Convergence of formal solutions of 1st order singular partial differential equations of nilpotent type

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Introduction and Known Results

Let $X = (x_1, \ldots, x_n) \in \mathbb{C}^n$.

We consider the following 1st order PDE;

$$(1) \qquad \left(\sum_{i,j=1}^{n} a_{ij} x_i \partial_{x_j} + c \right) u(X) = \sum_{i=1}^{n} b_j x_j + f_2(X, u, \partial_X u)$$

$$u(0) = 0,$$

Introduction and Known Results

Let $X = (x_1, \ldots, x_n) \in \mathbb{C}^n$.

We consider the following 1st order PDE;

$$(2) \qquad \left(\sum_{i,j=1}^{n} a_{ij} x_i \partial_{x_j} + c \right) u(X) = \sum_{i=1}^{n} b_j x_j + f_2(X, u, \partial_X u)$$

$$u(0) = 0,$$

where $a_{ij}, b_j, c \in \mathbb{C}$, $f_2(tX, t\xi, \eta) = O(t^2)$.

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$\boxed{\text{Problem}}$ In Theorem 1 (ii), if $\lambda_j = 0$ $({}^\forall j)$ and $c = 0$

$\Rightarrow$ Is $u(X)$ convergent?

A Typical Example

Let $X = (x, y, z) \in \mathbb{C}^3$. We consider the following simple operator of nilpotent type;

$$P := y\partial_x - z\partial_y = (x, y, z) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}.$$

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For this operator, we research on the conditions of the mapping on the convergent power series of the form

$$u(X) = \sum_{i,j=0}^{\infty} u_{ij}(x) y^i z^j \in \mathcal{O}_x\{y, z\} \text{ (or } \mathcal{O}_X).$$

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We put

$$\mathcal{O}_x[y, z]_p = \left\{ \sum_{i+2j=p} f_{ij}(x) y^i z^j \; ; \; f_{ij}(x) \in \mathcal{O}_x \right\}.$$

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$\boxed{\text{Remark}}$ The degree $i+2j$ is important.

$y\partial_x : u_{ij}(x) y^i z^j \mapsto u'_{ij}(x) y^{i+1} z^j$ (deg$= i+2j+1$)

$z\partial_y : u_{ij}(x) y^i z^j \mapsto i u_{ij}(x) y^{i-1} z^{j+1}$ (deg$= i+2j+1$)

$P : \mathcal{O}_x[y,z]_p \to \mathcal{O}_x[y,z]_{p+1}, \quad p = 0,1,2,\ldots.$

Result

For the mapping $P : \mathcal{O}_X \to \mathcal{O}_X$, we have

Theorem 2

$Pu(X) = f(X)$, we have
(i) $f(X) \in \mathrm{Im}(P; \mathcal{O}_X) \iff f(x,0,0) \equiv 0$ and

$$\sum_{k=0}^{n+1} \frac{(2k-1)!!}{(2n+1)!!} D_x^{n+1-k} f_{2k,n+1-k}(x) \equiv 0$$

for $n = 0, 1, 2, \ldots$, where
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$(2k-1)!! = 1 \cdot 3 \cdots (2k-1)$, $(-1)!! := 1$.

(ii) $\mathrm{Ker}(P; \mathcal{O}_X) \cong \mathcal{K}$ where

$$\mathcal{K} := \left\{ v(y,z) = \sum_{n=0}^{\infty} \sum_{k=0}^{m} C_{2n-2k,k} y^{2n-2k} z^{k} \in \mathcal{O}_{y,z} \right\}$$

$\boxed{\text{Remark}}$ Theorem 2 (i) $\Leftrightarrow$

$$\mathrm{Coker}(P;\mathcal{O}_X) \cong \mathcal{F} := \left\{ f(x,y) = \sum_{n=0}^{\infty} f_{2n,0}(x) y^{2n} \right\}$$

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Define

$$\begin{aligned} g(X) &= \sum_{i+2j=\mathrm{even}} g_{ij}(x) y^i z^j + \sum_{i+2j=\mathrm{odd}} g_{ij}(x) y^i z^j \\ &=: g_e(X) + g_o(X) \end{aligned}$$

