrightarrow \; x\frac{d\mathbf{w}}{dx} = L_0\mathbf{w}$$

(keep the linear part)

In a region with two sing. points $(0, \mathbf{0}), (p_1, \mathbf{0})$:

$$x(x - p_1)\frac{d\mathbf{u}}{dx} = (L_0 + xL_1)\mathbf{u} + \mathbf{f} \Leftrightarrow x(x - p_1)\frac{d\mathbf{w}}{dx} = (L_0 + xL_1)\mathbf{w} + \psi_0(\mathbf{w})$$

(keep linear part + some nonlinear terms)

In a region with region with three sing. points $(0, \mathbf{0}), (p_1, \mathbf{0}), (p_2, \mathbf{0})$:

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What was proved and what was not proved

- **Formal results**: find series for
  - correction $\phi(x,\mathbf{u}) = \sum_{\mathbf{m}} \mathbf{p_m}(x)\mathbf{u^m}$ with $\mathbf{p_m}(x)$ polynomials, and
  - equiv. map $\mathbf{h}(x,\mathbf{w}) = \sum_{\mathbf{m}} \mathbf{h_m}(x)\mathbf{w^m}$, with $\mathbf{h_m}(x)$ analytic at all p_j

  True under non-resonance conditions. Challenging!
  For this we construct matrix-valued generalizations of Jacobi polynomials, and of multiple-orthogonal polynomials.
  (RDC: Nonlin. 2008, J.Diff.Eq. 2009, JAT 2009, 2009, 2010)
- **Analytic results**: when are these series convergent?
  - Theorem: $\phi, \mathbf{h}$ converge in the commutative case, for two singularities, eigenvalues with positive real parts. (RDC, Nolin. 2008)
    Difficult proof! Steepest descent $\rightsquigarrow$ small denominators $\rightsquigarrow$ improvement of a rapidly convergent algorithm.
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A glimpse at the some proofs

A change of variables $\mathbf{u} = \mathbf{h}(x, \mathbf{w})$ provides a linearization iff

$$(**) \qquad \partial_x \mathbf{h} + d_{\mathbf{w}} \mathbf{h}\, A(x)\mathbf{w} = A(x)\mathbf{h} + \frac{1}{Q(x)} \left[\mathbf{f}(x, \mathbf{w} + \mathbf{h}) - \phi(x, \mathbf{w} + \mathbf{h})\right]$$

Power series in $\mathbf{w}$: denote by $\mathbf{h}_n$ the homogeneous part degree n of $\mathbf{h}(x, \mathbf{w})$:

$$\mathbf{h}_n(x, \mathbf{w}) = \sum_{|\mathbf{m}|=n} \mathbf{h}_{\mathbf{m}}(x)\mathbf{w}^{\mathbf{m}}, \quad (n \geq 2), \quad \text{similarly } \mathbf{f}_n,\ \phi_n$$

(**) splits into blocks of systems of ordinary differential equations for $\{\mathbf{h}_{\mathbf{m}}\}_{|\mathbf{m}|=n}$:

$$\partial_x \mathbf{h}_n + \mathrm{d}_{\mathbf{w}} \mathbf{h}_n\, A(x)\mathbf{w} - A(x)\mathbf{h}_n = \frac{1}{Q(x)} \mathbf{R}_n(x, \mathbf{w}), \qquad n \geq 2$$

where $\mathbf{R}_n = \mathbf{f}_n - \phi_n + \tilde{\mathbf{R}}_n$ with $\tilde{\mathbf{R}}_n$ a polynomial in $\phi_{\mathbf{m}}$, $\mathbf{h}_{\mathbf{m}}$, $\mathbf{f}_{\mathbf{m}}$ with $|\mathbf{m}| < n$, and $\tilde{\mathbf{R}}_2 = 0$.

Each $\mathbf{h}_n$ and ϕ_n are to be determined inductively on n.

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A glimpse at the some proofs

A change of variables $\mathbf{u} = \mathbf{h}(x, \mathbf{w})$ provides a linearization iff

$$(**) \qquad \partial_x \mathbf{h} + d_{\mathbf{w}}\mathbf{h}\, A(x)\mathbf{w} = A(x)\mathbf{h} + \frac{1}{Q(x)}\, [\mathbf{f}(x, \mathbf{w}+\mathbf{h}) - \phi(x, \mathbf{w}+\mathbf{h})]$$

Power series in $\mathbf{w}$: denote by $\mathbf{h}_n$ the homogeneous part degree n of $\mathbf{h}(x, \mathbf{w})$:

$$\mathbf{h}_n(x, \mathbf{w}) = \sum_{|\mathbf{m}|=n} \mathbf{h}_{\mathbf{m}}(x)\mathbf{w}^{\mathbf{m}}, \quad (n \geq 2), \quad \text{similarly } \mathbf{f}_n,\ \phi_n$$

(**) splits into blocks of systems of ordinary differential equations for $\{\mathbf{h}_{\mathbf{m}}\}_{|\mathbf{m}|=n}$:

$$\partial_x \mathbf{h}_n + \mathrm{d}_{\mathbf{w}}\mathbf{h}_n\, A(x)\mathbf{w} - A(x)\mathbf{h}_n = \frac{1}{Q(x)}\, \mathbf{R}_n(x, \mathbf{w}), \qquad n \geq 2$$

where $\mathbf{R}_n = \mathbf{f}_n - \phi_n + \tilde{\mathbf{R}}_n$ with $\tilde{\mathbf{R}}_n$ a polynomial in $\phi_{\mathbf{m}}$, $\mathbf{h}_{\mathbf{m}}$, $\mathbf{f}_{\mathbf{m}}$ with $|\mathbf{m}| < n$, and $\tilde{\mathbf{R}}_2 = 0$.
Each $\mathbf{h}_n$ and ϕ_n are to be determined inductively on n.