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$$=: g_e(X) + g_o(X)$$

$\boxed{\text{Remark}}$ The isomorphism for $\mathrm{Ker}(P;\mathcal{O}_X)$ is given by

$$\mathcal{K} \ni v(y,z) \mapsto u(X) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} C_{2n-2k,k} (y^2+2xz)^{n-k} z^k$$

Theorem 2 will be proved by showing the unique solvability of the following Cauchy problem.

$$(3) \qquad \begin{cases} Pu(X) \equiv f(X) \pmod{\mathcal{F}}, \\ u_e(0, y, z) = v(y, z) \in \mathcal{K}. \end{cases}$$

Sketch of Proof

For $P = y\partial_x - z\partial_y : u(X) \mapsto f(X)$, since $P : \mathcal{O}_x[y,z]_p \to \mathcal{O}_x[y,z]_{p+1}$ $(p = 0, 1, 2, \ldots)$, we have

$$\mathcal{O}_x[y,z]_0 \ni f(x,0,0) \equiv 0.$$

Therefore, we put

$$f(X) = \sum_{p=1}^{\infty} f_p(x,y,z), \quad f_p(x,y,z) \in \mathcal{O}_x[y,z]_p.$$

where $f_p(x,y,z) = \sum_{i+2j=p} f_{ij}(x) y^i z^j \in \mathcal{O}_x[y,z]_p$.

By substituting $u(X) = \sum_{p \geq 0} u_p(X)$ and $f(X) = \sum_{p \geq 1} f_p(X)$ into $Pu(X) = f(X)$, we have the following recurrence formula:

$$(4) \qquad u'_{i,j}(x) - (i+2)u_{i+2,j-1}(x) = f_{i+1,j}(x).$$

This recurrence formula is expressed by using the matrix as follows;

The case for $p = 2n$,

$$\begin{pmatrix} D_x & 0 & \cdots & \cdots & 0 \\ -2n & D_x & 0 & \cdots & 0 \\ & -2n+2 & D_x & & 0 \\ & & \ddots & \ddots & \vdots \\ & & & -2 & D_x \end{pmatrix} \begin{pmatrix} u_{2n,0} \\ u_{2n-2,1} \\ u_{2n-4,2} \\ \vdots \\ u_{0,n} \end{pmatrix}$$

$$= \begin{pmatrix} f_{2n+1,0} \\ f_{2n-1,1} \\ \vdots \\ f_{1,n} \end{pmatrix}$$

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$$= \begin{pmatrix} f_{2n+1,0} \\ f_{2n-1,1} \\ \vdots \\ f_{1,n} \end{pmatrix}$$

This is uniquely solvable if we give initial values $\{u_{2n-2k,k}(0)\}$. (We put $C_{ij} = u_{ij}(0)$)

The case for $p = 2n+1$,

$$\begin{pmatrix} D_x & 0 & \cdots & \cdots & 0 \\ -2n-1 & D_x & 0 & \cdots & 0 \\ & -2n+1 & D_x & & 0 \\ & & \ddots & \ddots & \vdots \\ & & & -3 & D_x \\ & & & & -1 \end{pmatrix} \begin{pmatrix} u_{2n,0} \\ u_{2n-2,1} \\ u_{2n-4,2} \\ \vdots \\ u_{3,n-1} \\ u_{1,n} \end{pmatrix}$$

$$= \begin{pmatrix} f_{2n+2,0} \\ f_{2n,1} \\ \vdots \\ f_{2,n} \\ f_{0,n+1} \end{pmatrix}$$

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$$= \begin{pmatrix} f_{2n+2,0} \\ f_{2n,1} \\ \vdots \\ f_{2,n} \\ f_{0,n+1} \end{pmatrix}$$

The coefficient of above equation is $(n+2, n+1)$-rectangle matrix.
$\rightarrow$ We need some conditions.

Compatibility condition

In order that $f(X)$ $(= Pu(X))$ belongs to $\mathrm{Im}(P; \mathcal{O}_X)$, the given function $f(X)$ must satisfy the following compatibility conditions for $n = 0, 1, 2, \ldots$ from the bottom side.

$$\sum_{k=0}^{n+1} \frac{(2k-1)!!}{(2n+1)!!} D_x^{n+1-k} f_{2k,n+1-k}(x) \equiv 0 \tag{5}$$

This is Theorem 2 (i) in formal sense.