Difficulties

$$(*) \qquad \partial_x \mathbf{h}_n + \mathrm{d}_{\mathbf{w}}\mathbf{h}_n \, A(x)\mathbf{w} - A(x)\mathbf{h}_n = \frac{1}{Q(x)}\,\mathbf{R}_n(x,\mathbf{w}), \qquad n \geq 2$$

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Main difficulties

- For systems with 1 sing., or for scalar equations: proving convergence, due to small denominators.
- For systems with two or more sing.: (*) cannot be solved explicitly in the non-commutative case.

The second difficulty: we need to find conditions for (*) to have an analytic solution.

Was tackled as follows.

Tackling the recurrent Heun systems

$$(*)\quad \partial_x \mathbf{h}_n + \mathrm{d}_{\mathbf{w}}\mathbf{h}_n\, A(x)\mathbf{w} - A(x)\mathbf{h}_n = \frac{\mathbf{R}_n(x,\mathbf{w})}{Q(x)}, \quad (n \geq 2),\ Q(x) = \prod_{j=0}^{S+1}(x-p_j)$$

where $\mathbf{h}_n = \mathbf{h}_n(x;\mathbf{w}) \in \mathcal{P}_n$ are $\mathbb{C}^d$-val. homog. polyn. in $\mathbf{w}$, of degree n.

Let $B(x)$ be lin. op. on $\mathcal{P}_n$: $B(x)\mathbf{h}_n = \mathrm{d}_{\mathbf{w}}\mathbf{h}_n\, A(x)\mathbf{w} - A(x)\mathbf{h}_n$

$$A(x) = \sum_{j=0}^{S+1} \frac{1}{x-p_j} A_j \quad \Longrightarrow \quad B(x) = \sum_{j=0}^{S+1} \frac{1}{x-p_j} B_j$$

where

$$B_j \in \mathcal{L}(\mathcal{P}_n), \quad B_j\mathbf{q} = \mathrm{d}_{\mathbf{w}}\mathbf{q}\, A_j\mathbf{w} - A_j\mathbf{q}$$

Systems (*) are non-homogeneous Fuchsian systems!

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A Fundamental Lemma

The results follow using iteratively:

Lemma

$$\mathbf{y}'(x) + B(x)\,\mathbf{y}(x) = \frac{\mathbf{g}(x)}{Q(x)}$$

a Fuchsian equation with a non-homogeneous term ($\mathbf{y} \in \mathbb{C}^N$) where

$B(x) = \sum_{j=0}^{S+1} \frac{1}{x - p_j} B_j$, and $Q(x) = \prod_{j=0}^{S+1}(x - p_j)$. Let $D \ni p_0, \ldots p_{S+1}$.

Assume non-resonance: $k + B_j$ are invertible for all j.

Then for any function $\mathbf{g}(x)$ analytic on D there exists a unique $\phi(x) \in \mathbb{C}^N[x]$, $\deg\phi \leq S$ so that the corrected equation

$$\mathbf{y}'(x) + B(x)\,\mathbf{y}(x) = \frac{\mathbf{g}(x) - \phi(x)}{Q(x)}$$

has a solution $\mathbf{y}(x)$ analytic on D.

Idea of the proof

The simplest case: scalar, two singularities:

$$(*) \qquad y'(x) + \left(\frac{\alpha}{x-1} + \frac{\beta}{x+1}\right) y(x) = \frac{g(x)}{(x+1)(x-1)}$$

If $g(x) = P_k^{(\alpha-1,\beta-1)}$ then the unique analytic solution is $y(x) = P_{k-1}^{(\alpha,\beta)}$.

Then expand $g(x)$ in Jacobi polynomials: $g(x) = \sum_{k=0}^{\infty} g_k P_k^{(\alpha-1,\beta-1)}$.

Equation (*) has an analytic solution at both $x = \pm 1$ iff $g_0 = 0$.
In this case, the analytic solution is

$$y(x) = \sum_{k=1}^{\infty} g_k P_{k-1}^{(\alpha,\beta)}$$

For scalar equations, three singularities: use expansions in Jacobi-Angelesco polynomials.

For higher dimensions: appropriate generalizations to matrix-valued polynomials is done!

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Matrix-valued generalizations of Jacobi polynomials

Let $B(x) = \sum_{j=0}^{S+1} \frac{1}{x - p_j} B_j$ Fuchsian matrix. Denote $Q(x) = \prod_{j=0}^{S+1} (x - p_j)$.
Let $W(x)$ be a fundamental matrix for $W'(x) = W(x)B(x)$.

Theorem (RDC)

$P_k(x) = W(x)^{-1} \frac{d^k}{dx^k} \left(Q(x)^k W(x) x^j \right)$ are matrix-valued polynomials.

$P_n(x),\ n = (S+1)m + i,\ m \in \mathbb{N},\ i = 0, 1, \ldots, S$ forms a complete set in $\mathcal{M}_d[x]$ and are independent over $\mathcal{M}$.

Like classical orthogonal polynomials, they satisfy a three-term relation, orthogonality-like relations, Rodrigues' formula.

Further questions

- Which linear equations are equivalent, in finite regions, to Fuchsian ones?
- Classification of linear equations.
- Classification of nonlinear perturbations of second order, and higher, differential equations.
- Study the Henon-Heiles system and other Hamiltonian systems with polynomial potential: for which parameters can they be linearized in selected regions?