Next we research on $\mathrm{Ker}(P; \mathcal{O}_X)$, that is, we consider the equation

$$Pu(X) = 0 \iff Pu_p(X) = 0 \text{ for } p = 0, 1, \ldots.$$

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$$Pu(X) = 0 \ \Leftrightarrow \ Pu_p(X) = 0 \text{ for } p = 0, 1, \ldots.$$

In the case $p =$ odd, by the matrix representation, we obtain

$$u_{2n+1}(X) \equiv 0 \ \text{ for all } n.$$

Because, we can solve the system of equation from the bottom side step by step.

In the case $p =$ even, we calculate carefully from the top side, we have the following relations.

$$
\begin{aligned}
&u_{2n,0}(x) = C_{2n,0} \\
&u_{2n-2,1}(x) = C_{2n-2,1} + 2nC_{2n,0}x \\
&\qquad\qquad\qquad \vdots \\
&u_{2n-2k,k}(x) = \sum_{\ell=0}^{k} C_{2n-2\ell,\ell} \frac{(n-\ell)!}{(n-k)!(k-\ell)!} (2x)^{k-\ell}
\end{aligned}
$$

$\uparrow$

The coefficient of $y^{2n-2k} z^k$.

Therefore,

$$
\begin{aligned}
& u_{2n}(X) \\
& = \sum_{k=0}^{n} u_{2n-2k,k}(x) y^{2n-2k} z^k \\
& = \sum_{k=0}^{n} \sum_{\ell=0}^{k} C_{2n-2\ell,\ell} \frac{(n-\ell)!}{(n-k)!(k-\ell)!} (2x)^{k-\ell} y^{2n-2k} z^k \\
& = \text{careful calculation} \\
& = \sum_{\ell=0}^{n} C_{2n-2\ell,\ell} (y^2 + 2xz)^{n-\ell} z^{\ell}.
\end{aligned}
$$

Hence, the basis of $\mathrm{Ker}(P; \mathcal{O}_X)$ is

$$\{(y^2 + 2xz)^{n-\ell} z^\ell \ ; \ 0 \le \ell \le n, \ n = 0, 1, \ldots\}.$$

This implies that the kernel of P is given by

$$u(X) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} C_{2n-2\ell,\ell} (y^2 + 2xz)^{n-\ell} z^\ell.$$

This is Theorem 2 (ii) in formal sense.

Convergence of $u(X)$

We put $U(X) = u(X) - v(y, z)$ as a new unknown function.

$U(X)$ satisfies the following equation:

(6) $$\begin{cases} PU(X) \equiv F(X) \pmod{\mathcal{F}} \\ U_e(0) = 0 \end{cases}$$

Convergence of $u(X)$

We put $U(X) = u(X) - v(y,z)$ as a new unknown function.
$U(X)$ satisfies the following equation:

$$(7) \qquad \begin{cases} PU(X) \equiv F(X) \pmod{\mathcal{F}} \\ U_e(0) = 0 \end{cases}$$

$\boxed{\text{Remark}}$ $v(y,z)$ is any convergent even function. Then Pv is an odd function. The compatibility conditions for $f(X)$ are the conditions for the even part. Therefore, in this equation, the compatibility conditions for $F(X)$ are the same one as the original equation.

By the matrix representation, we have

$$U_{2n}(X) = \sum_{k=0}^{n} \sum_{\ell=0}^{k} 2^{k-\ell} \frac{(n-\ell)!}{(n-k)!} D_x^{\ell-k-1} F_{2(n-\ell)+1,\ell}(x) y^{2n-2k} z^k.$$

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We remark that $F(X)$ is holomorphic in a neighborhood of the origin. Therefore,

$$\sup_{|x|\le r} |F_{jk}(x)| \le CA^{j+2k} \quad ({}^{\exists}C, A > 0)$$

In this case, we have

$$\left| D_x^{\ell-k-1} F_{2(n-\ell)+1,\ell}(x) \right| \le CA^{2n+1} \frac{|x|^{k-\ell+1}}{(k-\ell+1)!}$$

By using this estimate, we obtain

$$|U_{2n}(X)| \leq CA^{2n+1}|x|(|y|^2 + 2|xz| + |z|)^n.$$

Hence, $U_e(X) = \displaystyle\sum_{n=0}^{\infty} U_{2n}(X)$ is convergent in a neighborhood of the origin.

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Hence, $U_e(X) = \sum_{n=0}^{\infty} U_{2n}(X)$ is convergent in a neighborhood of the origin.

$\boxed{\text{Remark}}$ The estimate of even part $U_e(X)$ is dominated by the integrations of the coefficients of $F(X)$.

For the odd part, we have

$$U_{2n+1}(X)$$

$$= \sum_{k=0}^{n} \sum_{\ell=0}^{k} \frac{(2(k-\ell)-1)!!}{(2k+1)!!} D_x^{\ell} F_{2(k-\ell),n-k+\ell+1}(X) y^{2k+1} z^{n-k}$$

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$$= \sum_{k=0}^{n} \sum_{\ell=0}^{k} \frac{(2(k-\ell)-1)!!}{(2k+1)!!} D_x^\ell F_{2(k-\ell),n-k+\ell+1}(X) y^{2k+1} z^{n-k}$$

By $\sup_{|x|\le r} |F_{ij}(x)| \le CA^{i+2j}$, we have for $r' < r$,

$$\left| D_x^\ell F_{2(k-\ell),n-k+\ell+1}(x) \right|$$

$$= \left| \frac{\ell!}{2\pi i} \oint_{|z-x|=r-r'} \frac{F_{2(k-\ell),\ell}(\xi)}{(\xi - x)^{\ell+1}} d\xi \right|$$

$$\le CA^{2n+2} B^\ell \ell! \quad \text{(where } \; B = 1/(r-r')).$$

By using this estimate, we obtain

$$
\begin{aligned}
&|U_{2n+1}(X)| \\
&= \left| \sum_{k=0}^{n} \sum_{\ell=0}^{k} \frac{(2(k-\ell)-1)!!}{(2k+1)!!} D_x^{\ell} F_{2(k-\ell),n-k+\ell+1}(X) y^{2k+1} z^{n-k} \right| \\
&\le \sum_{k=0}^{n} \sum_{\ell=0}^{k} \frac{(2k-2\ell-1)!!\ell!}{(2k+1)!!} \frac{CA^{2n+2}B^{k+1}}{B-1} |y|^{2k+1} |z|^{n-k} \\
&\le \sum_{k=0}^{n} \frac{CA^{2n+2}B^{k+1}}{B-1} |y|^{2k+1} |z|^{n-k} \\
&= \frac{CA^{2n+2}B|y|}{B-1} (B|y|^2 + |z|)^n
\end{aligned}
$$

Therefore, $U_o(X) = \sum_{n=0}^{\infty} U_{2n+1}(X)$ is convergent in a neighborhood of the origin.

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$\boxed{\text{Remark}}$ The estimate of odd part $U_o(X)$ is dominated by the differentiations of the coefficients of $F(X)$. This is quite different situation of the case for even part.

Problem

We consider the following operator with perturbation terms;

$$\tilde{P} = (y + a(X))\partial_x + (z + b(X))\partial_y + c(X)\partial_z\,,$$

where $a(X)$, $b(X)$ and $c(X)$ are holomorphic in a neighborhood of the origin, and assume

$$a(X), b(X), c(X) = O(|X|^2).$$

Moreover, we assume that

$$\sigma(a) \geq 1, \quad \sigma(b) \geq 2, \quad \sigma(c) \geq 3.$$

where $\sigma(f)$ is defined as follows:

$$f(X) = \sum f_{ij}(x) y^i z^j \;\Rightarrow\; \sigma(f) := \min\{i+2j \,;\, f_{ij}(x) \neq 0\}.$$

Moreover, we assume that

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Under these conditions, we have the following results:

For $\mathrm{Im}(\tilde{P}, \mathcal{O}_x[[y,z]])$ $\rightarrow$ similar compatibility conditions.

For $\mathrm{Ker}(\tilde{P}, \mathcal{O}_x[[y,z]])$ $\rightarrow$ similar conditions.

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I am not successful to prove convergence yet.
I am trying now.

Thank you.